Title: Classical and Quantum Chaos 2021/2022 - Lecture 3

Speakers: Meenu Kumari

Collection: Classical and Quantum Chaos 2021/2022

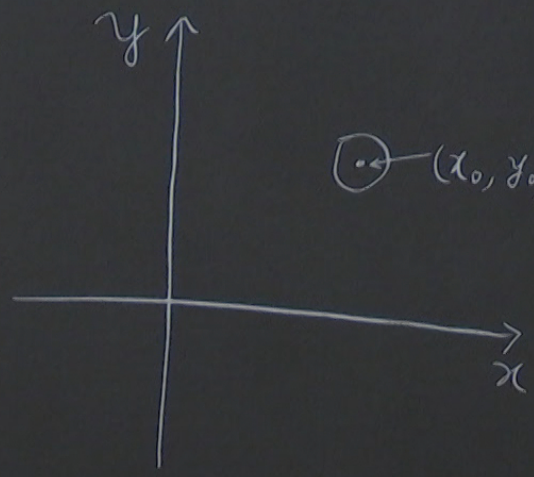
Date: March 04, 2022 - 11:30 AM

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$\dot{x} = P(x, y)$

$\dot{y} = Q(x, y)$

$X^* = (x_0, y_0)$ is eq. pt.



$x = x_0 + \xi$

$y = y_0 + \eta$

$$\begin{bmatrix} \dot{\xi} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{bmatrix}_{(x_0, y_0)} \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = J|_{(x_0, y_0)}$$

$\det(J$

$\xi(t) = Ae$

$\eta(t) = Ce$

$\lambda_{1,2} =$

$\det(J|_{(x_0,y_0)}) \neq 0$

Case 1

(x_0, y_0)

x

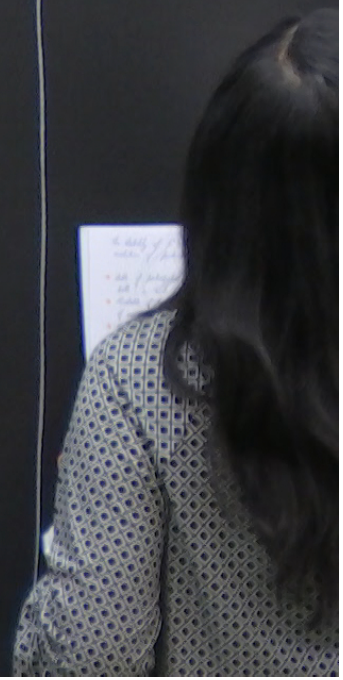
$$\xi(t) = A\exp(\lambda_1 t) + B\exp(\lambda_2 t)$$

$$\eta(t) = C\exp(\lambda_1 t) + D\exp(\lambda_2 t)$$

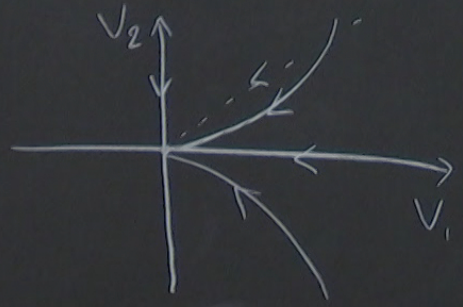
$$\lambda_{1,2} = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

$$\begin{bmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{bmatrix}_{(x_0,y_0)} \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

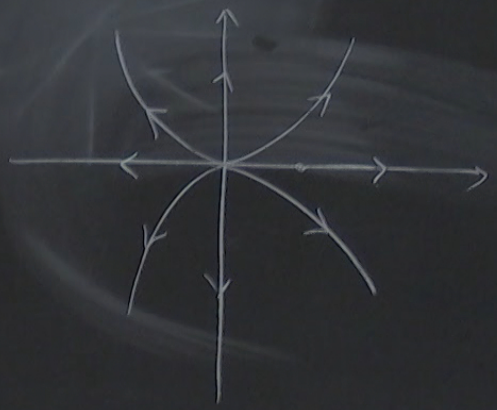
$$= J|_{(x_0,y_0)} \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$



Case 1 - $\lambda_{1,2} < 0$ - Stable node



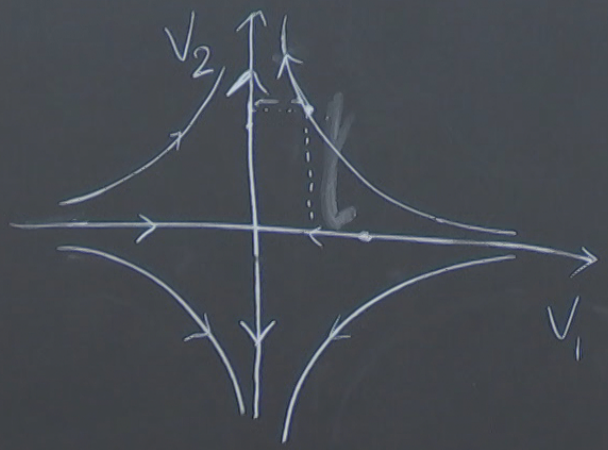
Case 2 - $\lambda_{1,2} > 0$ Unstable node



