Title: Classical and Quantum Chaos 2021/2022 - Lecture 5

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Collection: Classical and Quantum Chaos 2021/2022

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Lorenz model

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz$$

Parameters $\sigma, r, b > 0$

Equilibrium points. -

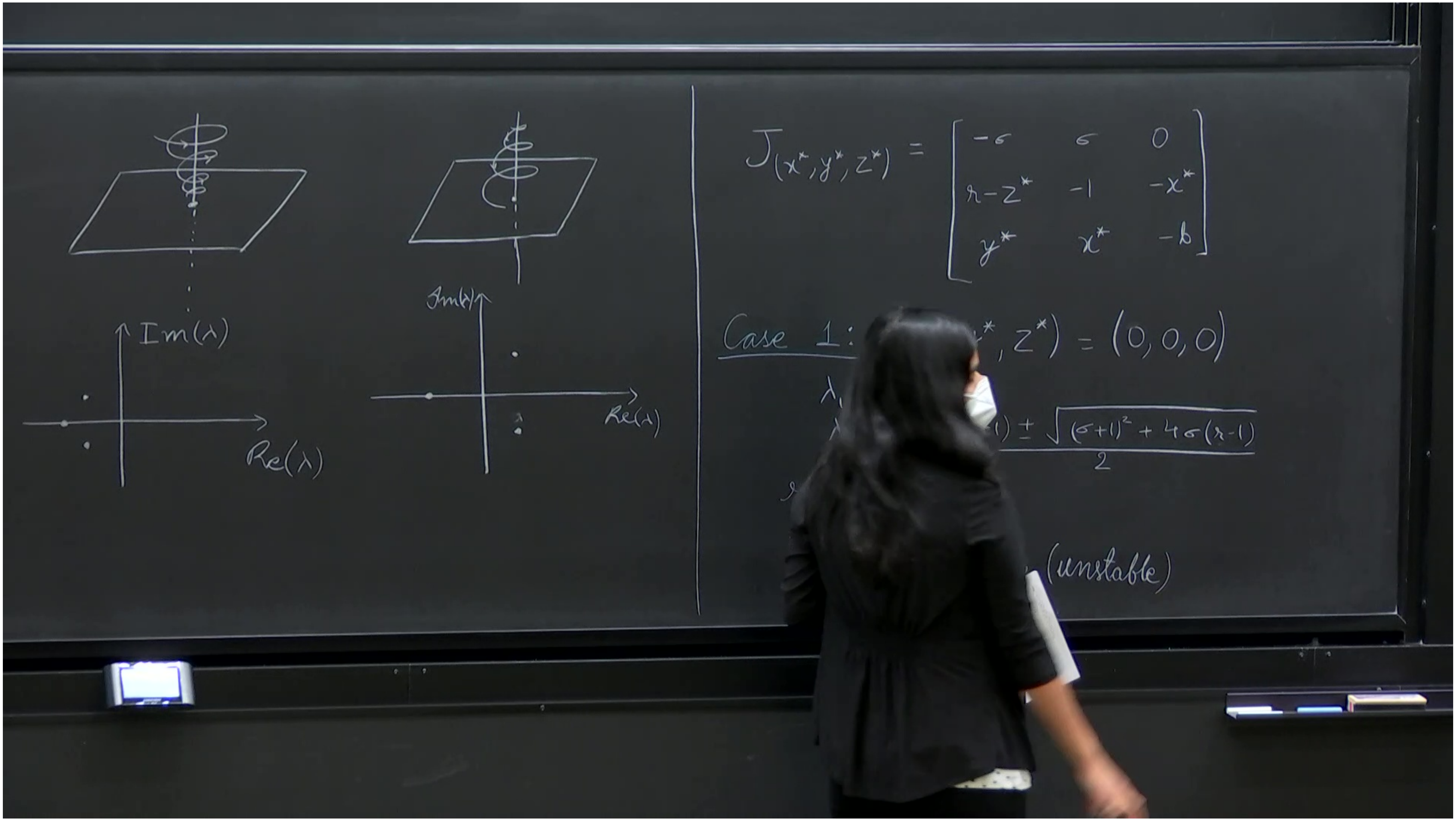
$\dot{x} = \dot{y} = \dot{z} = 0$

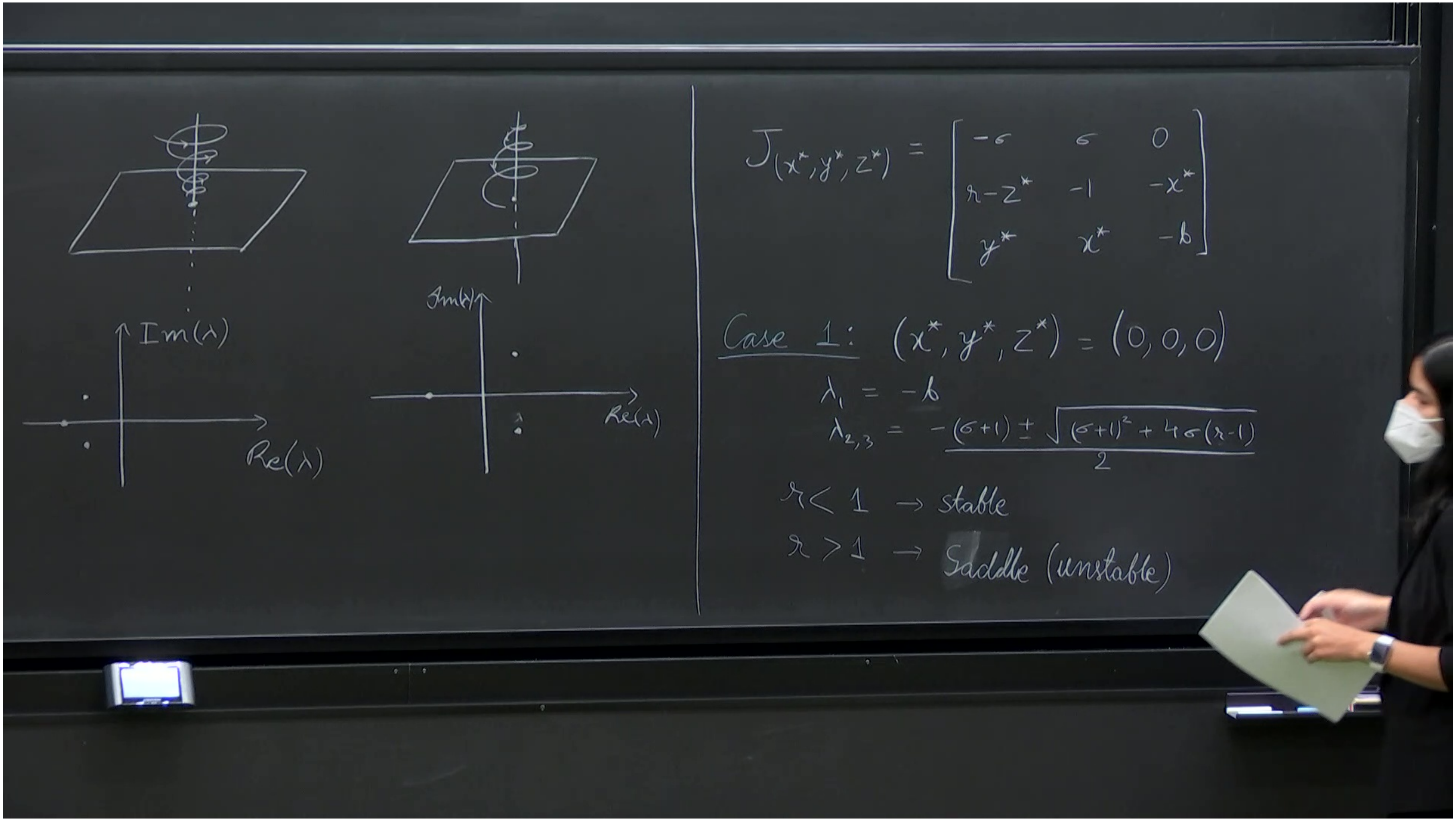
Solutions (x^*, y^*, z^*)

$$C_0 = (0,0,0)$$
$$C_{\pm} = \left(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1\right)$$

For $r < 1$, only C_0 is the eq. pt.

For $r > 1$, all 3 exist.


