Title: Classical and Quantum Chaos 2021/2022 - Lecture 2

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Collection: Classical and Quantum Chaos 2021/2022

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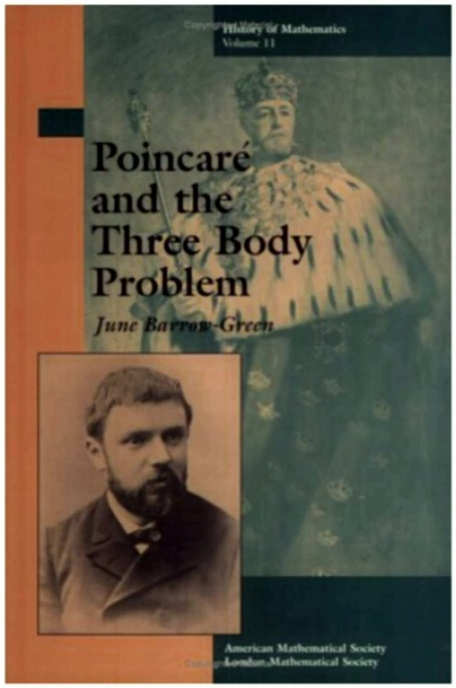
History of chaos theory

- The three body problem: Henri Poincaré
- Nonlinear amplifiers: Mary Cartwright and John E. Littlewood
- Weather modelling: Edward Norton Lorenz

I. The Three Body Problem

"Three particles move in space under their mutual gravitational attraction; given their initial conditions, determine their subsequent motion."



Henri Poincaré
(1854-1912)

I. The Three Body Problem

"Let us take, for example, the Three-Body Problem: is it not possible to ask whether one of the bodies will remain forever in some region of the sky or whether it will possibly get away indefinitely; whether the distance beween two bodies will increase, or decrease indefinitely, or whether it will stay bounded between some limits? Is it not possible to ask a thousand similar questions, which will all be solved as soon as one is able to construct qualitatively the trajectories of the three bodies?"

- Mémoire sur les courbes définies par une équation différentielle, Henri Poincaré, 1881 (English translation)

II. Nonlinear Amplifiers

Mary Cartwright
(1900-1998)

John E. Littlewood
(1885-1977)

Annals of Mathematics
Vol. 48, No. 2, April, 1947

ON NON-LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

II. The Equation

$$\ddot{y} + kf(y, \dot{y} + g(y, k) = p(t) = p_1(t) + kp_2(t);\ k > 0,\ f(y) \geqq 1$$

By M. L. Cartwright and J. E. Littlewood

(Received November 13, 1946)

§1. The present paper is mainly a study of the above equation in real variables in the case when the damping factor $kf(y)$ is always positive. We suppose that

$$f(y),\ g(y, k),\ p(t),\ p_1(t),\ p_2(t)$$

*"…. When I was a young student in England in 1942, I heard **Mary Cartwright** (see Chapter 15) lecture about the pathological behavior of nonlinear amplifiers. The radars in World War II were driven by amplifiers, which behaved badly when pushed to high power levels. The Royal Air Force blamed the manufacturers and sent the radars back for repair. Cartwright showed that the manufacturers were not to blame. The Van der Pol equation was to blame. The Van der Pol equation is the standard equation describing a nonlinear amplifier. Cartwright studied the solutions of the Van der Pol equation and discovered the unexpected phenomenon that is now called chaos. As the power is increased, the periodic solutions go through an infinite sequence of period doublings and finally become aperiodic. The aperiodic solutions have disastrous effects on the radar, but have a beautifully intricate topological structure. These discoveries were published at the end of the war, but nobody paid much attention to her papers and she went on to other things. She became famous as a pure mathematician. Twenty years later, chaos was rediscovered by the meteorologist Ed Lorenz and became one of the most fashionable parts of physics……"*

- Freeman J. Dyson

Foreword in the book "Out of the Shadows: Contributions of Twentieth-Century Women to Physics"

