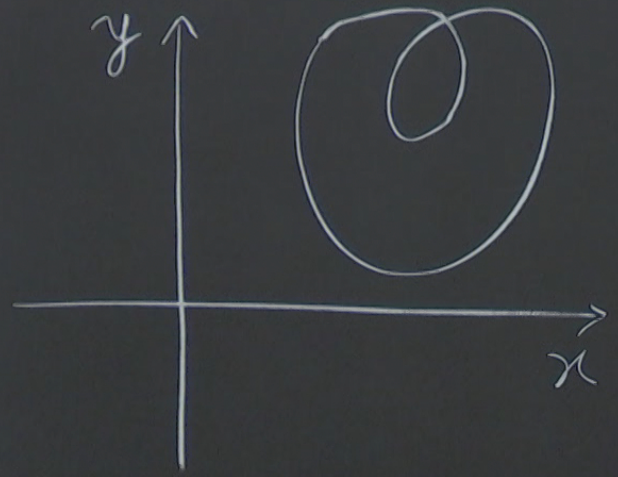
Title: Classical and Quantum Chaos 2021/2022 - Lecture 4

Speakers: Meenu Kumari

Collection: Classical and Quantum Chaos 2021/2022

Date: March 08, 2022 - 9:00 AM

URL: https://pirsa.org/22030049




1) Autonomous system

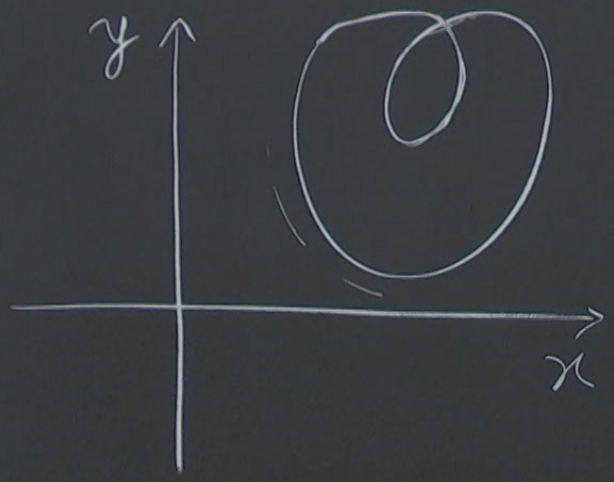
$$\dot{x} = P(x, y)$$

$$\dot{y} = Q(x, y)$$

2) Non-autonomous system

$$\dot{x} = P(x, y, t)$$

$$\dot{y} = Q(x, y, t)$$




1) Autonomous system

$$\dot{x} = P(x, y)$$

$$\dot{y} = Q(x, y)$$

2) Non-autonomous system

$$\dot{x} = P(x, y, t)$$

$$\dot{y} = Q(x, y, t)$$

$$z = t$$

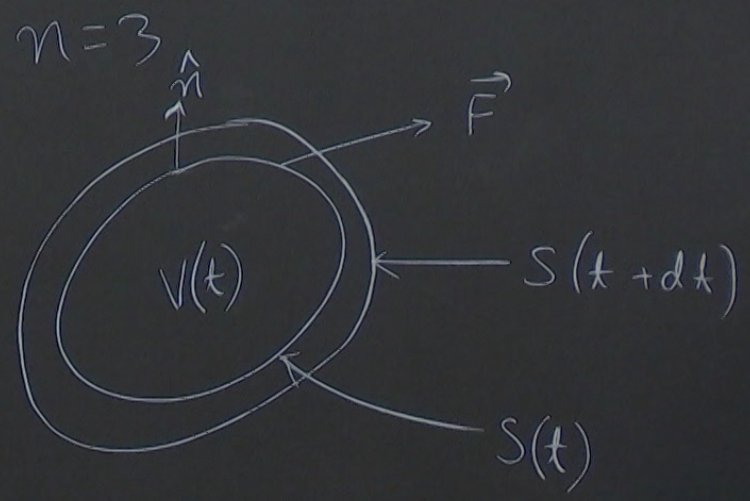
$$\dot{z} = 1$$

$$\left(P(x_A, y_A, t_1), Q(x_A, y_A, t_1)\right) \neq \left(P(x_A, y_A, t_2), Q(x_A, y_A, t_2)\right)$$

$\vec{\nabla} \cdot \vec{f}$

$\dot{X} = F(X), \quad X \in \mathbb{R}^n$

$F = (f_1, f_2 \ldots f_n)$




$\hat{n}$ = outward normal

$\vec{F}$ gives ins. Vel. of points

$\vec{F} \cdot \hat{n}$

$\vec{F}$

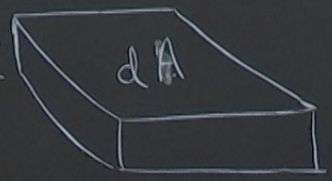
$S(t+dt)$

$S(t)$

...rnal

...el of points

$\vec{F}\cdot\hat{n}\,dt$




$\text{vol} = (\vec{F}\cdot\hat{n}\,dt)\,dA$

$V(t+dt) = V(t) + \text{vol}$ swept by tiny patches of surface integrated over all patches