Im(λ)
Re(λ)
Im(λ)
Re(λ)
J(x*, y*, z*) = [-σ σ 0 ; r-z* -1 -x* ; y* x* -b]
Case 1:
, z*) = (0, 0, 0)
± √((σ+1)² + 4σ(r-1))
2
(unstable)


Im(λ)
Re(λ)
Im(λ)
Re(λ)
$J_{(x^*, y^*, z^*)} = \begin{bmatrix} -\sigma & \sigma & 0 \\ r - z^* & -1 & -x^* \\ y^* & x^* & -b \end{bmatrix}$
Case 1: $(x^*, y^*, z^*) = (0,0,0)$
$\lambda_1 = -b$
$\lambda_{2,3} = \frac{-(\sigma+1) \pm \sqrt{(\sigma+1)^2 + 4\sigma(r-1)}}{2}$
$r < 1 \rightarrow$ stable
$r > 1 \rightarrow$ Saddle (unstable)

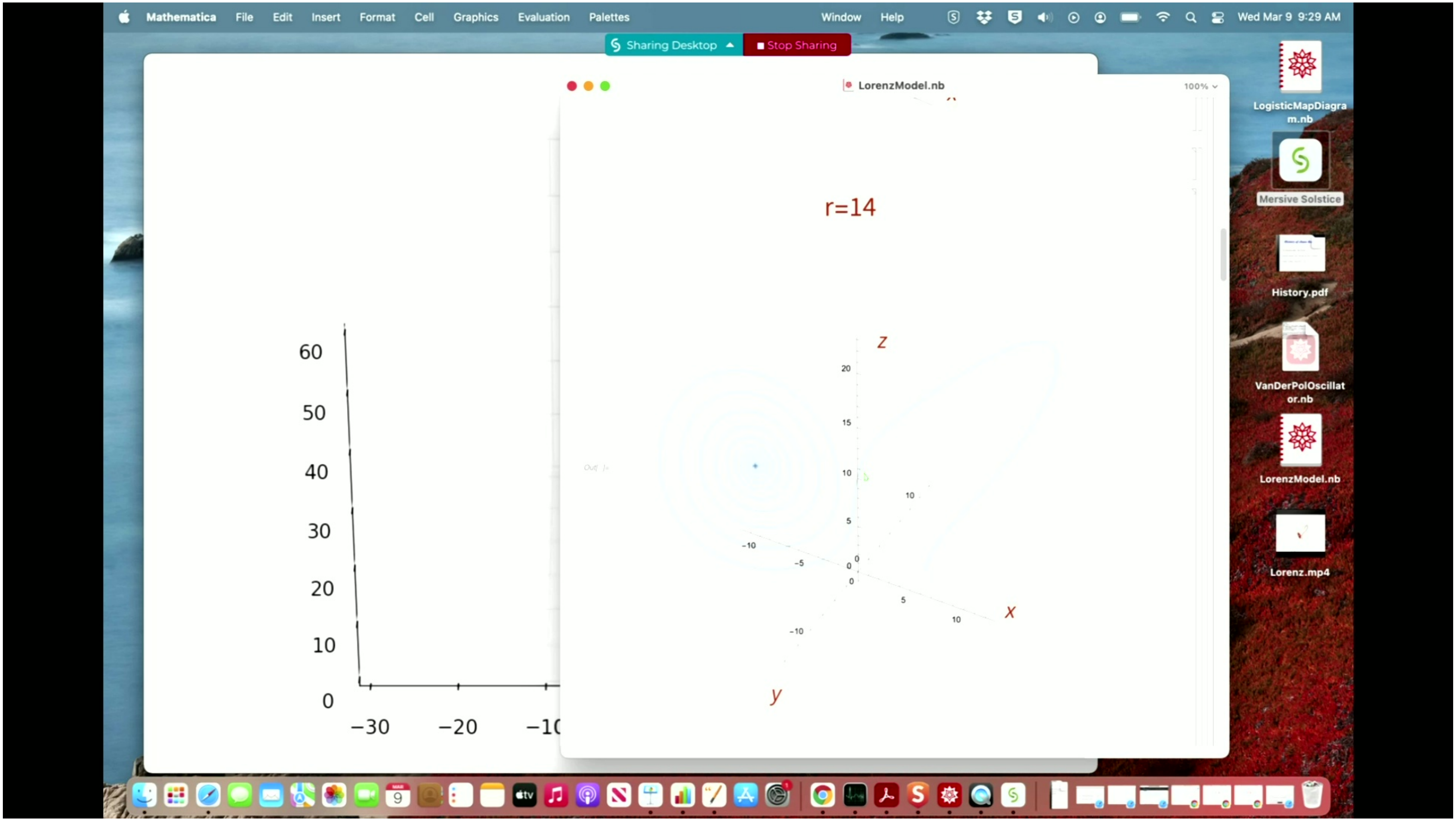
Case 2 :- $(x^*, y^*, z^*) = \left(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1 \right)$
$= C_\pm$

Stable for $r < r_c = \dfrac{\sigma(\sigma+b+3)}{\sigma-b-1}$

$\sigma = 10, \quad b = 8/3$

$r_c = \dfrac{470}{19} \approx 24.74$

So, $C_\pm$ stable for $1 < r \lesssim 24.74$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Wed Mar 9 9:29 AM
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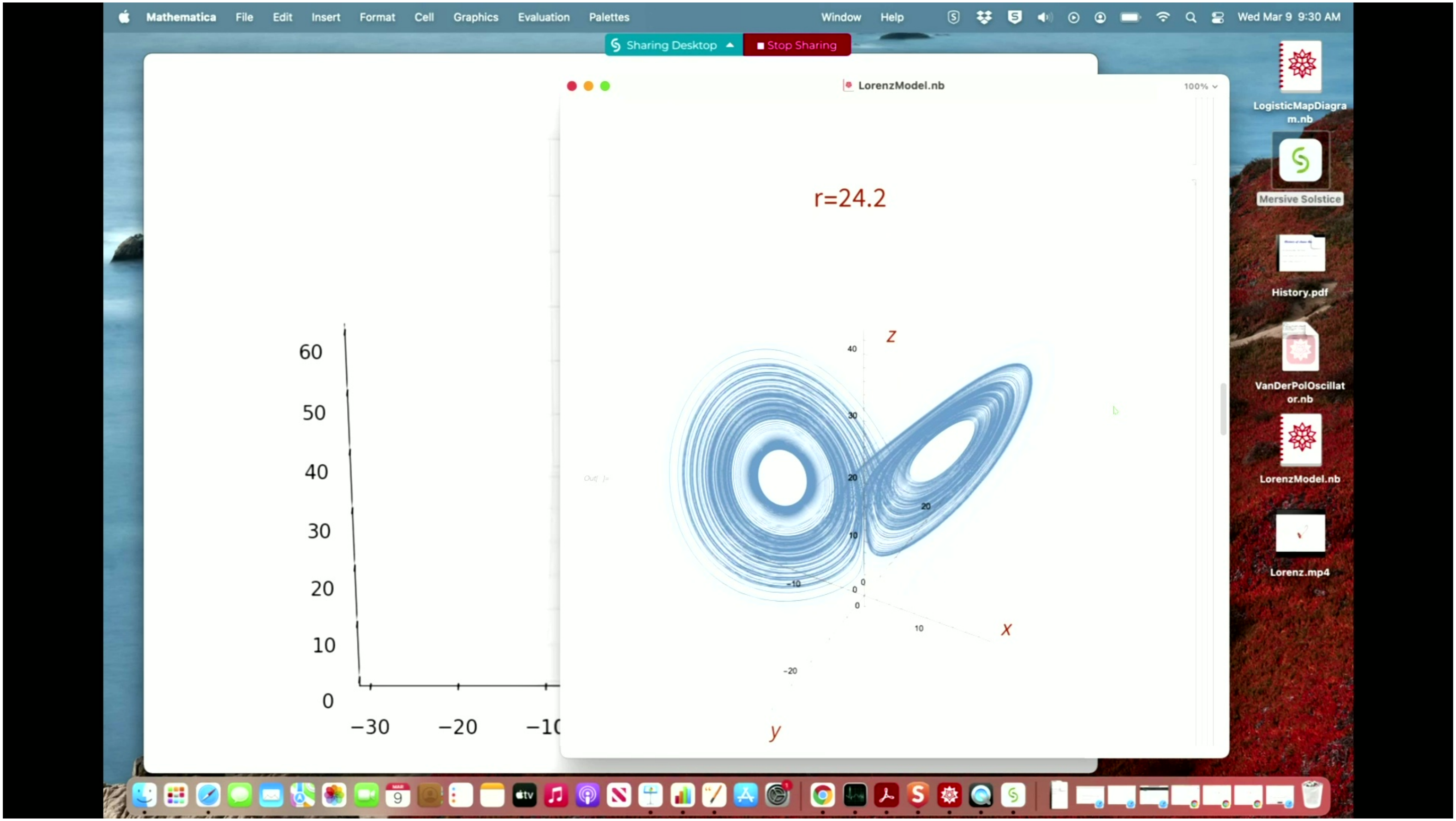
LorenzModel.nb
100%
r=14
z
20
15
10
5
0
-10
-5
0
0
5
10
10
-10
x
y
60
50
40
30
20
10
0
−30
−20
LogisticMapDiagram.nb
Mersive Solstice
History.pdf
VanDerPolOscillator.nb
LorenzModel.nb
Lorenz.mp4