> 0 – Stable node

> 0 Unstable node

Case 3: $\lambda_1 < 0 < \lambda_2$ Saddle



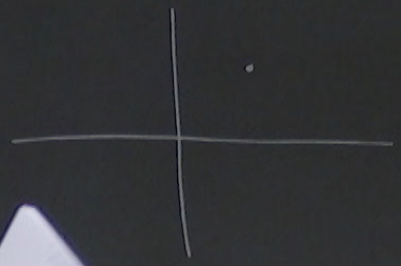
Case 4 :- Complex

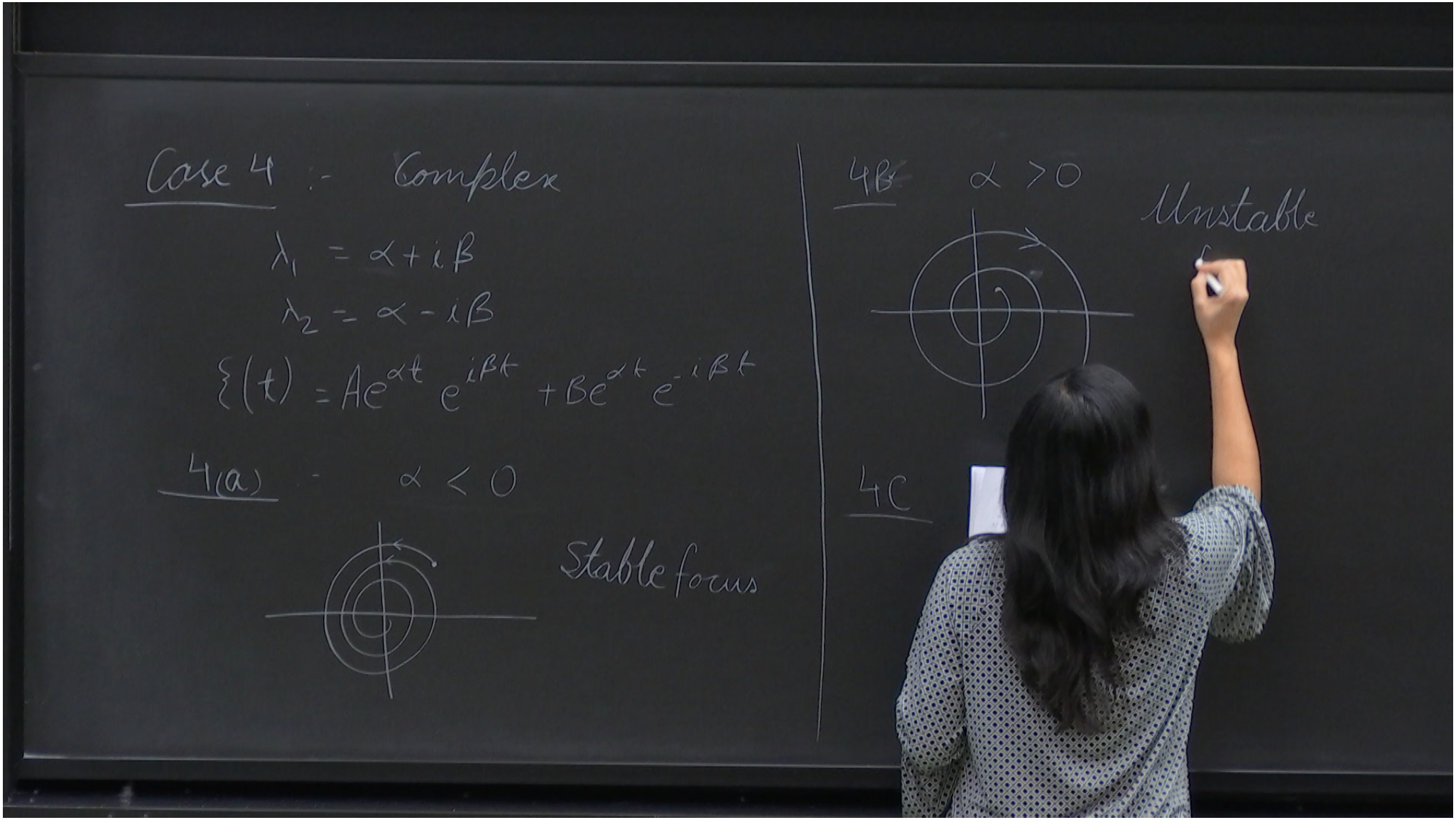
$\lambda_1 = \alpha + i\beta$

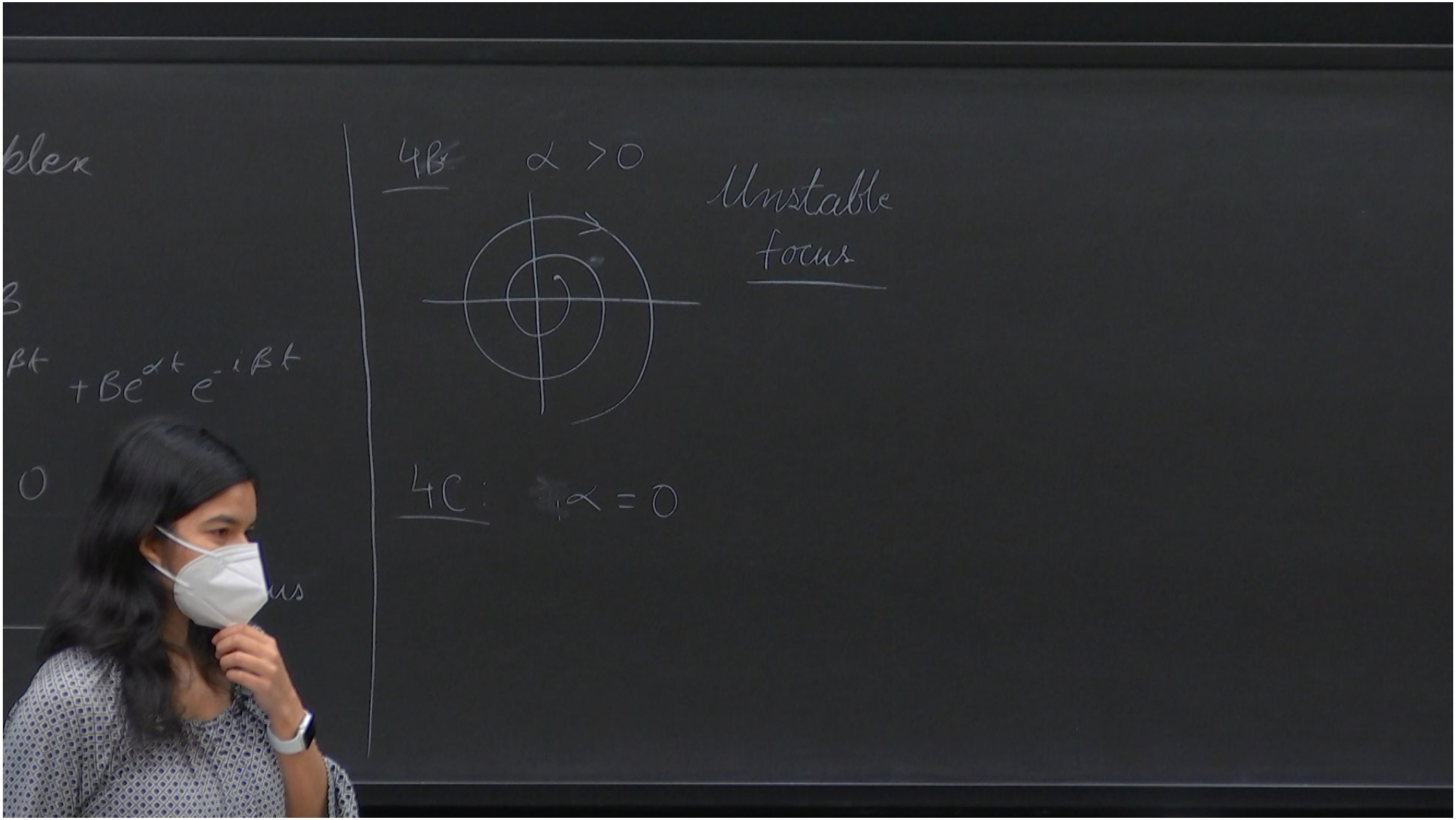
$\lambda_2 = \alpha - i\beta$

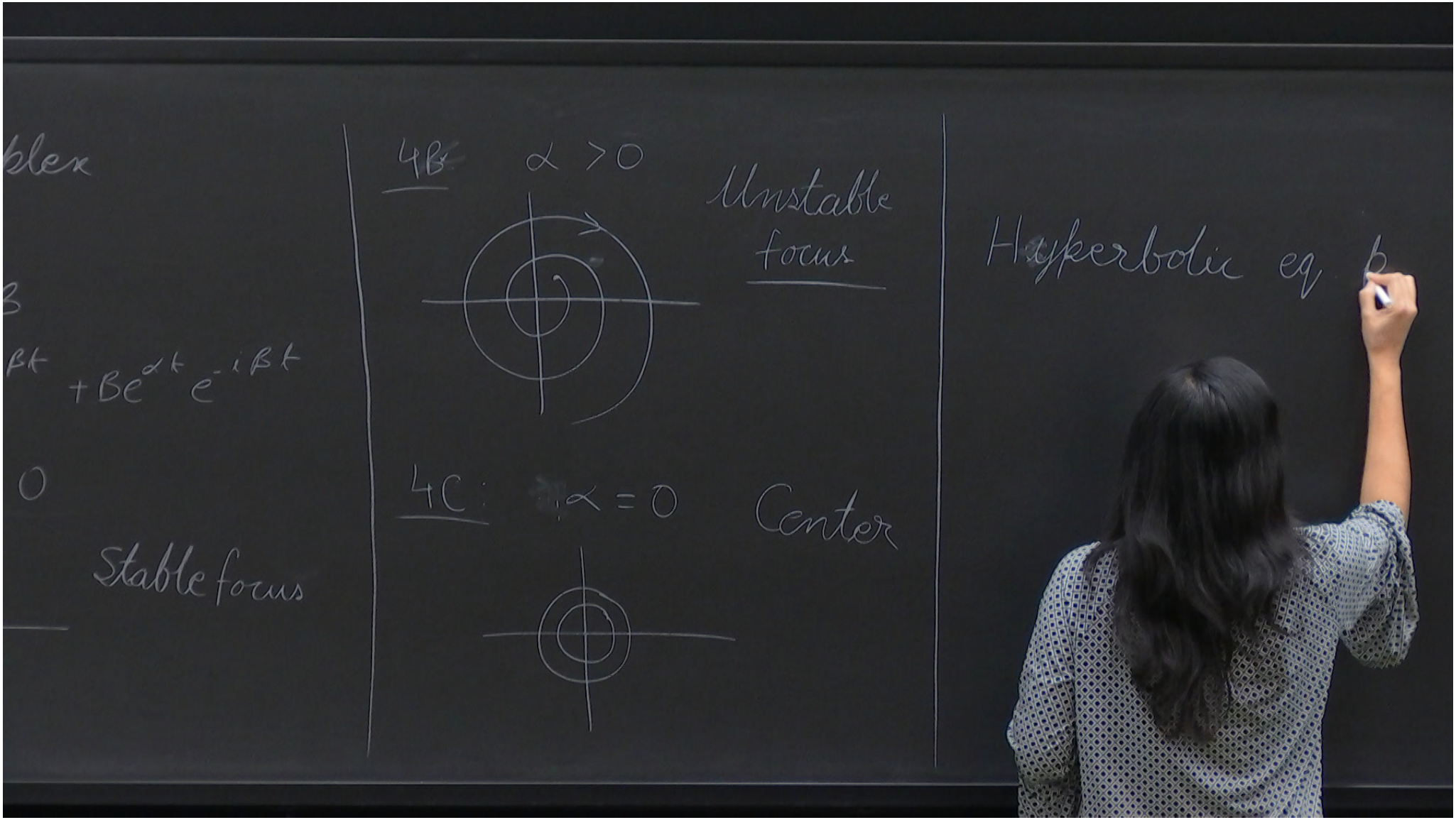
$\xi(t) = Ae^{\alpha t} e^{i\beta t} + Be^{\alpha t} e^{-i\beta t}$