III. Weather Modelling

Lorenz Model

$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = x(r - z) - y$$

$$\frac{dz}{dt} = xy - bz$$

Lorenz attractor
(https://en.wikipedia.org/wiki/Lorenz_system)

Edward Norton Lorenz
(1917-2008)

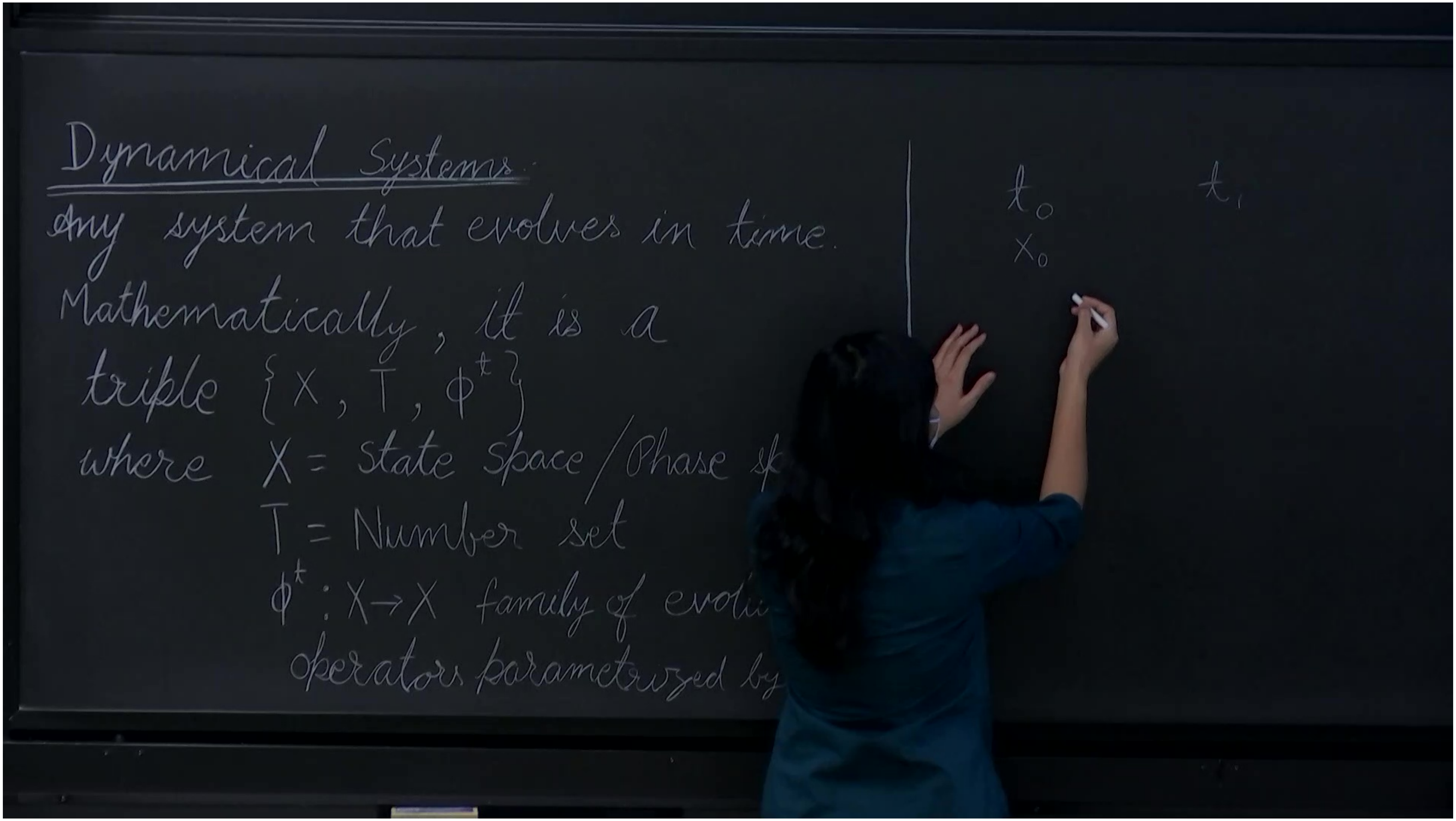
III. Weather Modelling

Circa January 1961: Lorenz and the Butterfly Effect, APS News, Jan 2003:

".....He kept a continuous simulation running on an extremely primitive computer, which would produce a day's worth of virtual weather every minute........"