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100%
r=24.2
z
x
y
60
50
40
30
20
10
0
−30
−20
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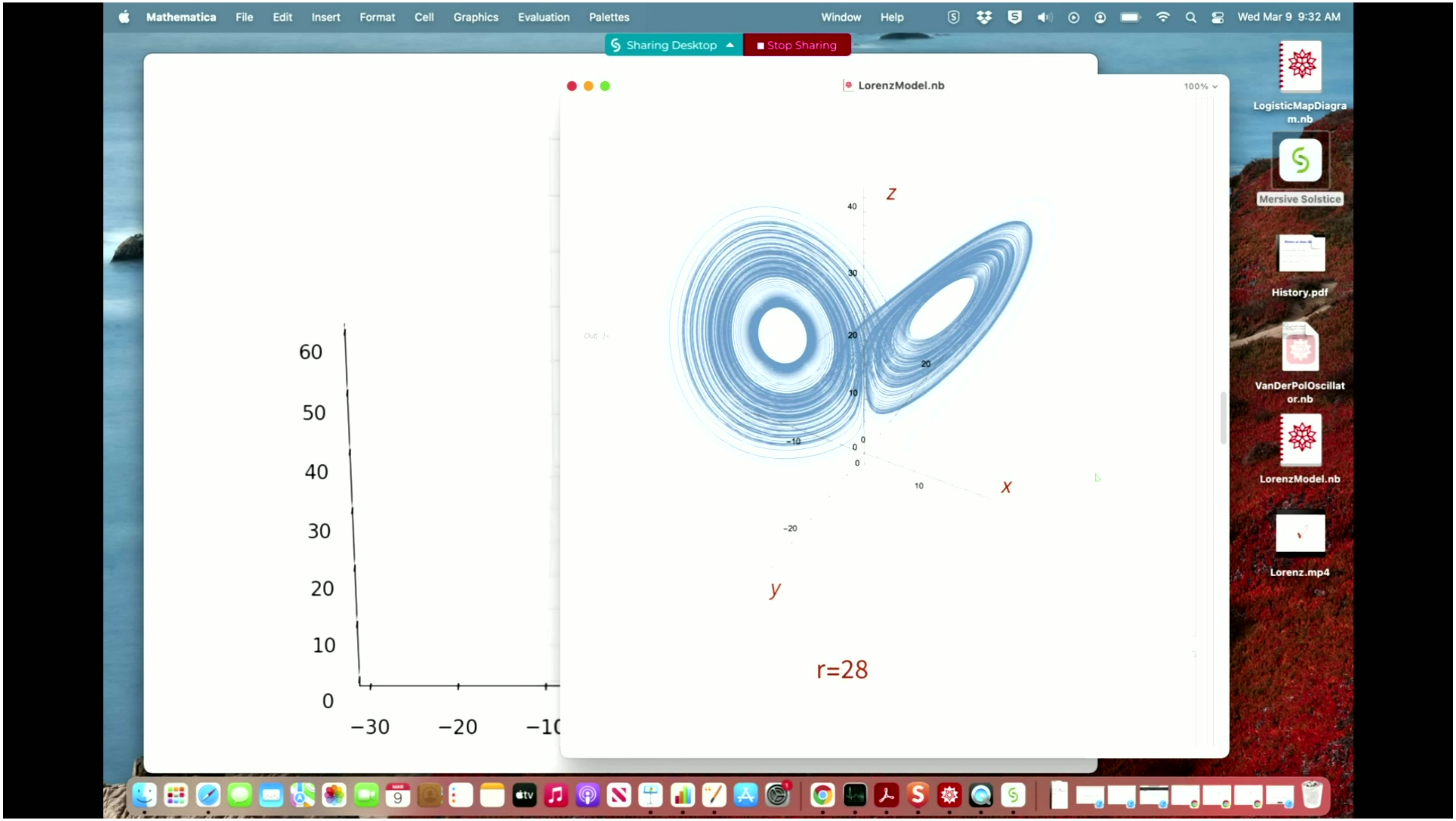
History.pdf
VanDerPolOscillator.nb
LorenzModel.nb
Lorenz.mp4



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LorenzModel.nb
100%
z
40
30
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10
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-10
0
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10
x
-20
y
r=28
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History.pdf
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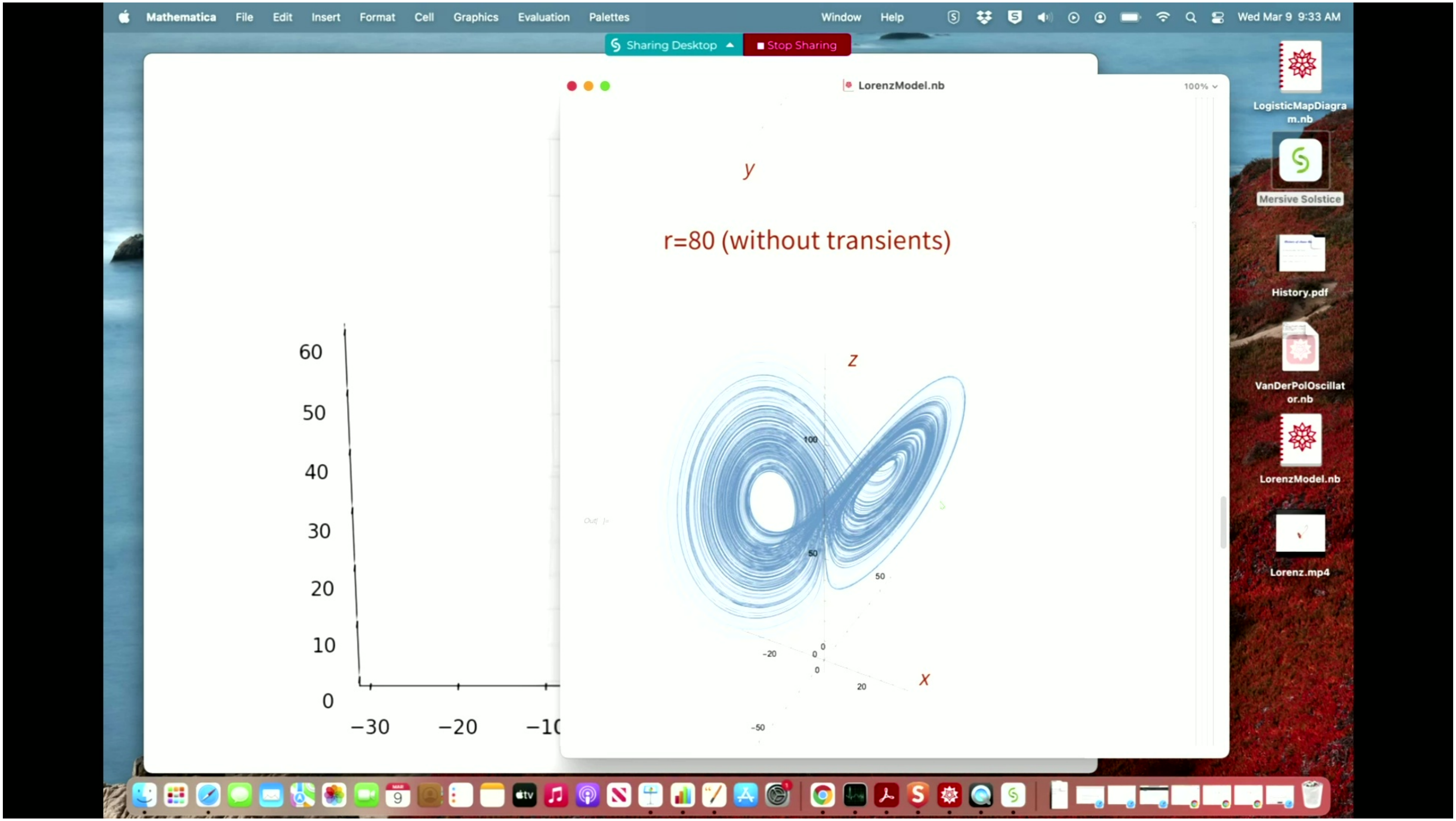
Lorenz.mp4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LorenzModel.nb
y
r=80 (without transients)
z
x
60
50
40
30
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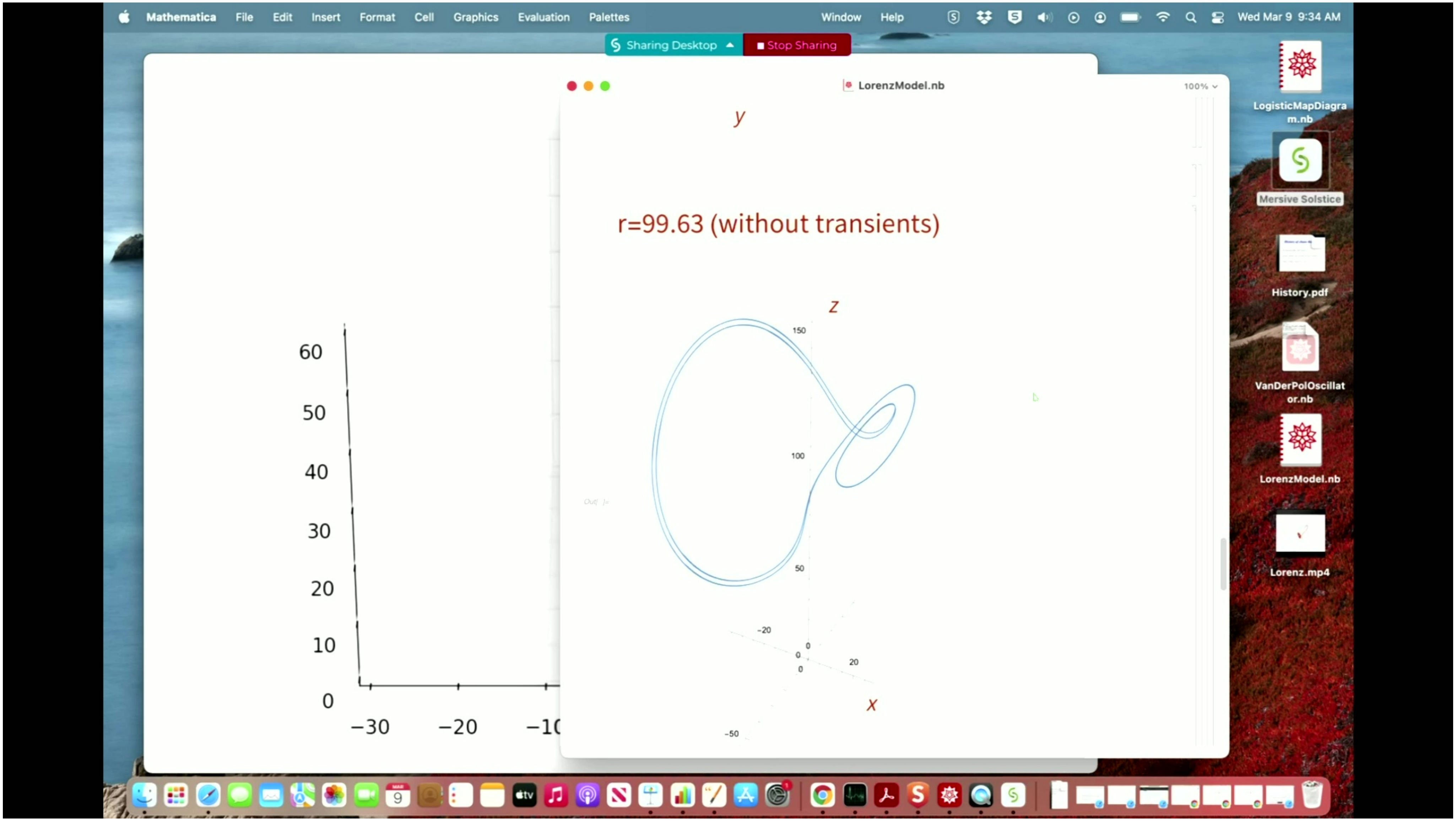
Mersive Solstice
History.pdf
VanDerPolOscillator.nb
LorenzModel.nb
Lorenz.mp4