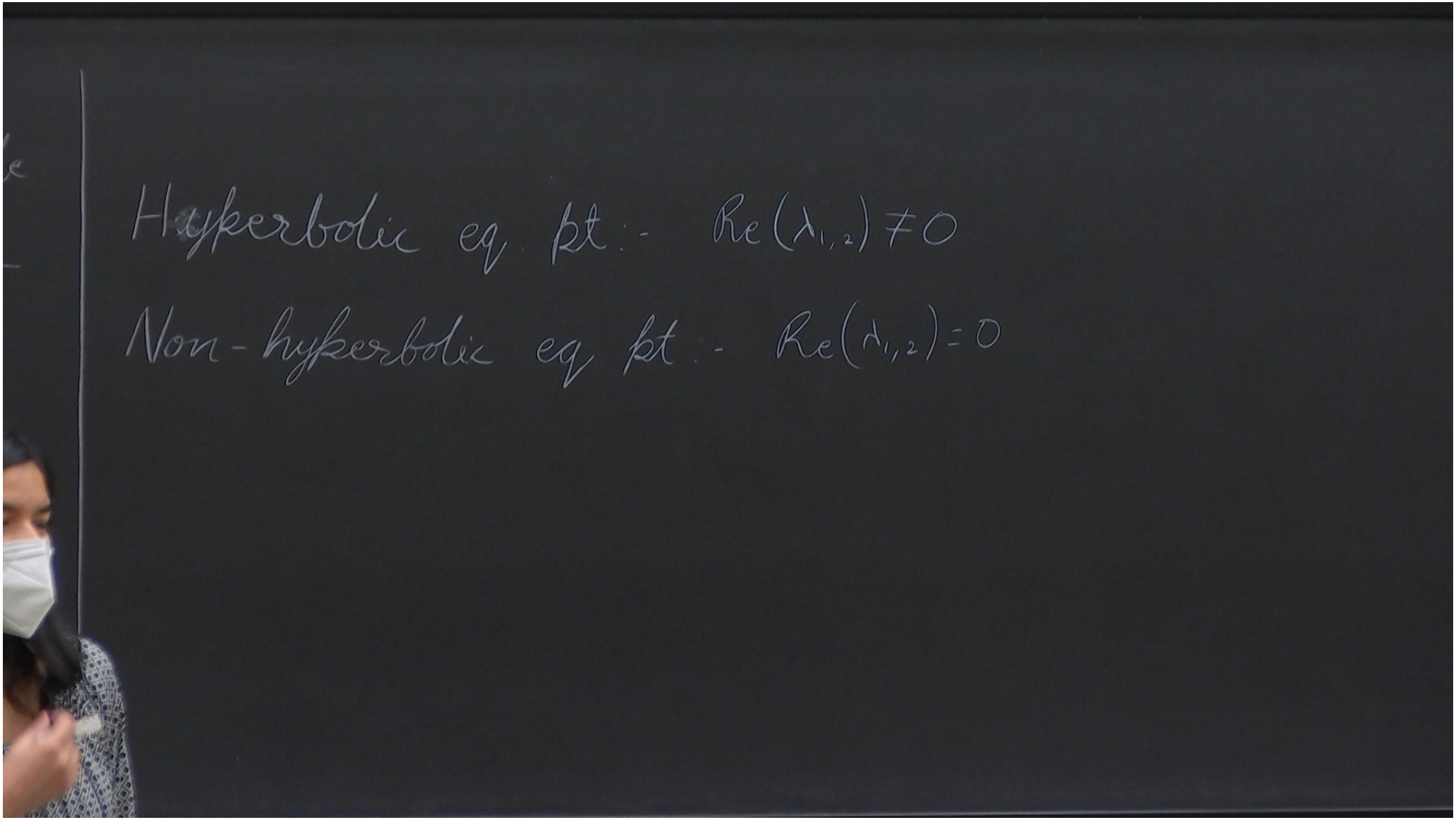
4(a) · $\alpha < 0$

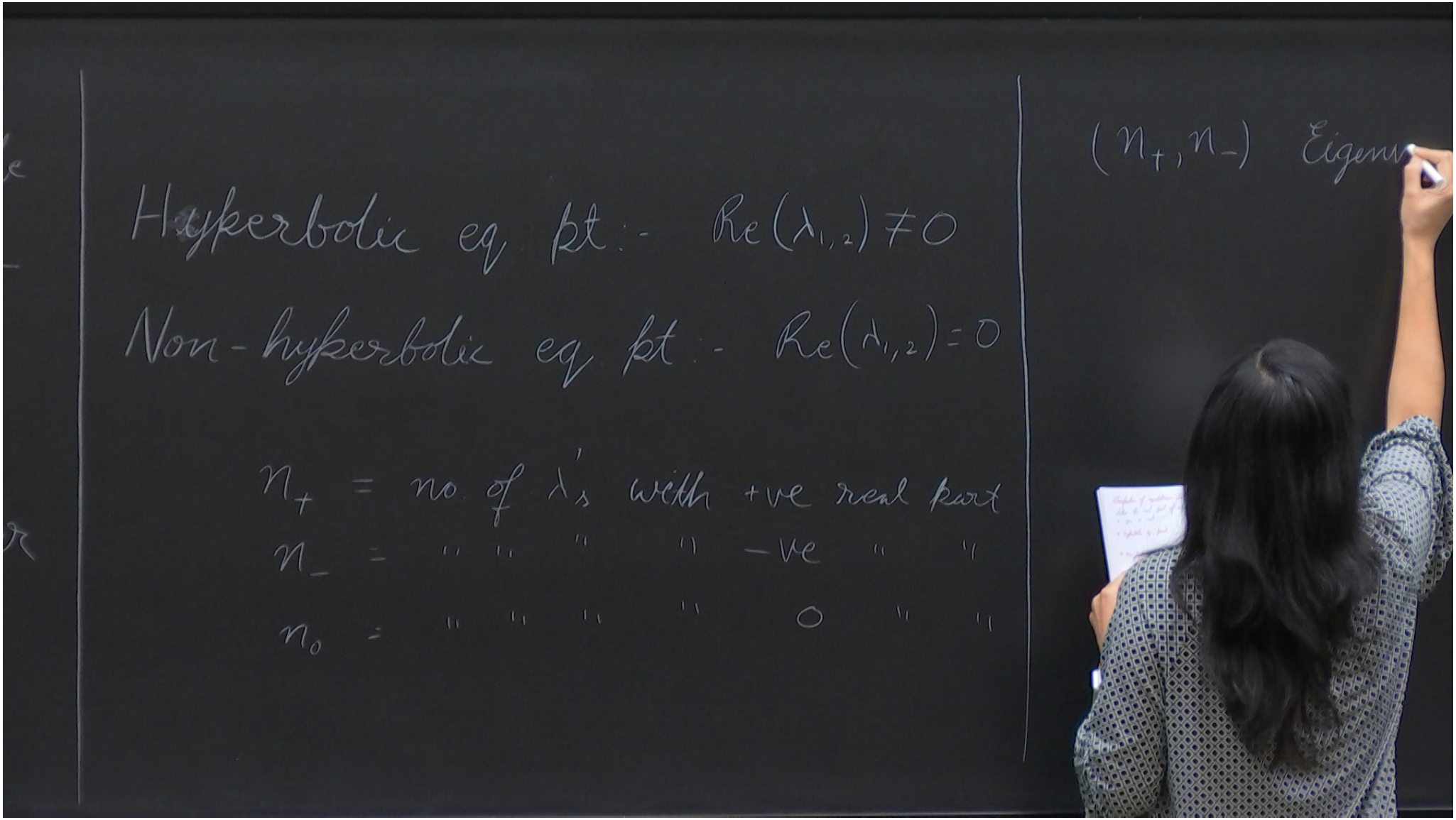


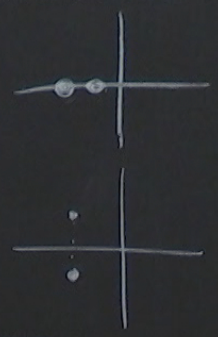
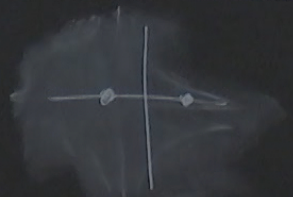
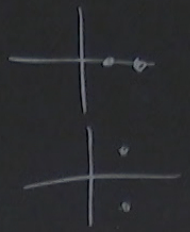
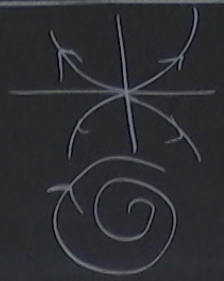

Case 4 :- Complex
$\lambda_1 = \alpha + i\beta$
$\lambda_2 = \alpha - i\beta$
$\xi(t) = Ae^{\alpha t} e^{i\beta t} + Be^{\alpha t} e^{-i\beta t}$
4(a) $\alpha < 0$
Stable focus
4B $\alpha > 0$
Unstable
4C


4B
α > 0
Unstable
focus
4C:
α = 0

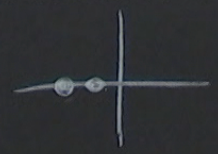
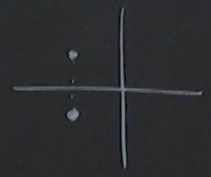
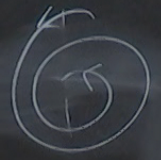
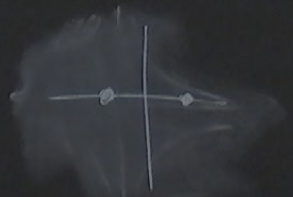
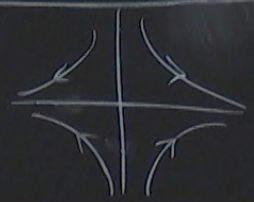
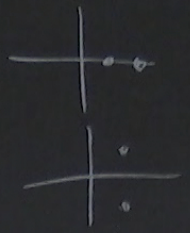
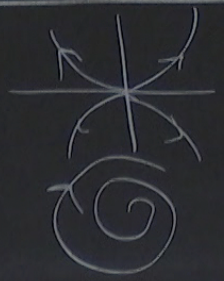

blex
Bt
+Be^{αt} e^{-iβt}
0
Stable focus
4B α > 0
Unstable focus
4C: α = 0
Center
Hyperbolic eq