"One day in the winter of 1961, Lorenz wanted to examine one particular sequence at greater length, but he took a shortcut. Instead of starting the whole run over, he started midway through, typing the numbers straight from the earlier printout to give the machine its initial conditions. Then he walked down the hall for a cup of coffee, and when he returned an hour later, he found an unexpected result. Instead of exactly duplicating the earlier run, the new printout showed the virtual weather diverging so rapidly from the previous pattern that, within just a few virtual "months", all resemblance between the two had disappeared."

"The problem lay in the numbers he had typed. Six decimal places were stored in the computer's memory: .506127. To save space on the printout, only three appeared: .506. Lorenz had entered the shorter, rounded-off numbers assuming that the difference one part in a thousand was inconsequential."


Dynamical Systems:
Any system that evolves in time.
Mathematically, it is a
triple {X, T, ϕ^t}
where X = State Space / Phase sp
T = Number set
ϕ^t : X → X family of evolu
operators parametrized by
t_0
x_0
t_1

III. Weather Modelling

"Deterministic Nonperiodic Flow", Edward N. Lorenz, Journal of atmospheric sciences 20 (1963)

Abstract: Finite systems of deterministic ordinary nonlinear differential equations may be designed to represent forced dissipative hydrodynamic flow. Solutions of these equations can be identified with trajectories in phase space. For those systems with bounded solutions, it is found that nonperiodic solutions are ordinarily unstable with respect to small modifications, so that slightly differing initial states can evolve into considerably different states. Systems with bounded solutions are shown to possess bounded numerical solutions.

A simple system representing cellular convection is solved numerically. All of the solutions are found to be unstable, and almost all of them are nonperiodic.

The feasibility of very-long-range weather prediction is examined in the light of these results.