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LorenzModel.nb
y
r=99.63 (without transients)
z
150
100
50
-20
0
0
0
20
x
-50
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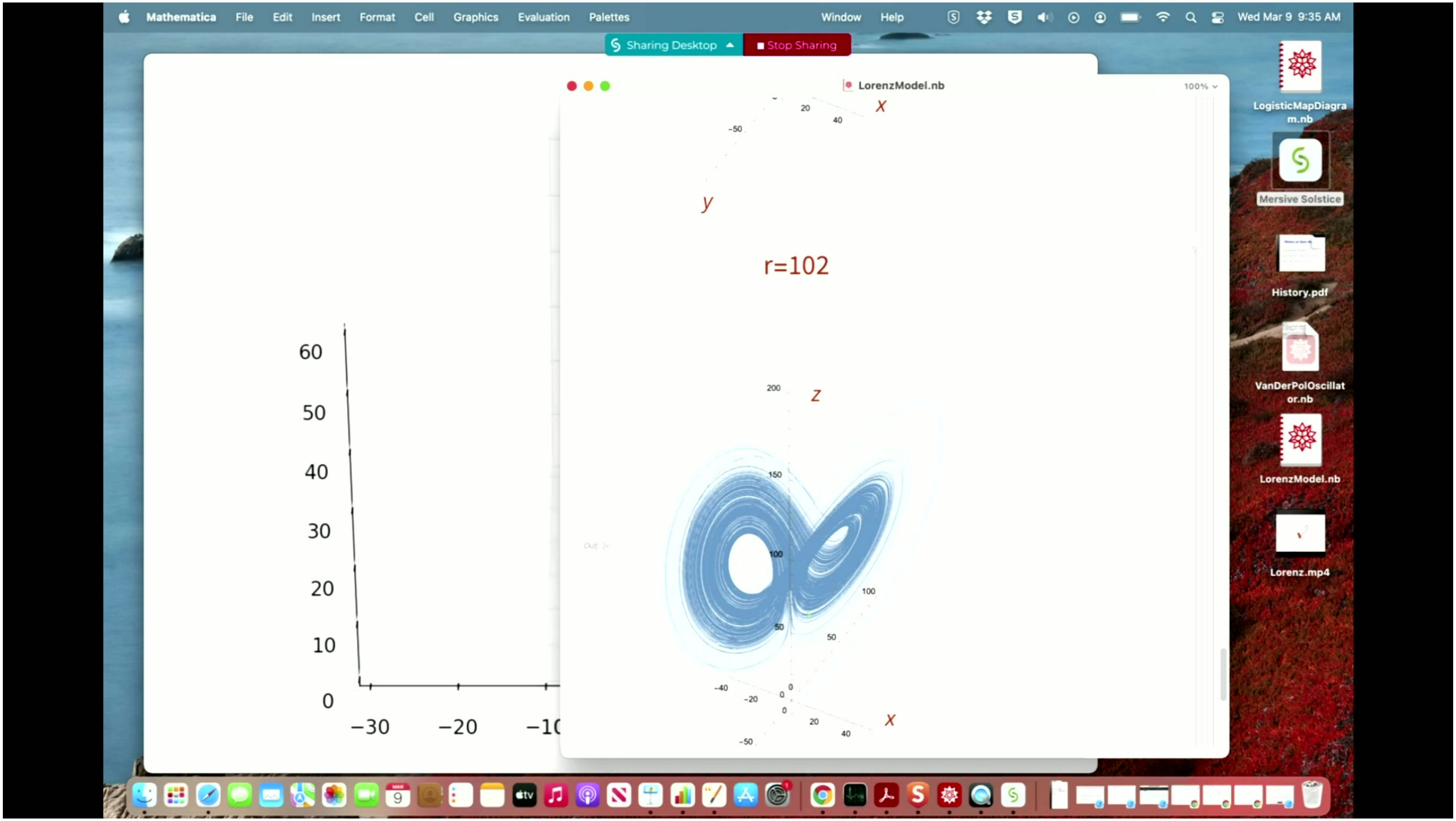
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LorenzModel.nb
r=102
x
y
z
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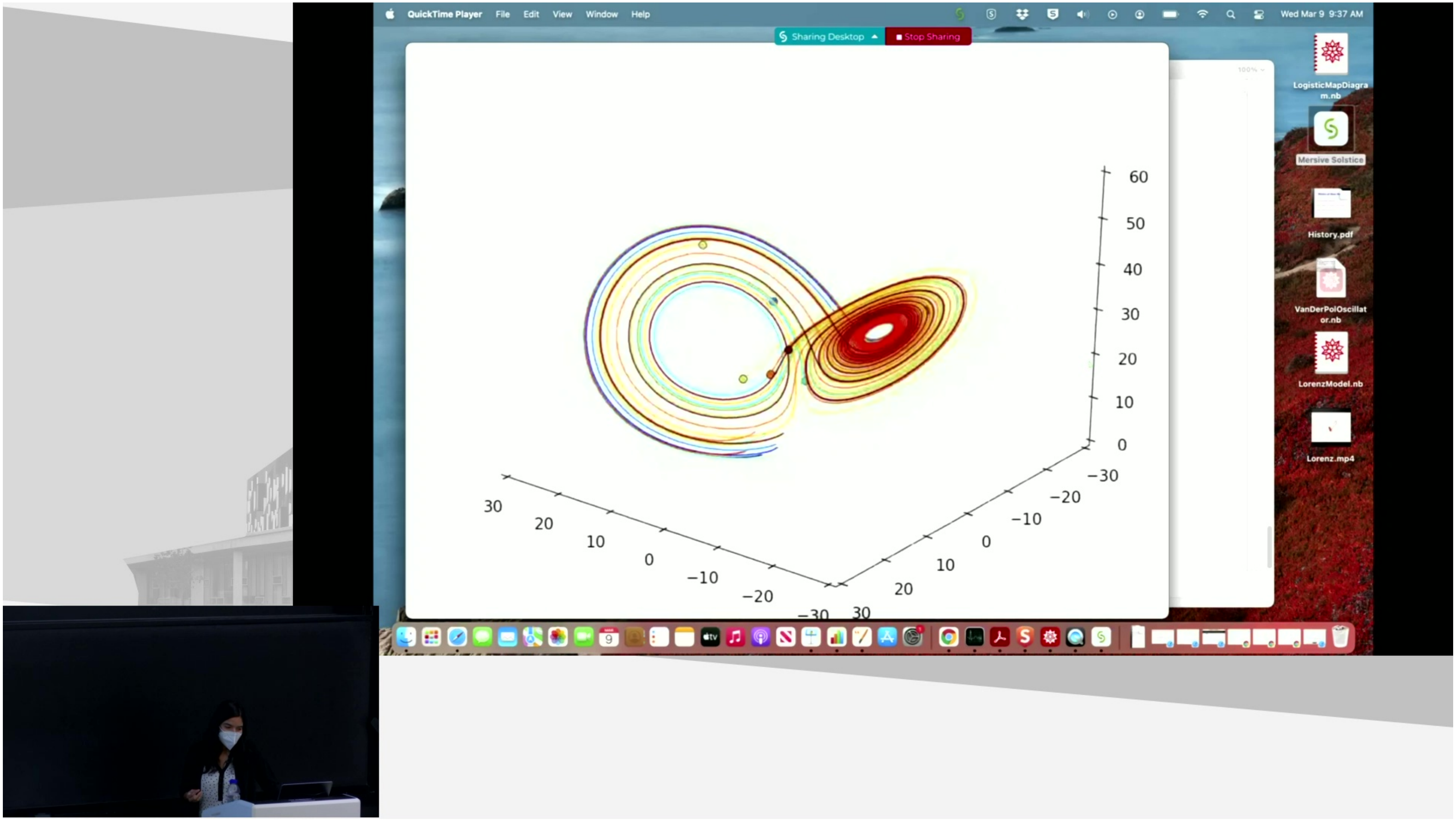
History.pdf
VanDerPolOscillator.nb
LorenzModel.nb
Lorenz.mp4