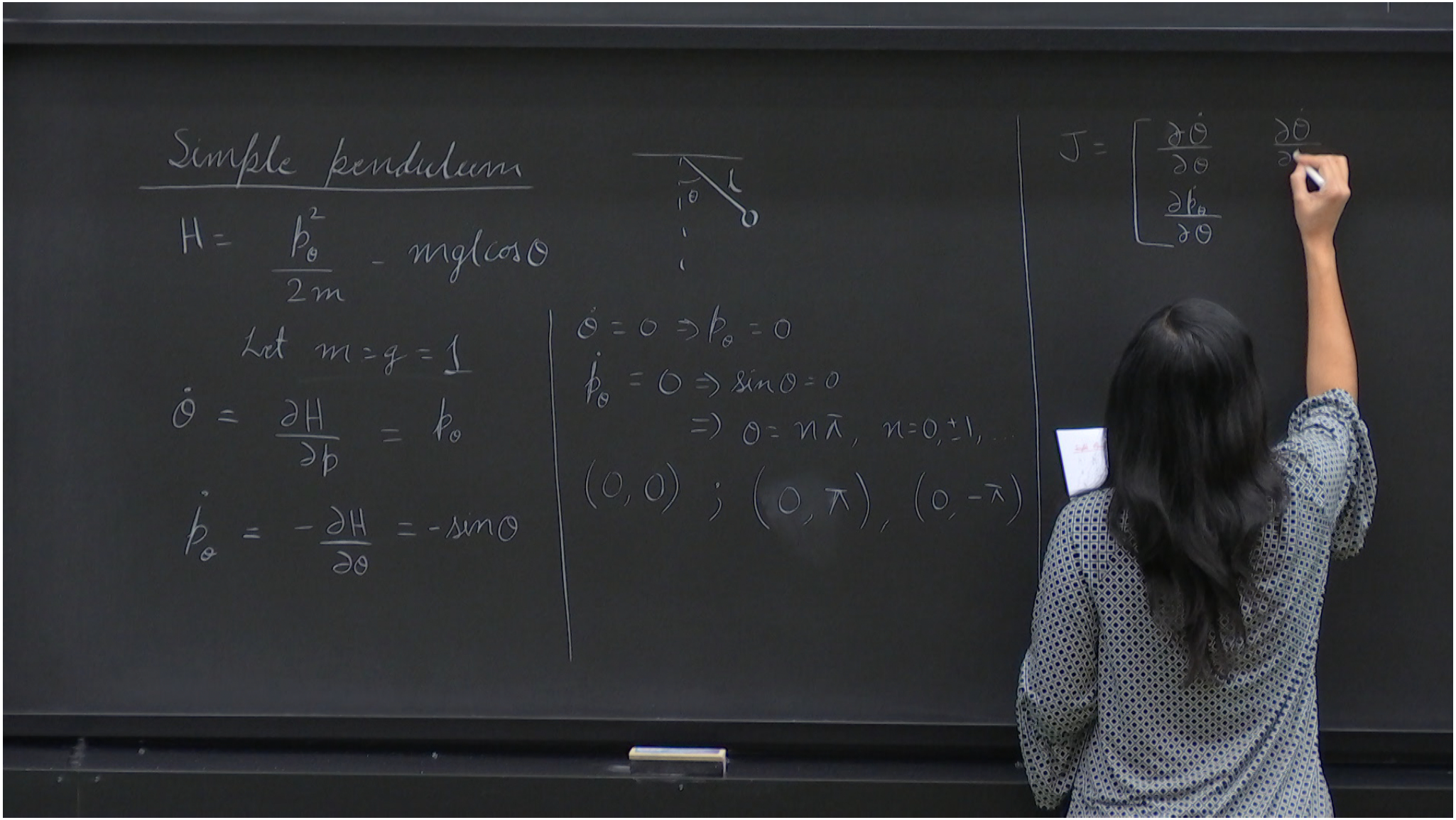

Hyperbolic eq pt :- $Re(\lambda_{1,2}) \neq 0$
Non-hyperbolic eq pt :- $Re(\lambda_{1,2}) = 0$


Hyperbolic eq pt :- $Re(\lambda_{1,2}) \neq 0$
Non-hyperbolic eq pt :- $Re(\lambda_{1,2}) = 0$
n_+ = no of λ's with +ve real part
n_- = " " " " -ve " "
n_0 = " " " " 0 " "
(n_+, n_-) Eigen

(n_+, n_-)	Eigenvalues	Phase portrait	Stability
$(0, 2)$		 node  Focus	Stable
$(1, 1)$		 Saddle	Unstable
$(2, 0)$		 node Focus	Unstable

Summary of hyperbolic eq. in 2-D

(n_+, n_-)	Eigenvalues	Phase portrait		Stability
$(0, 2)$			node	Stable
			Focus	
$(1, 1)$			Saddle	Unstable
$(2, 0)$			node	Unstable
			Focus	

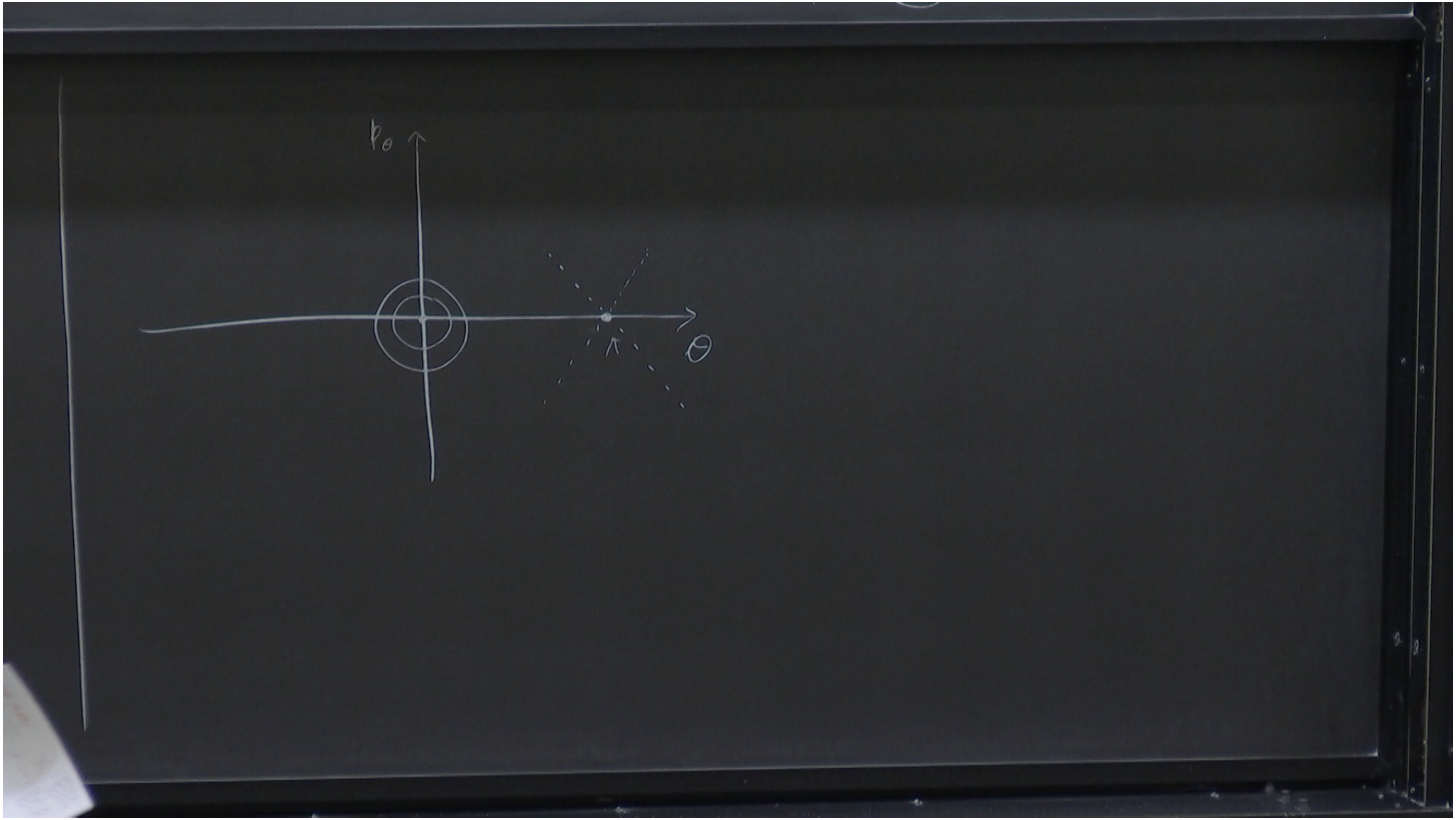

Simple pendulum
$H = \frac{p_\theta^2}{2m} - mgl\cos\theta$
Let $m = g = 1$
$\dot{\theta} = \frac{\partial H}{\partial p} = p_\theta$
$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -\sin\theta$
θ
l
$\dot{\theta} = 0 \Rightarrow p_\theta = 0$
$\dot{p}_\theta = 0 \Rightarrow \sin\theta = 0$
$\Rightarrow \theta = n\pi, \; n = 0, \pm 1, \ldots$
$(0,0)$; $(0,\pi)$, $(0,-\pi)$
$J = \begin{bmatrix} \frac{\partial\dot{\theta}}{\partial\theta} & \frac{\partial\dot{\theta}}{\partial} \\ \frac{\partial\dot{p}_\theta}{\partial\theta} & \end{bmatrix}$

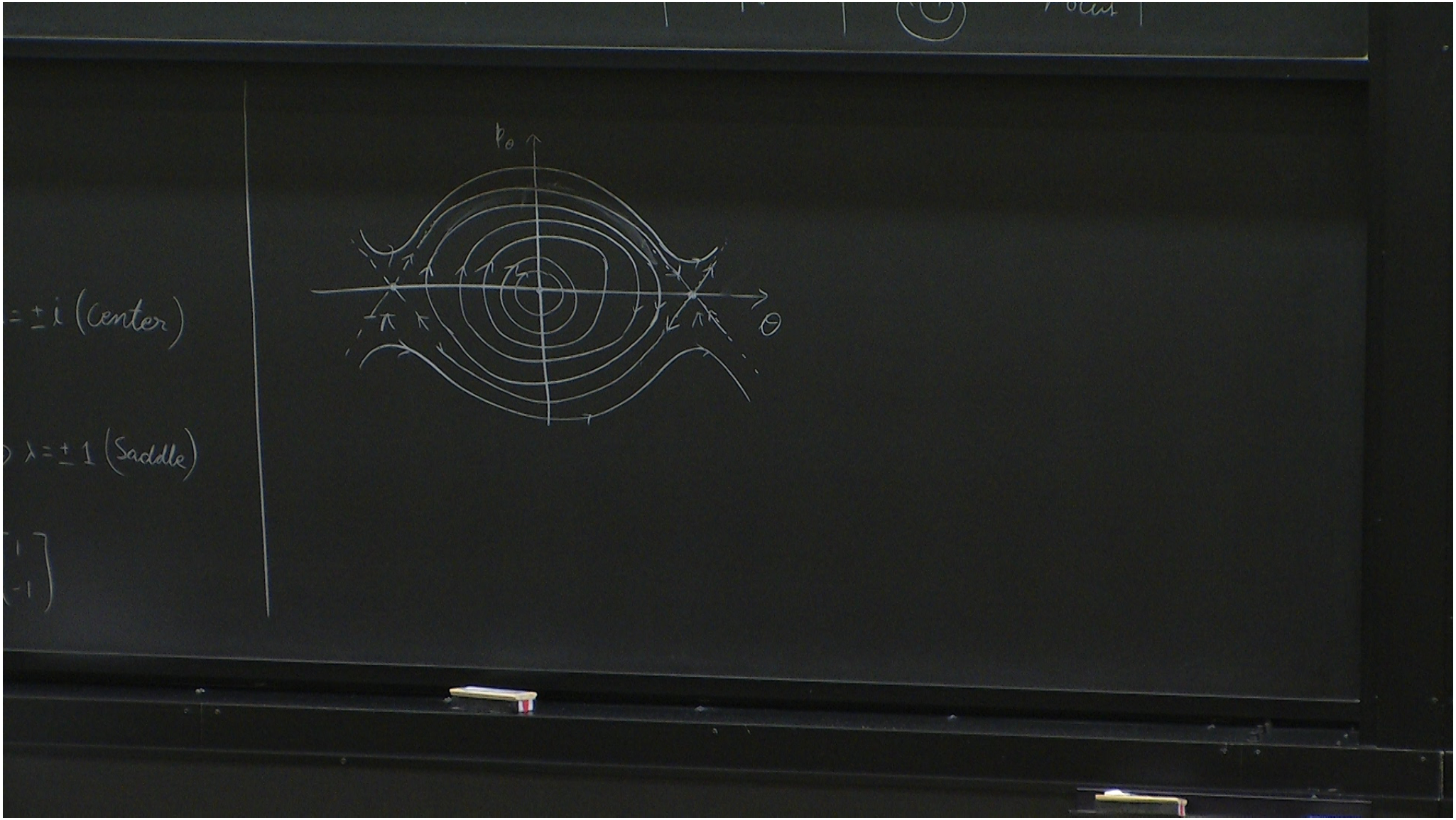
$$J = \begin{bmatrix} \frac{\partial \dot{\theta}}{\partial \theta} & \frac{\partial \dot{\theta}}{\partial p_\theta} \\ \frac{\partial \dot{p}_\theta}{\partial \theta} & \frac{\partial \dot{p}_\theta}{\partial p_\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\cos\theta & 0 \end{bmatrix}$$