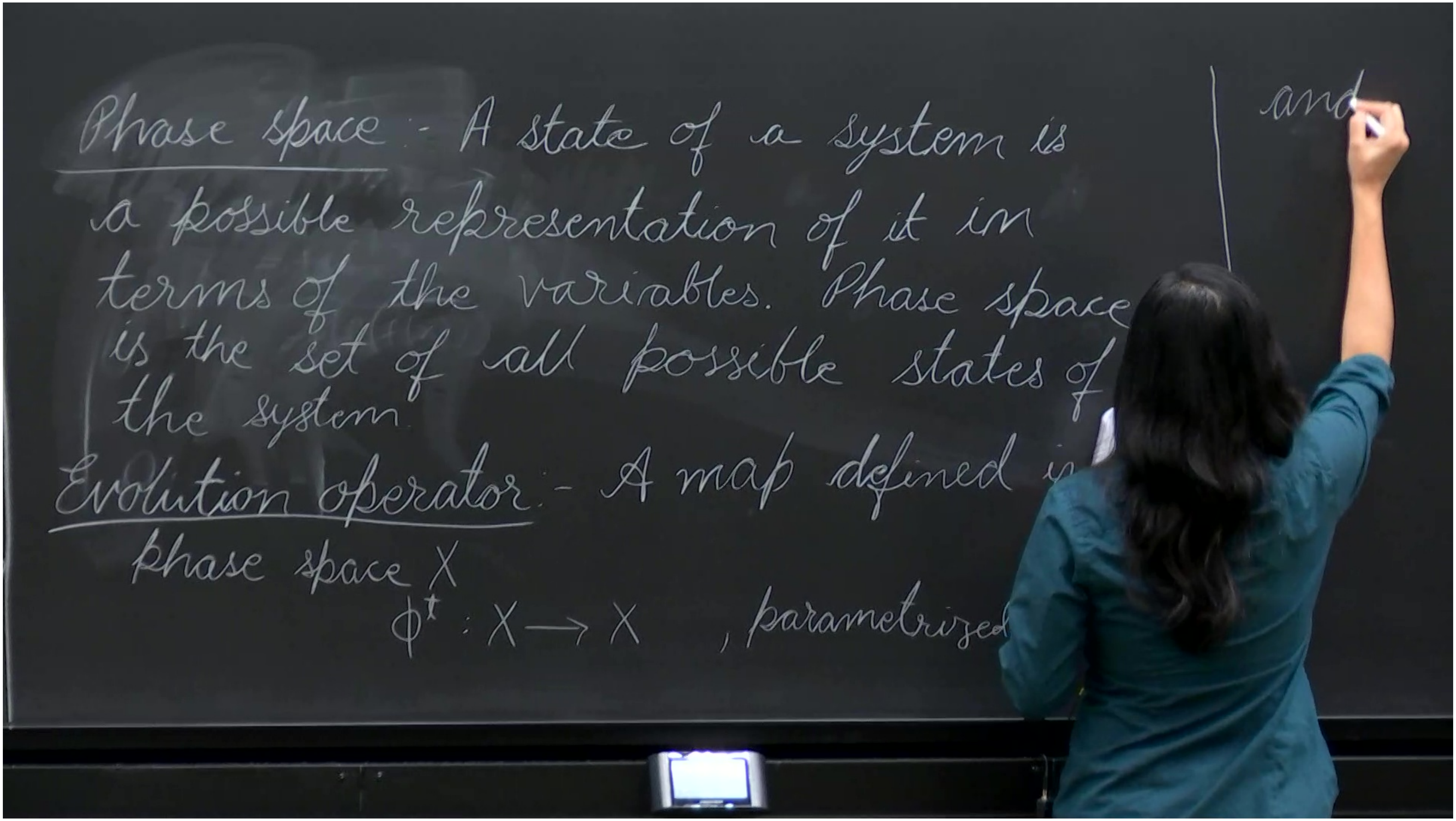
Tien-Yien Li, and James A. Yorke, "Period three implies **chaos**." The American Mathematical Monthly 82, No. 10 (1975).

$$x_{n+1} = f(x_n)$$

$$x^* = f(x^*) \rightarrow$$

$$x = f(f(x))$$

$$x = f(f(f(x)))$$


Phase space :- A state of a system is
a possible representation of it in
terms of the variables. Phase space
is the set of all possible states of
the system.
Evolution operator :- A map defined i
phase space X
$\phi^t : X \to X$, parametrized
and