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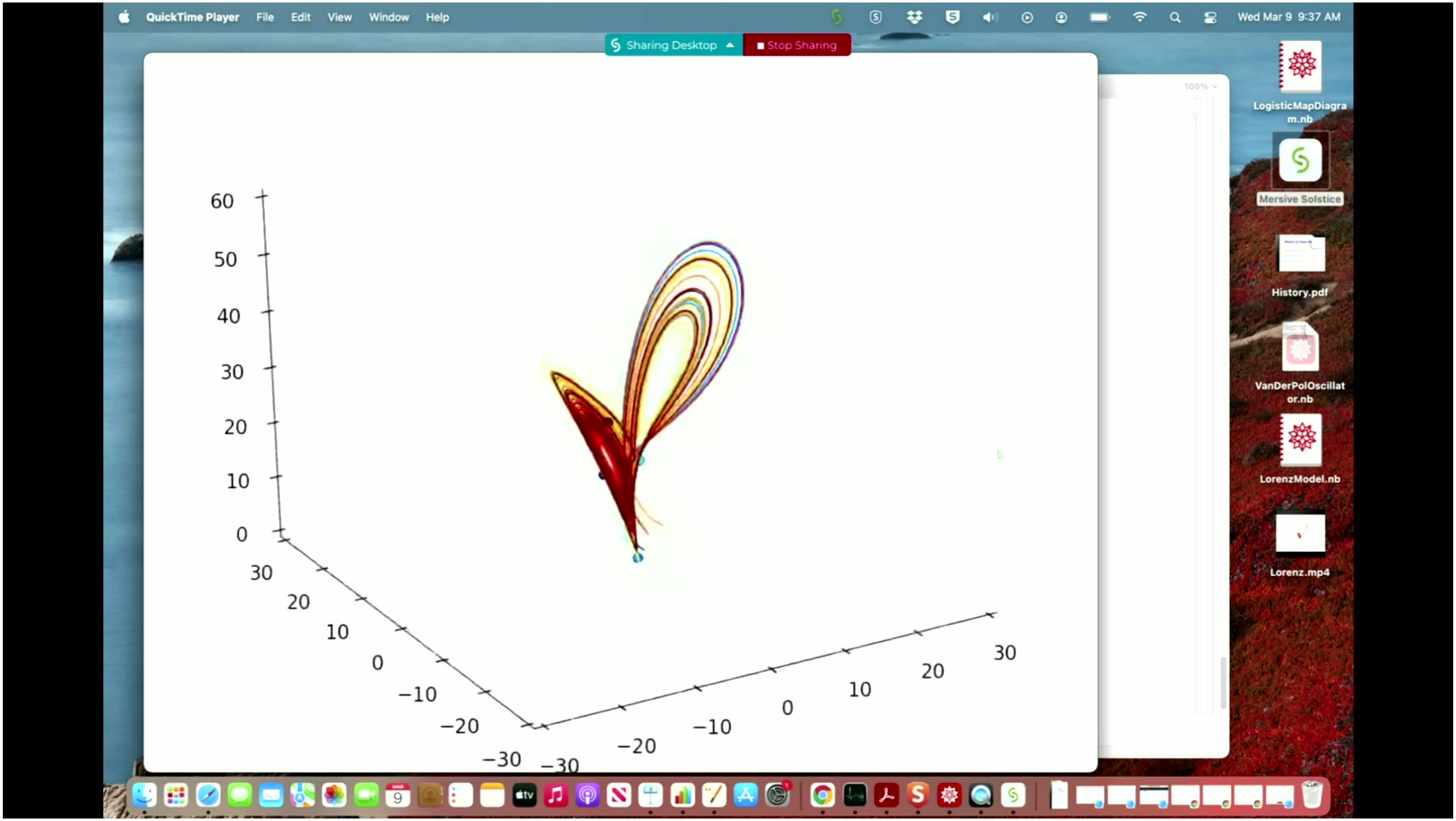
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