$$J_{|(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}; \quad \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i \ (\text{center})$$

$$J_{|(0,\pi)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1 \ (\text{Saddle})$$

$$\lambda = +1, \ \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad \lambda = -1 \ \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$


p_θ
θ


p_θ
θ
$= \pm i$ (center)
$\lambda = \pm 1$ (Saddle)

Node

Focus

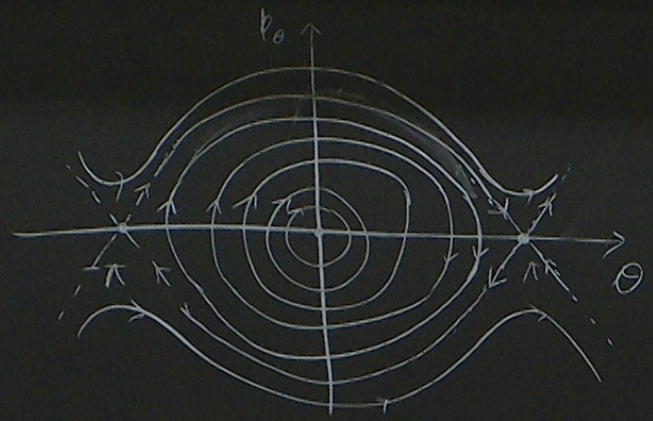
Unstable

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$0 \Rightarrow \lambda = \pm i$ (center)

$= 0 \Rightarrow \lambda = \pm 1$ (Saddle)

$-1 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$



Van der Pol oscillator

$$\dot{x} = y \quad = P(x,y)$$

$$\dot{y} = b(1-x^2)y - x = Q(x,y)$$

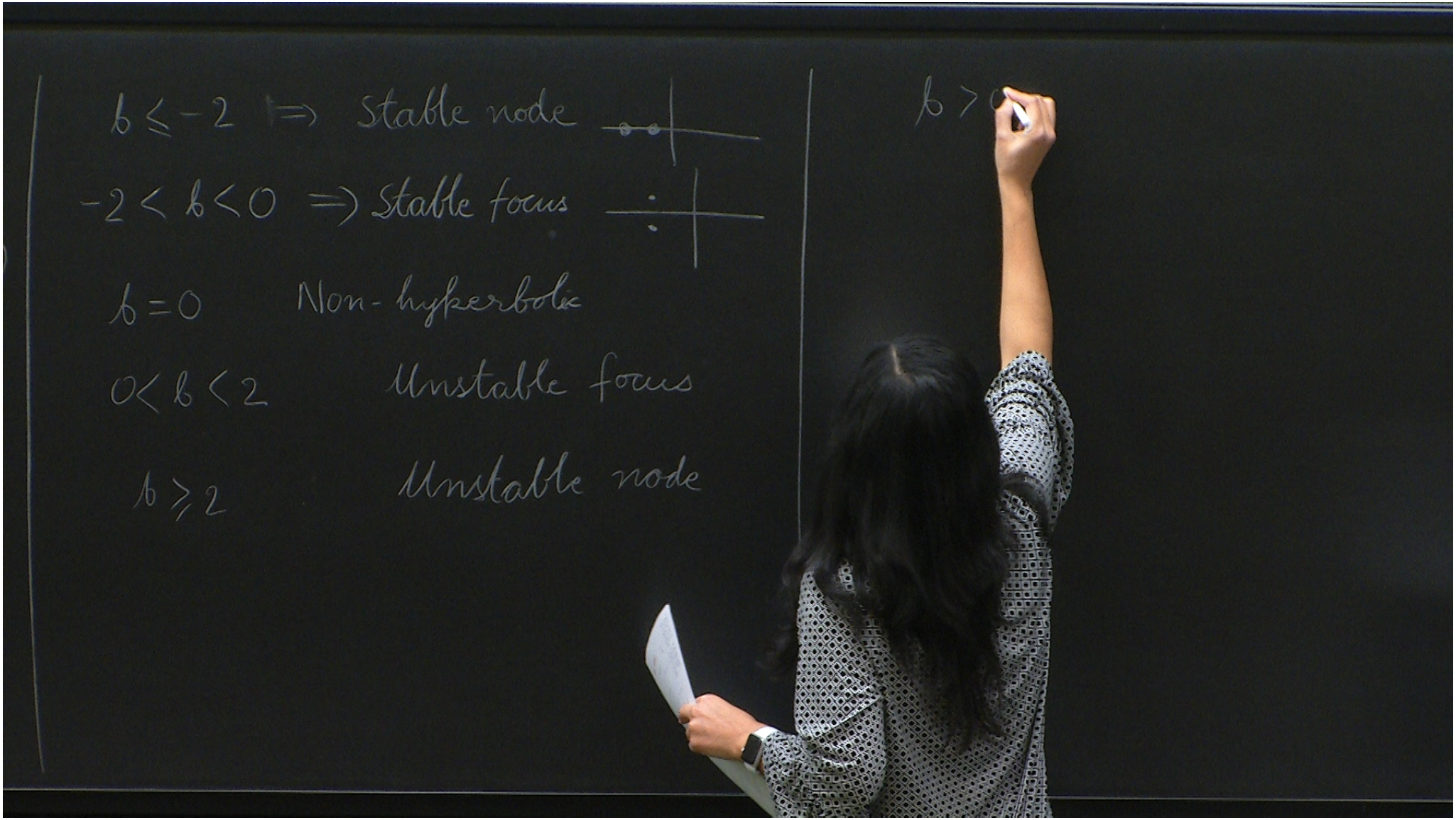
$$(x_0, y_0) = (0,0)$$

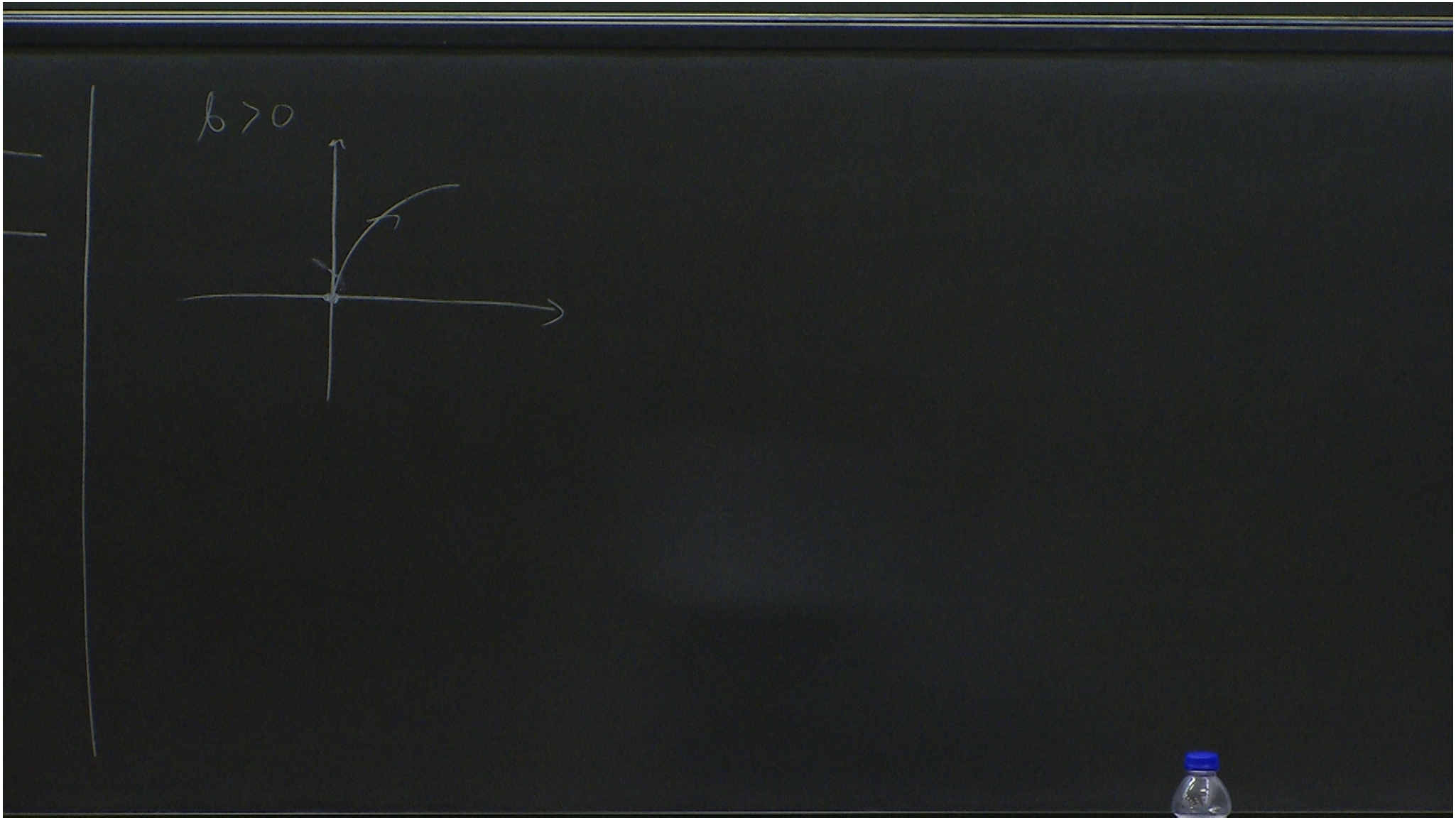
$$J|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & b \end{bmatrix}$$