$= V(t) + \int_S (\vec{F}\cdot\hat{n}\,dt)\,dA$

$$\dot{V} = \frac{V(t+dt) - V(t)}{dt} = \int_S (\vec{F} \cdot \hat{n})\, dA$$

$$\Rightarrow \dot{V} = \int_V \vec{\nabla} \cdot \vec{F}\, dV \quad \text{(using div. theorem)}$$

If $\vec{\nabla} \cdot \vec{F} = \text{const} = C_1$

$$\Rightarrow \frac{dV}{dt} = \int C_1 V$$

$$\Rightarrow \int_{V(0)}^{V(t)} \frac{dV}{V} = c_1 \int dt$$

$$\Rightarrow \ln\left|\frac{V(t)}{V(0)}\right| = c_1 t$$

$$\Rightarrow V(t) = V(0)\, e^{c_1 t}$$

$c_1 = \vec{\nabla}\cdot\vec{F}$ $<0 \Rightarrow V(0)$ contract

$>0 \Rightarrow$ " expand

$=0 \Rightarrow V(0)$ remains const upon evolution

$$H = H(x_i, p_i, t)$$

$$\dot{x} = -\frac{\partial H}{\partial p_i}$$

$$\dot{y} = \frac{\partial H}{\partial x_i}$$

$$\vec{\nabla} \cdot \vec{F} = \sum_i \frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i} = \sum_i \frac{\partial^2 H}{\partial q_i \partial p_i} - \frac{\partial^2 H}{\partial p_i \partial q_i} = 0$$

1) Stable
   a) Focus
   b) node
   } Sinks

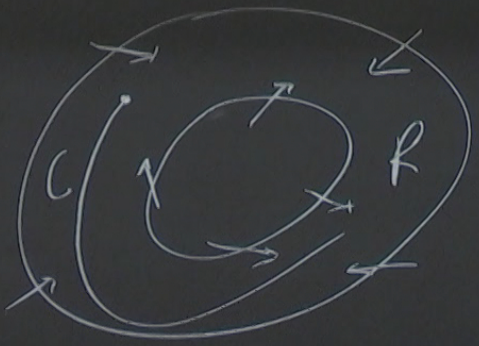
2) Unstable
   a) Focus
   b) node
   } Sources

3) Saddles —

4) Center —

inks

ources




Trapping region :- Closed connected set such that the vector fied points "inward" everywhere on the boundary of R.

## Non-existence

Bendixson's criterion :-

If $\frac{\partial P}{\partial x}$ and $\frac{\partial Q}{\partial y}$ are cont. in a region $R$

which is "simply connected" and

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \neq 0$$ at any point in $R$

then $\left( \dot{x} = P(x,y),\ \dot{y} = Q(x,y) \right)$ has no closed orbits in $R$

Attractors :- An attractor is a closed set A with following properties :-

1) A is an invariant set - any trajectory $X(t)$ that starts in A stays in A for all times.

2 A attracts an open set of initial conditions:- there exists an open set U contains A such that if $X(0) \in U$, then the distance from $X(t)$ to A tends to 0 as $t \to \infty$.

The largest $U$ is called "basin of attraction"

3 $A$ is minimal :- There is no proper subset of $A$ that satisfies 1 and 2

$$\dot{x} = x - x^3$$
$$\dot{y} = -y$$

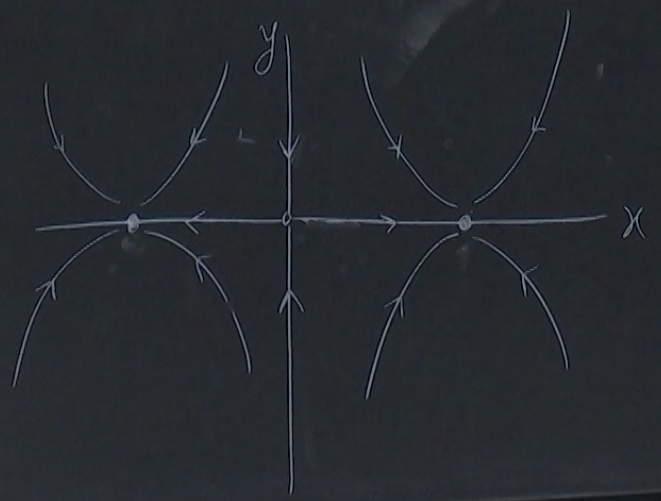
$\dot{x} = 0 \Rightarrow x = 0, \pm 1$

$\dot{y} = 0 \Rightarrow y = 0$

$(0,0)$ = unstable (saddle)

$(\pm 1, 0)$ = stable

$I : -1 \leq x \leq 1, y = 0$. Is $I$ an attractor.