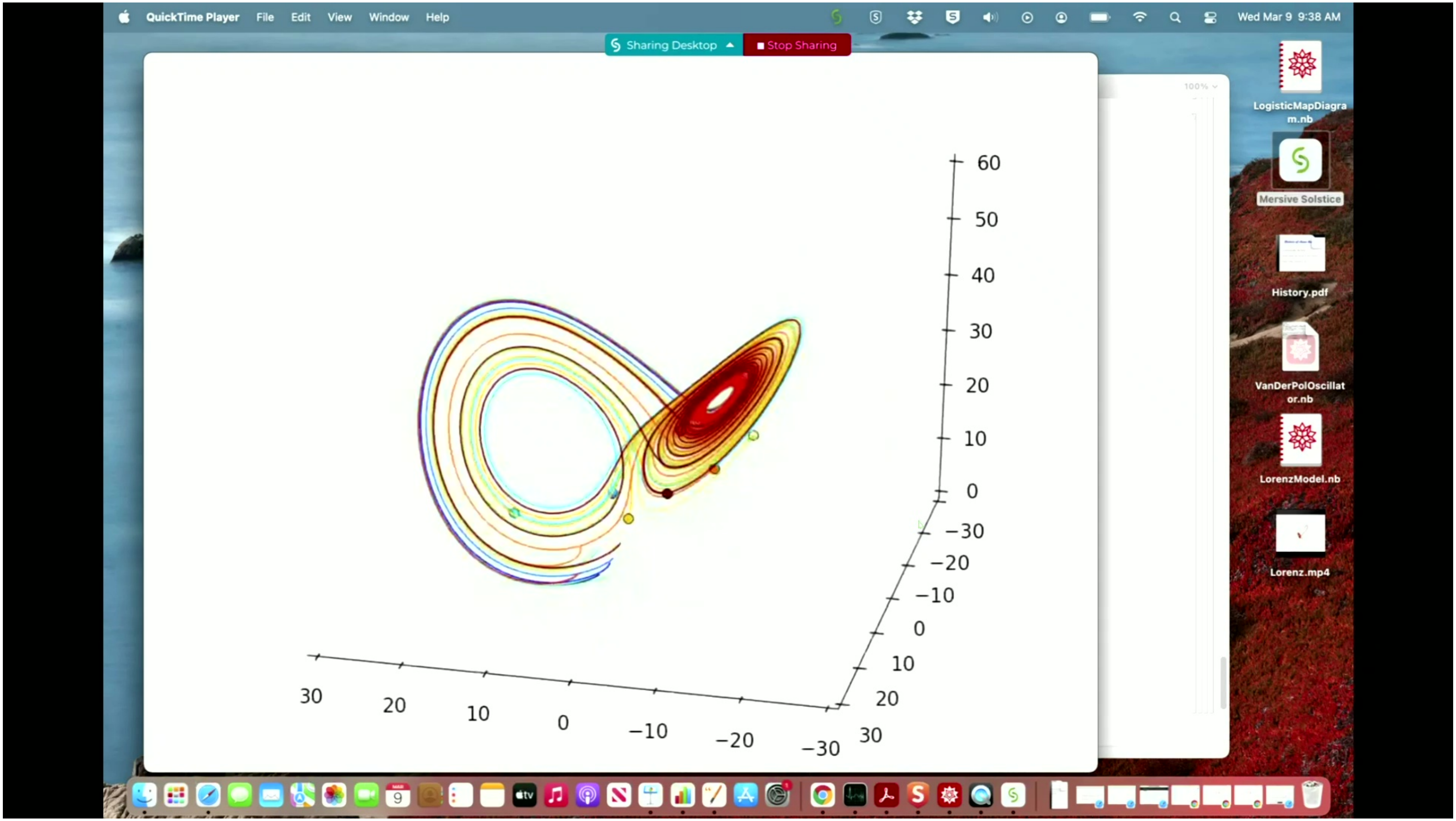
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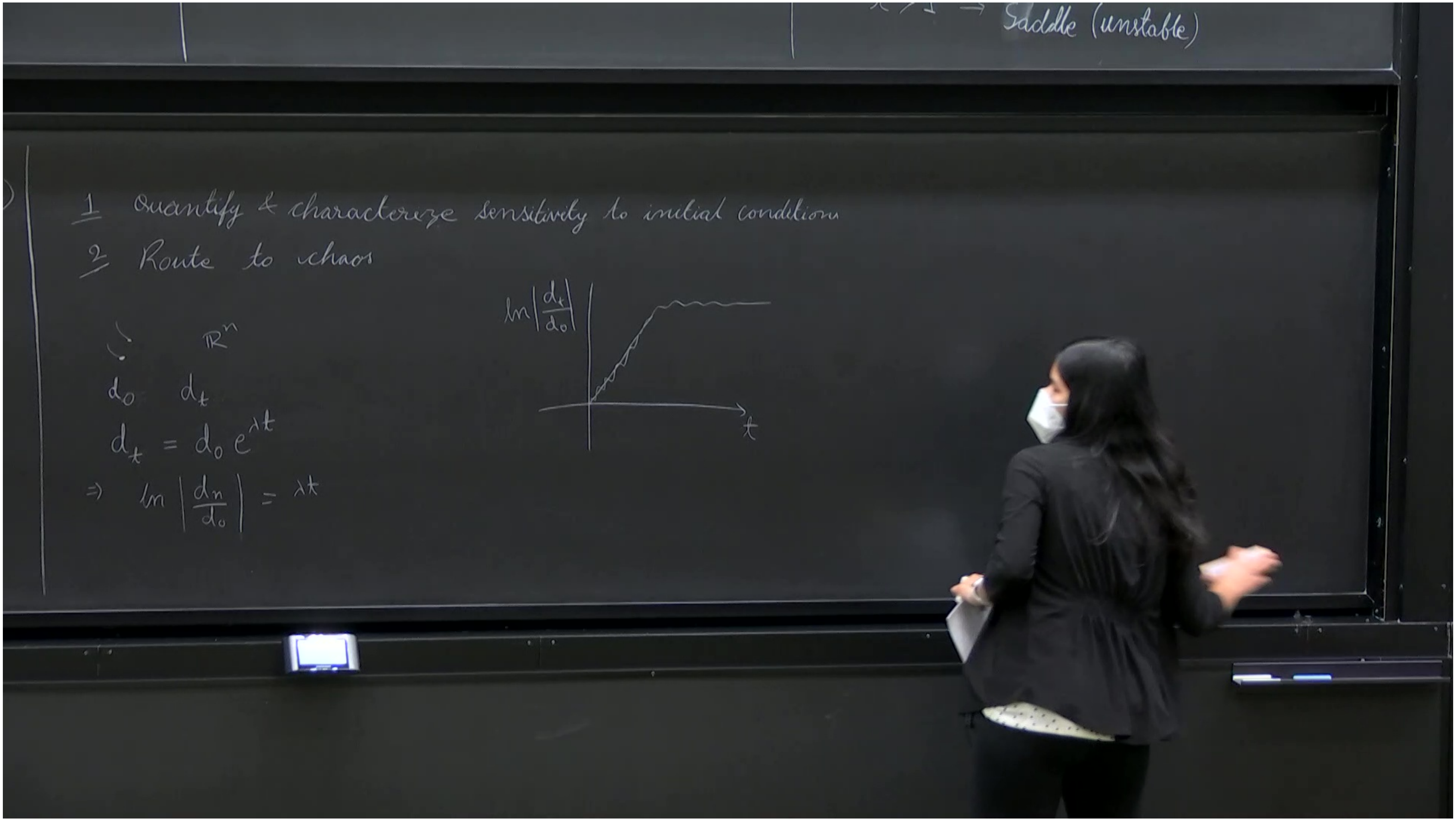