$$(-\lambda)(b-\lambda) + 1 = 0$$

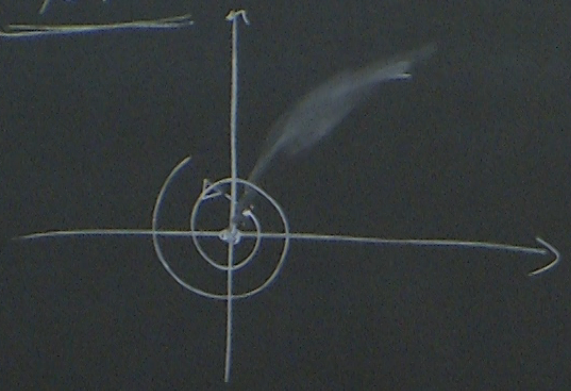
$$\lambda = \frac{b \pm \sqrt{b^2-4}}{2}$$

$b \leq -2 \Rightarrow$


$b \leq -2 \Rightarrow$ Stable node
$-2 < b < 0 \Rightarrow$ Stable focus
$b = 0$ Non-hyperbolic
$0 < b < 2$ Unstable focus
$b \geq 2$ Unstable node
$b > 0$


b > 0

$b > 0$



$$\vec{\nabla} \cdot (P, Q) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

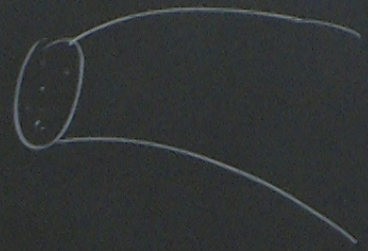
$$= 0 + b(1 - x^2)$$

$|x| > 1 \Rightarrow \vec{\nabla} \cdot (P, Q) < 0$

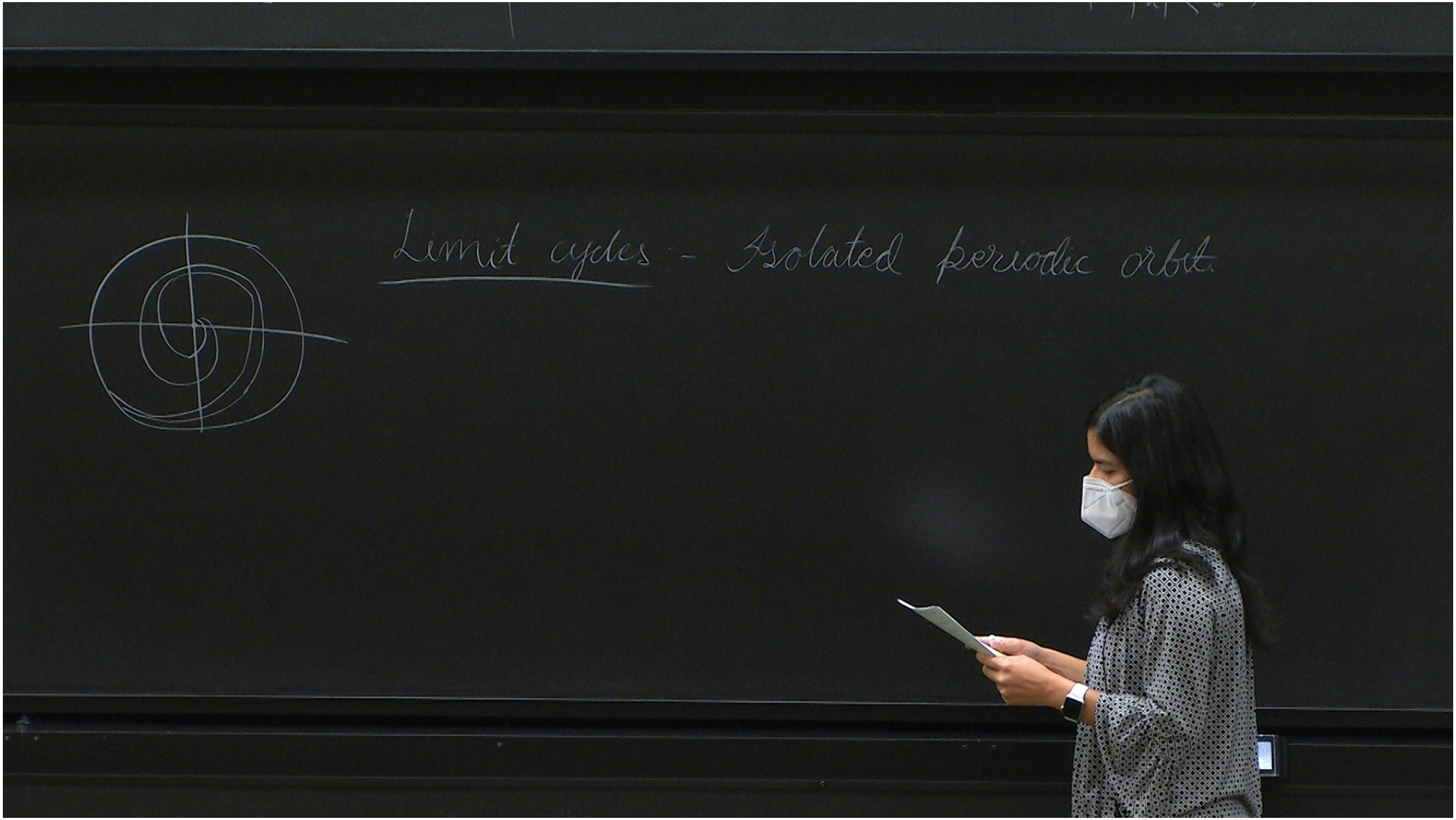
$|x| < 1 \Rightarrow \quad " \quad > 0$

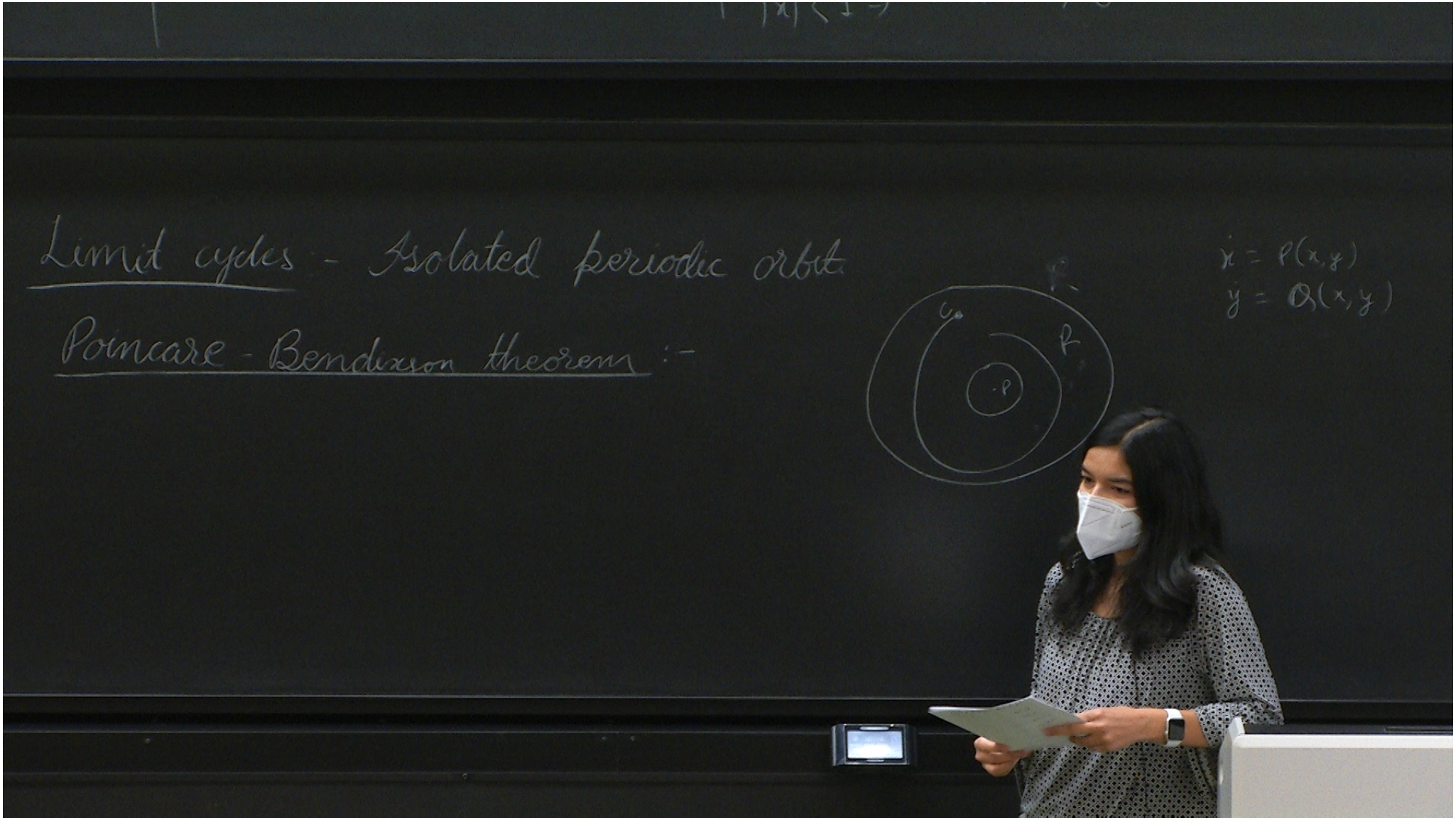


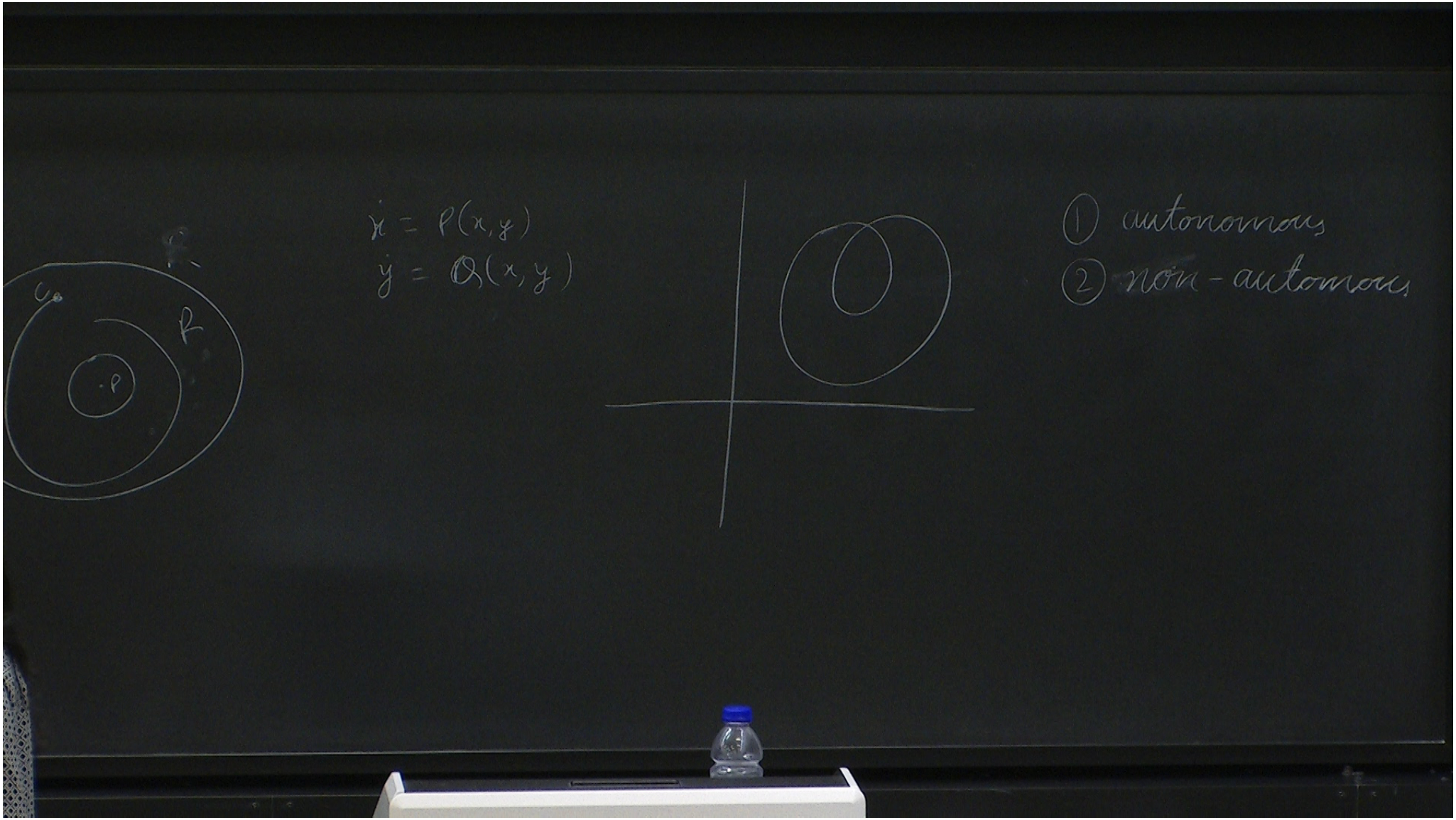
< 0

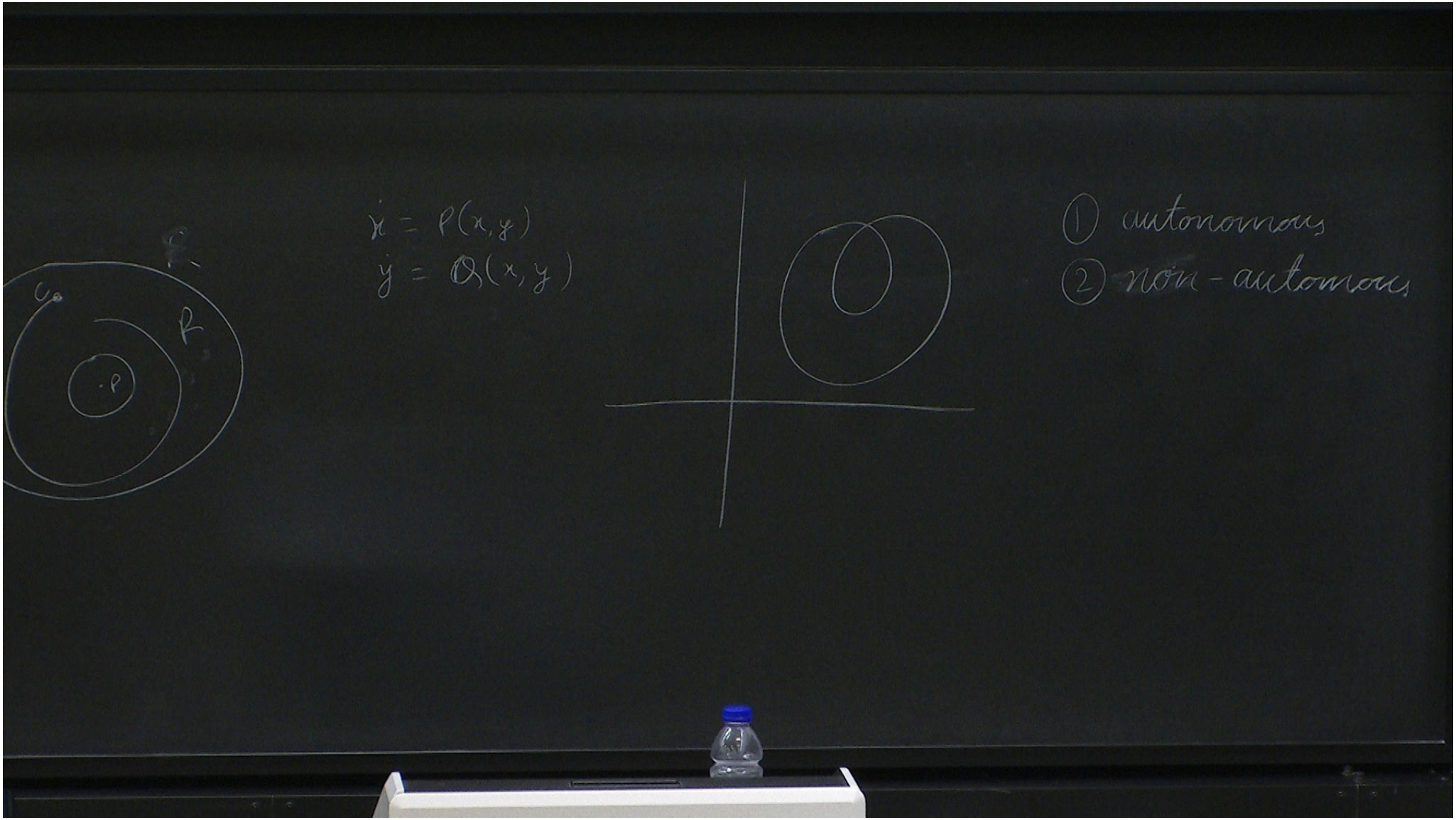