s

ace

of

in the

by time t

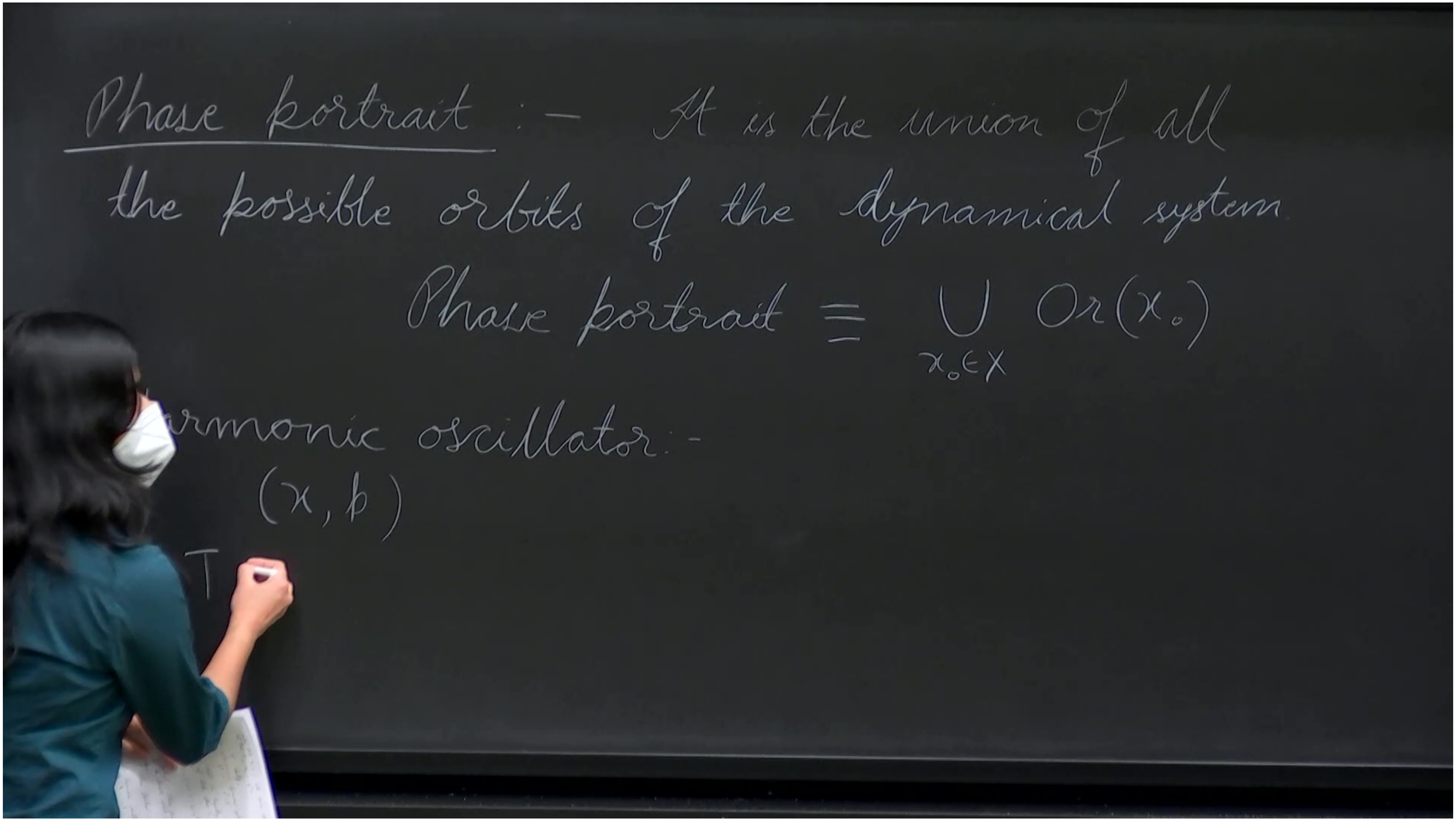
and ϕ^t transforms any initial state $x_0 \in X$ into some state $x_t \in X$ at time t.