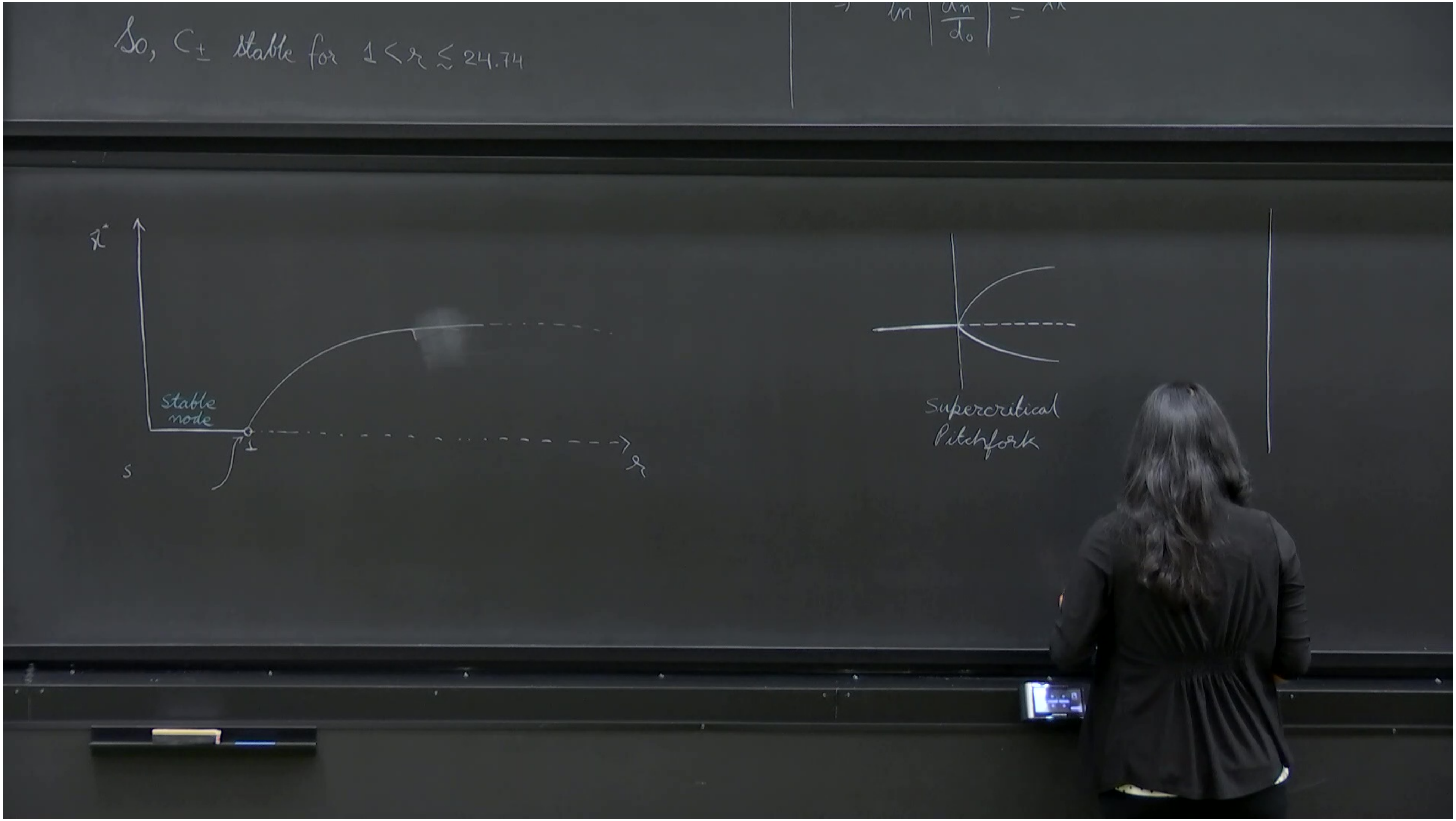
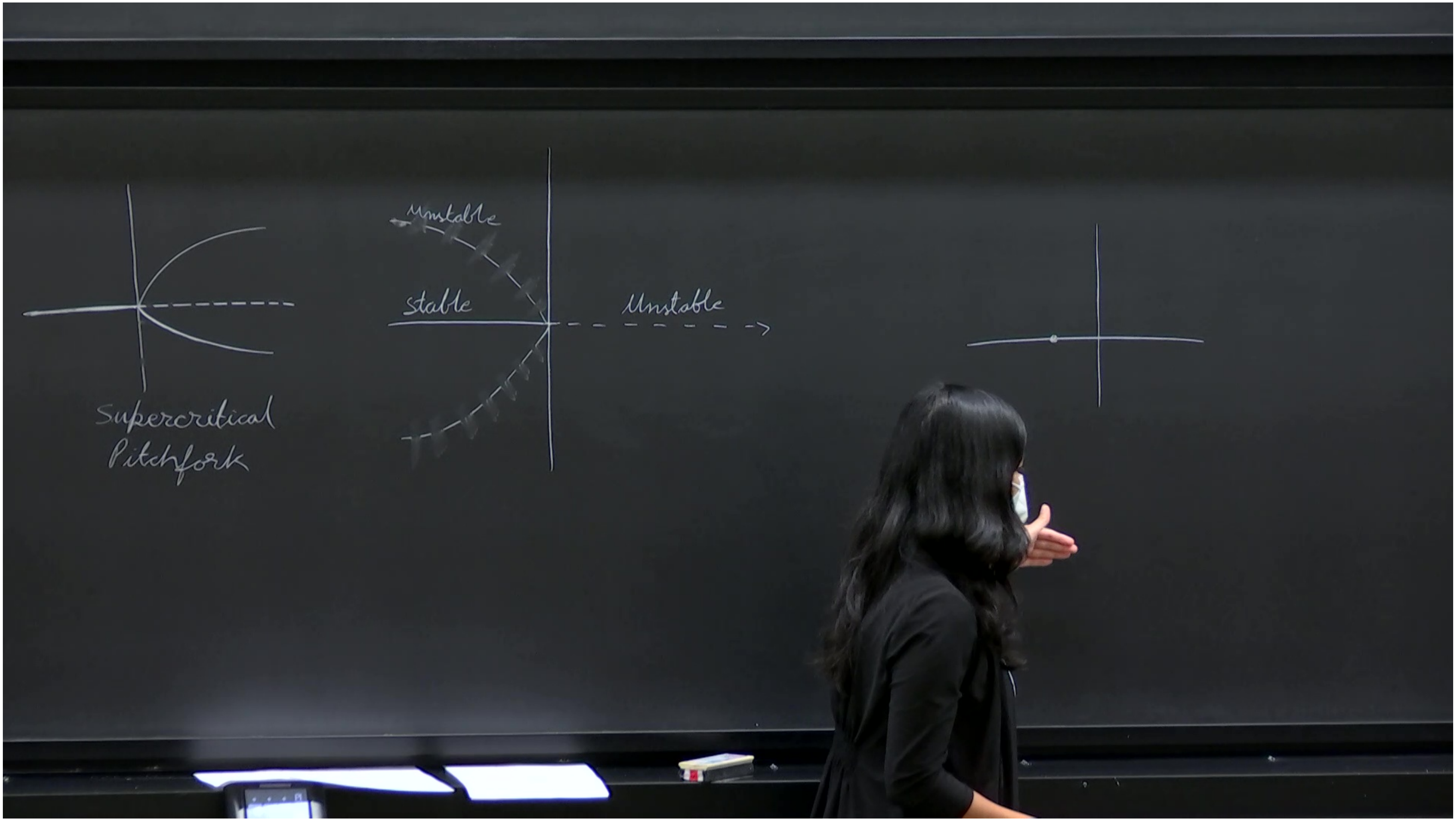
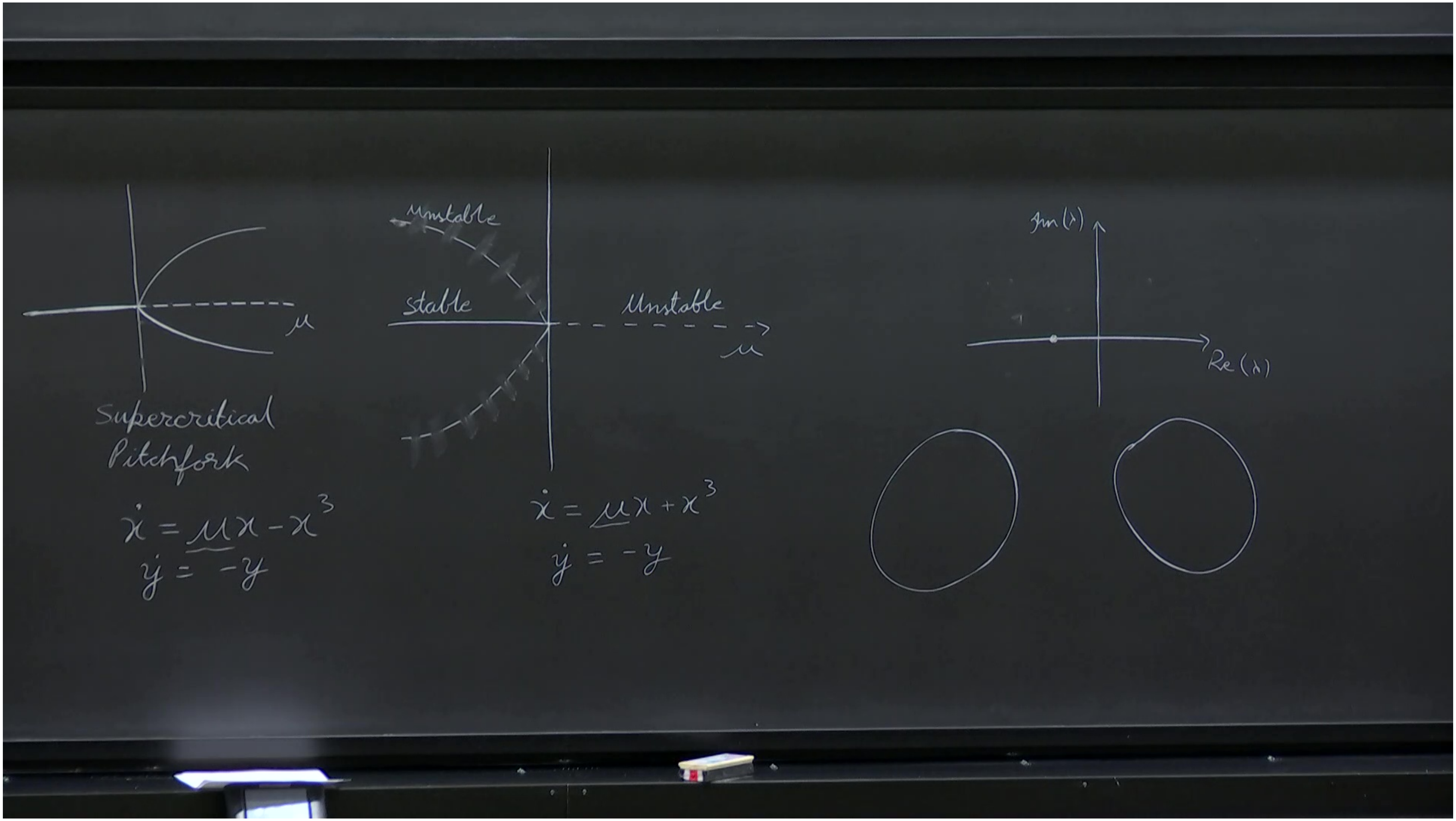
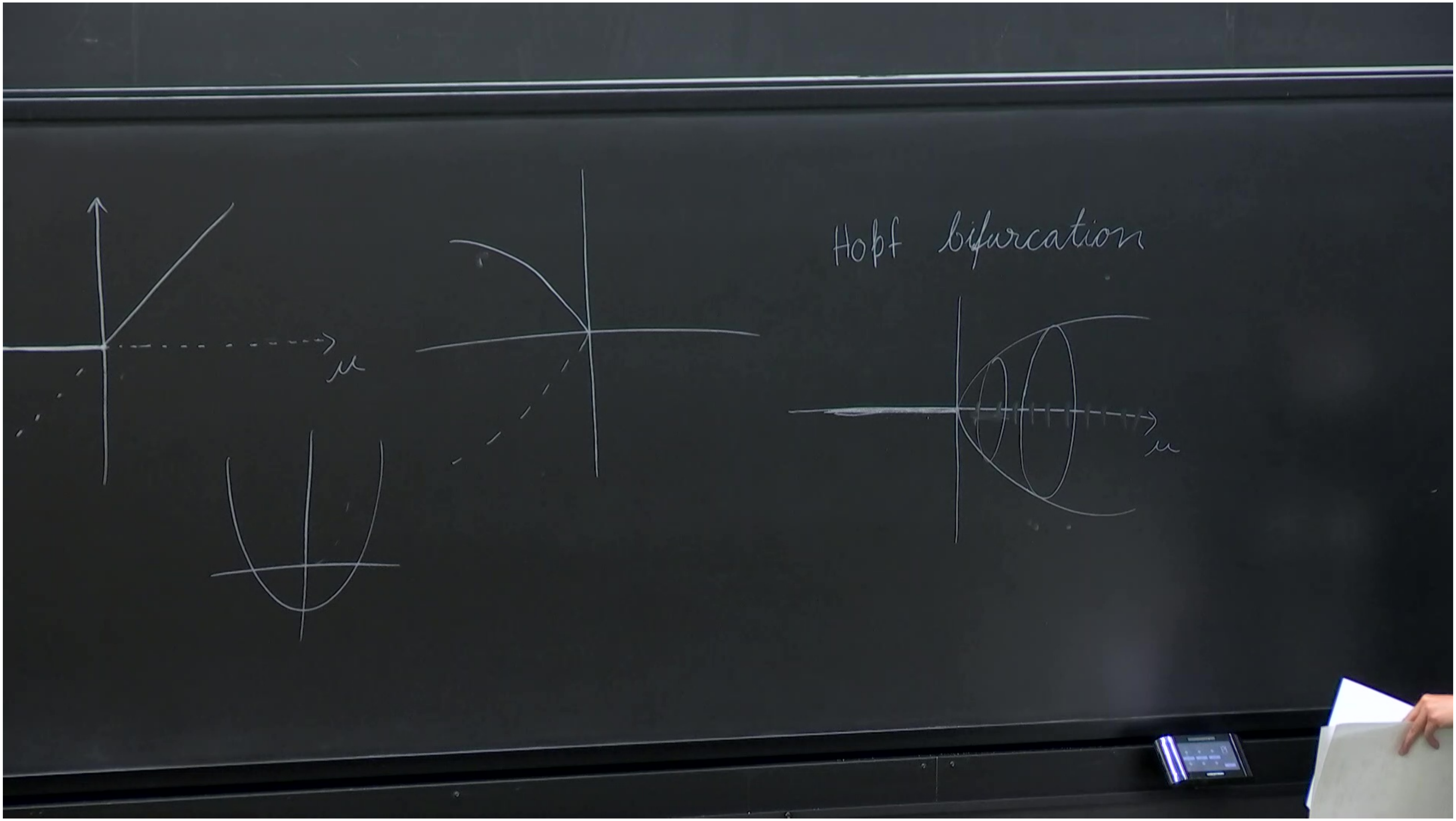