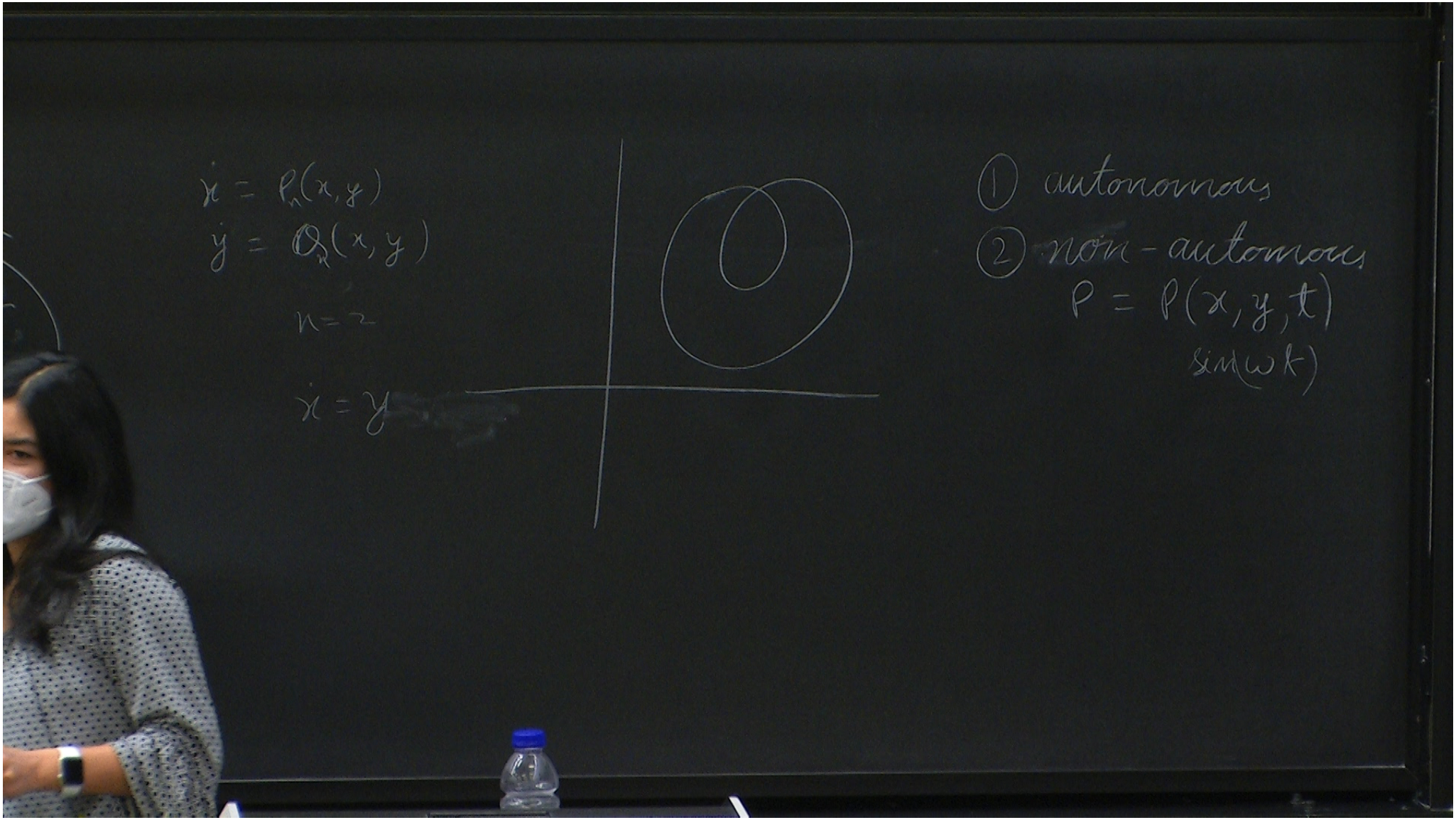
> 0

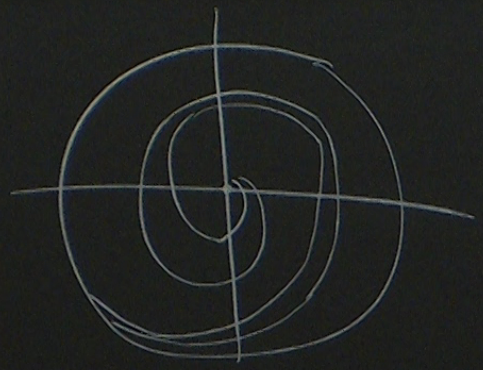

Limit cycles - Isolated periodic orbit


Limit cycles - Isolated periodic orbit
Poincare - Bendixson theorem :-
$\dot{x} = P(x,y)$
$\dot{y} = Q(x,y)$
P


$\dot{x} = P(x,y)$
$\dot{y} = Q(x,y)$
① autonomous
② non-autonomous


$\dot{x} = P(x,y)$
$\dot{y} = Q(x,y)$
① autonomous
② non-autonomous


$\dot{x} = P_n(x, y)$
$\dot{y} = Q_n(x, y)$
$n = 2$
$\dot{x} = y$
① autonomous
② non-autonomous
$P = P(x, y, t)$
$\sin(\omega t)$



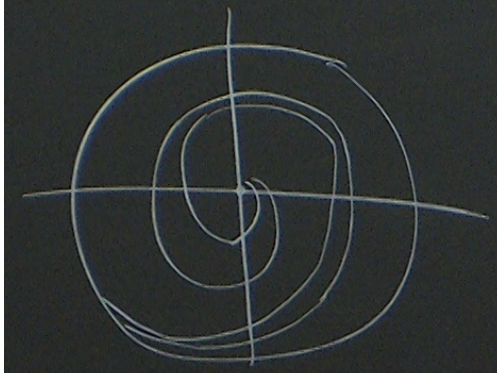
Limit cycles - Isolated periodic orbit

Poincare - Bendixson theorem :-

Suppose that :-

1) R is a closed, bounded subset of the 2-D plane