$(\pm 1, 0)$ = attractor

# Lorenz model

$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = rx - y - xz$$

$$\dot{z} = xy - bz$$

$\sigma, r, b > 0$

$\sigma$: Prandtl number

$r$: Rayleigh number

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z}$$

$$= -\sigma - 1 - b$$

$$V(t) = V(0)\, e^{-(\sigma + 1 + b)t}$$

Volumes in phase space shrink exponentially fast.

odel

$y - x)$

$-y - xz$

$z - bz$

$> 0$

— Rayleigh number

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z}$$

$$= -\sigma - 1 - b$$

$$V(t) = V(0)\, e^{-(\sigma + 1 + b)t}$$

Volumes in phase space shrink exponentially fast.

Fixed points, limit cycles — stable or saddle

odel

$y - x)$

$-y - xz$

$z - bz$

$> 0$

— Rayleigh number

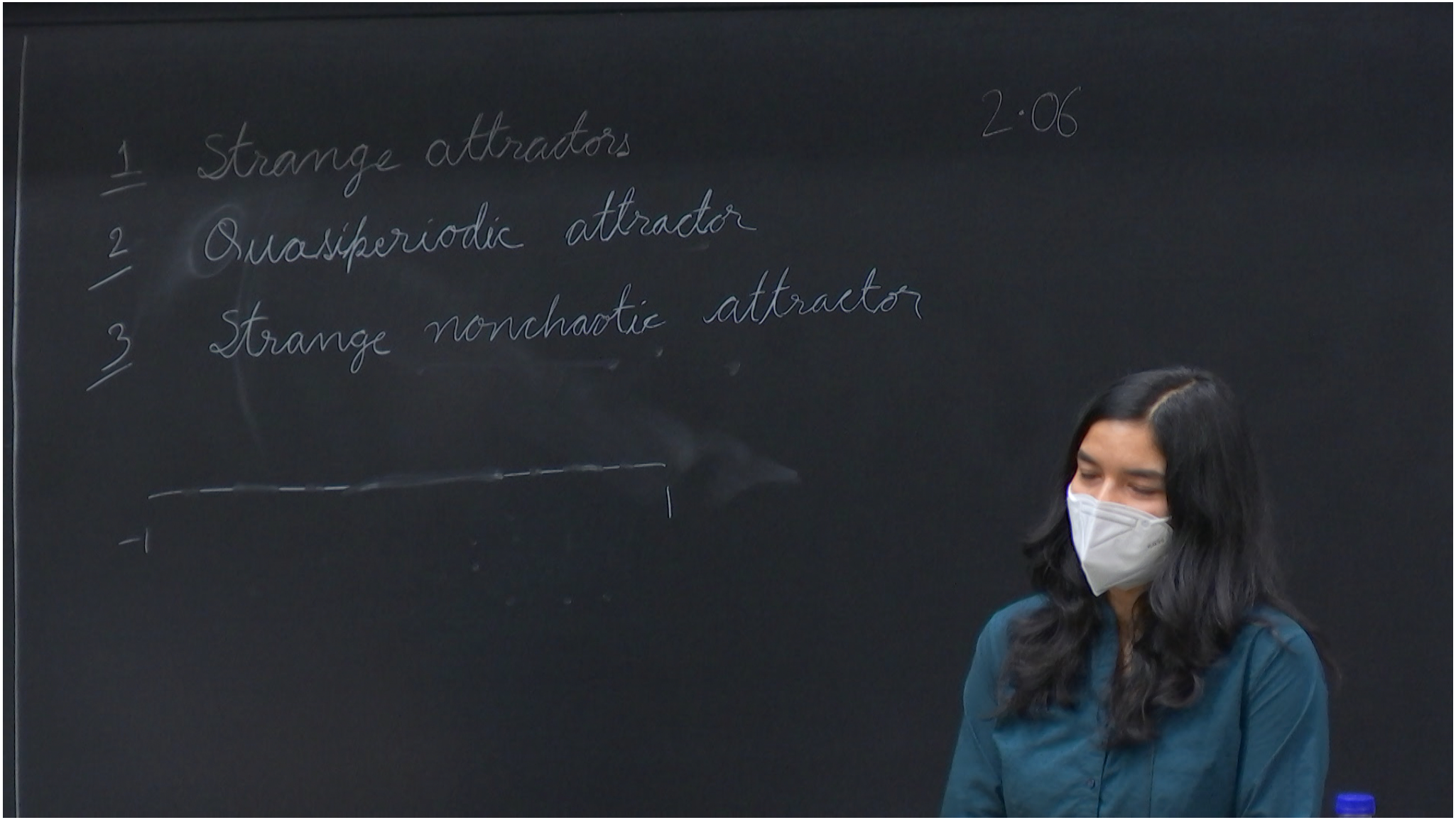
$$\vec{\nabla} \cdot \vec{F} = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z}$$

$$= -\sigma - 1 - b$$

$$V(t) = V(0)\, e^{-(\sigma + 1 + b)t}$$

Volumes in phase space shrink exponentially fast.

Fixed points, limit cycles — stable or saddle


1 Strange attractors
2 Quasiperiodic attractor
3 Strange nonchaotic attractor
2.06
-1
1