Saddle (unstable)
1 Quantify & characterize sensitivity to initial conditions
2 Route to chaos
$\mathbb{R}^n$
d_0 d_t
$d_t = d_0 e^{\lambda t}$
$\Rightarrow \ln\left|\frac{d_n}{d_0}\right| = \lambda t$
$\ln\left|\frac{d_t}{d_0}\right|$
t

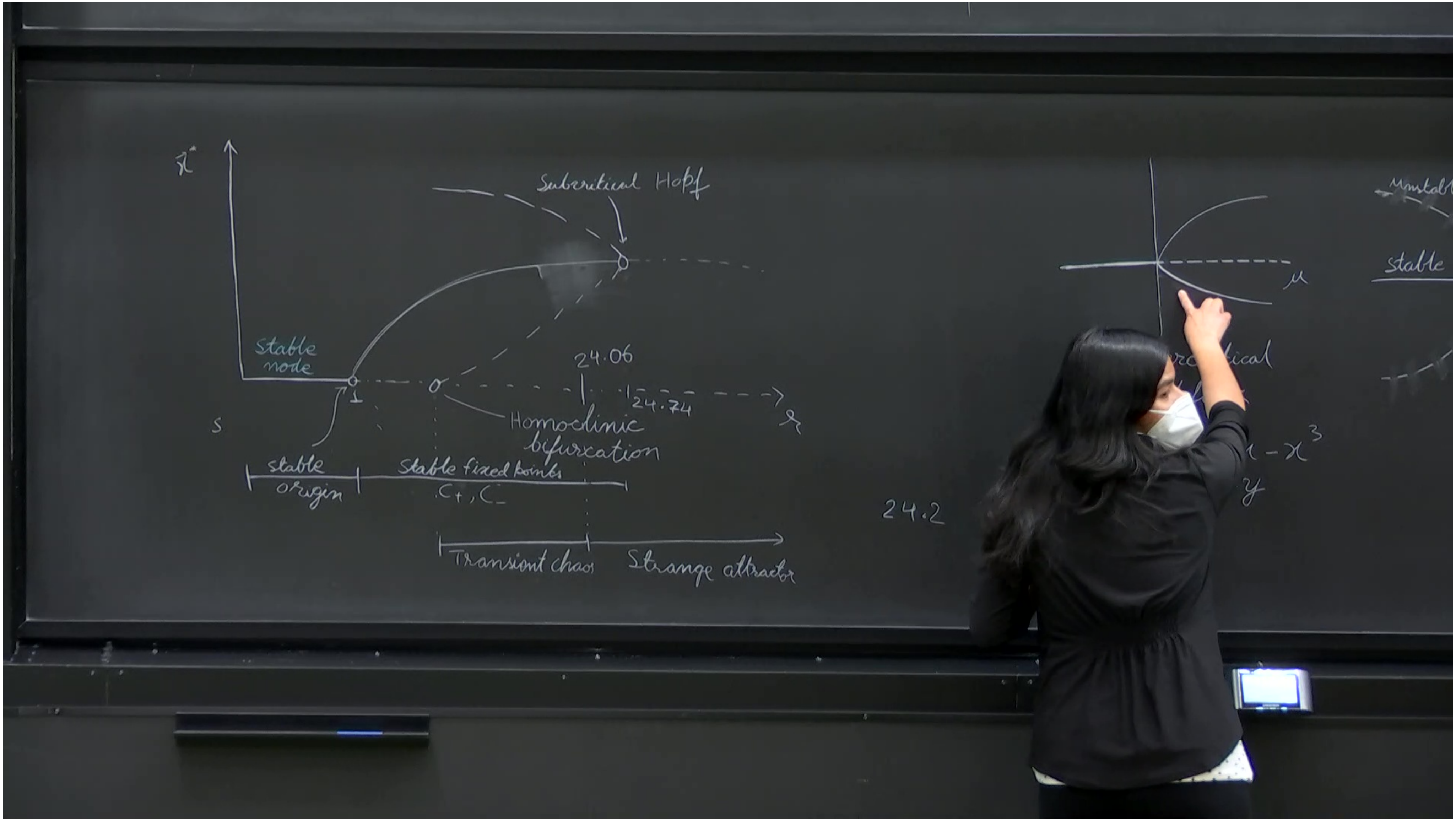

So, C± stable for 1 < r ≲ 24.74
stable node
Supercritical Pitchfork


Supercritical
Pitchfork
Unstable
stable
Unstable


μ
Supercritical
Pitchfork
$\dot{x} = \mu x - x^3$
$\dot{y} = -y$
Unstable
stable
Unstable
μ
$\dot{x} = \mu x + x^3$
$\dot{y} = -y$
$Im(\lambda)$
$Re(\lambda)$



Hopf bifurcation
μ
μ



x^*
Subcritical Hopf
Stable node
24.06
24.74
Homoclinic bifurcation
stable origin
Stable fixed points C_+, C_-
Transient chaos
Strange attractor
24.2
μ
stable