2) $\dot{x} =$



Limit cycles - Isolated periodic orbit

Poincare - Bendixson theorem :-

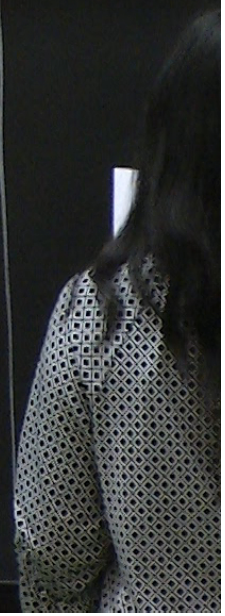
Suppose that :-

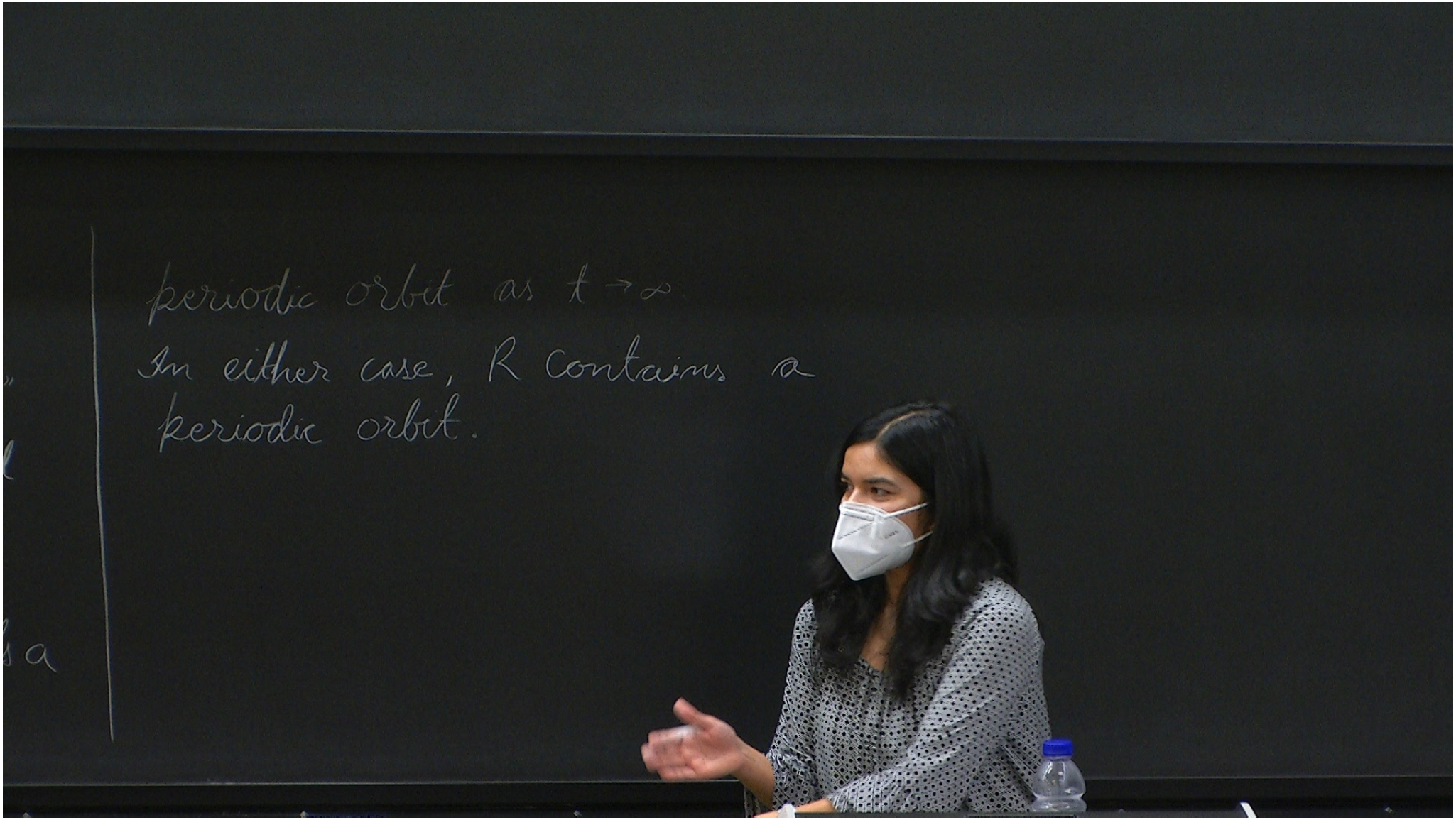
1) R is a closed, bounded subset of the 2-D plane
2) $\left(\dot{x} = P(x,y),\ \dot{y} = Q(x,y)\right)$ is a continuously differentiable vector field on an open set containing R

vector field on an open set containing R

3) R does not contain any fixed points, and
4) There exists a trajectory C that is "confined" in R, in the sense that it starts in R and stays in R for all future time.

Then, either C is a periodic orbit, or it spirals towards a




periodic orbit as $t \to \infty$
In either case, R contains a
periodic orbit.