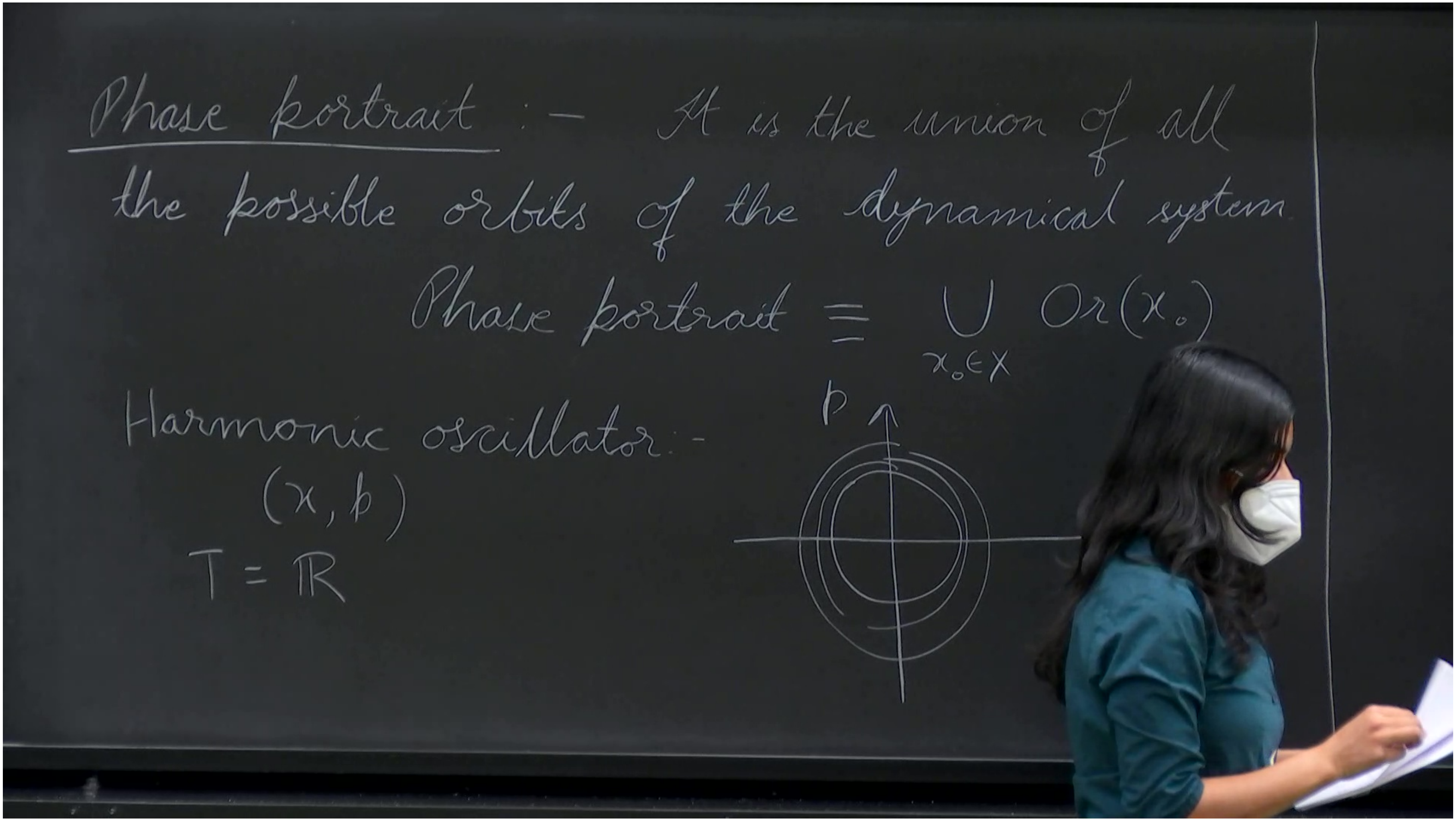
$$\phi^0 = \text{identity}$$

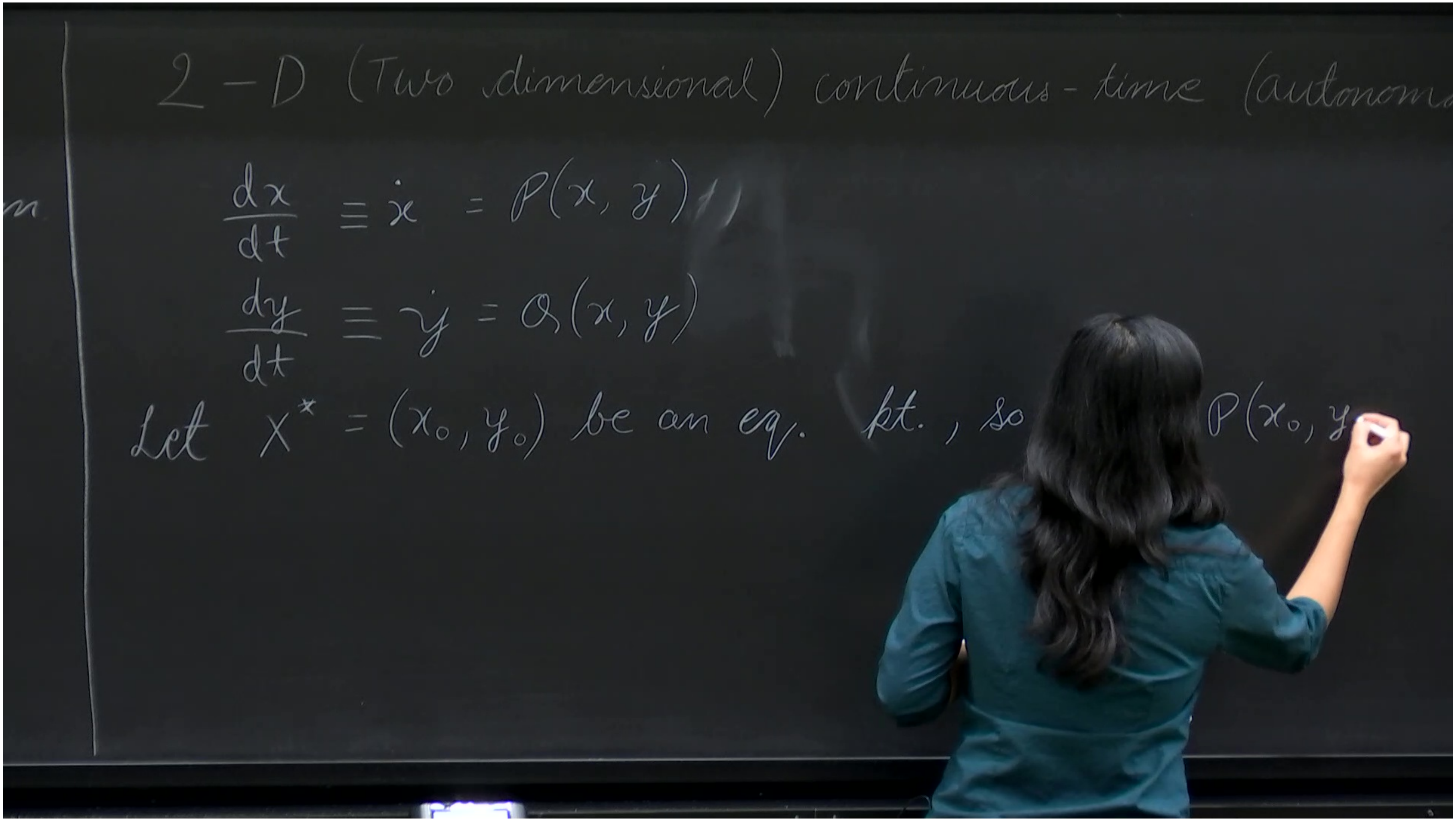
$$\phi^{t+s} = \phi^t \circ \phi^s$$

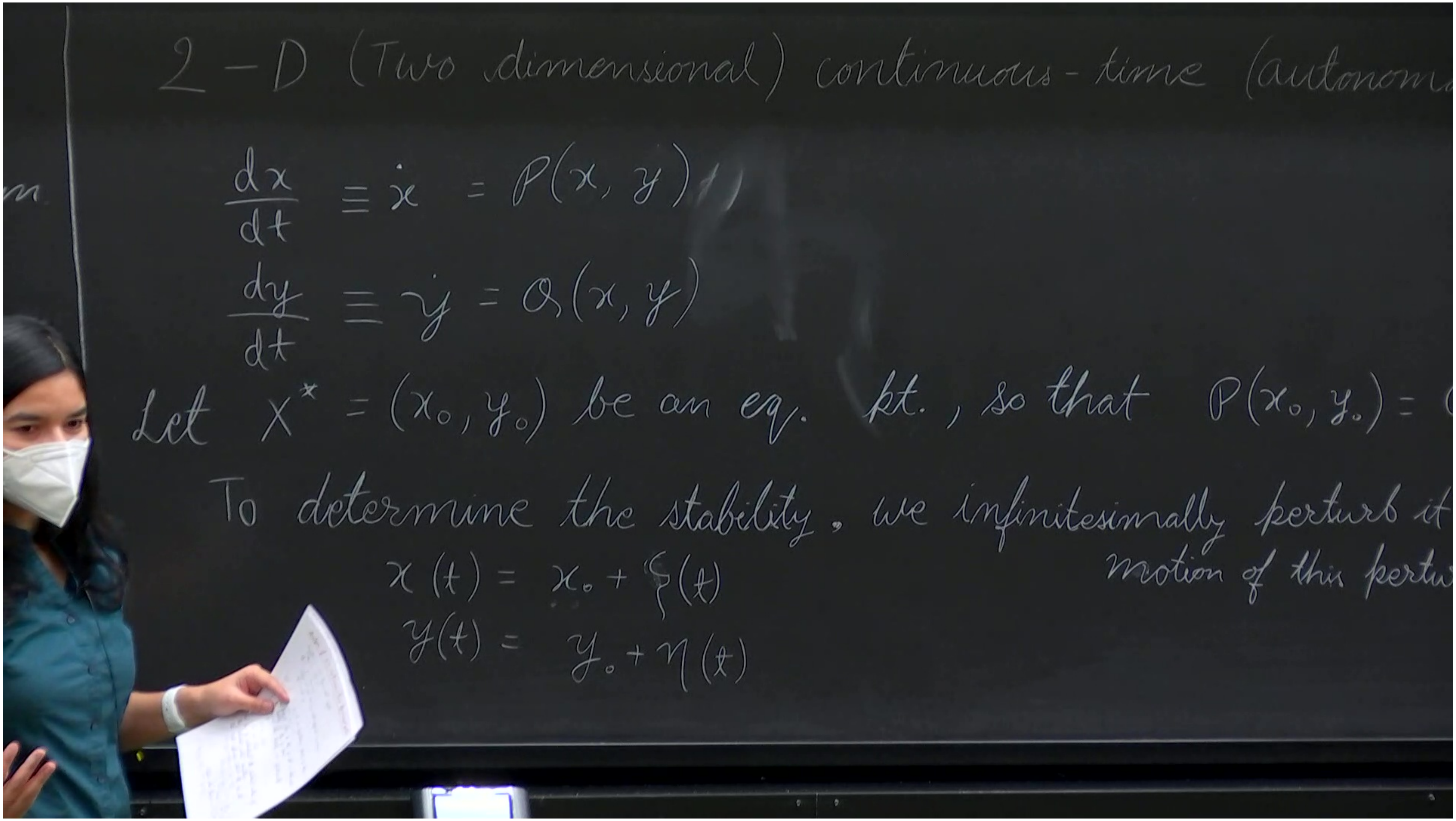
<u>Orbit</u> (trajectory) :- It is an ordered subset of the phase space X starting at $x_0 \in X$.

$$Or(X_0) = \{ x \in X : x = \phi^t x_0 \ \text{ for all } t \in T \text{ s.t. } \phi^t x_0 \text{ is defined}$$


Phase portrait :- It is the union of all
the possible orbits of the dynamical system
Phase portrait $\equiv \bigcup_{x_0 \in X} Or(x_0)$
armonic oscillator:-
(x, p)
T


Phase portrait :- It is the union of all the possible orbits of the dynamical system.
Phase portrait $\equiv \bigcup_{x_0 \in X} Or(x_0)$
Harmonic oscillator :-
(x, p)
$T = \mathbb{R}$
p


2-D (Two dimensional) continuous-time
dx/dt ≡ ẋ = P(x, y)
dy/dt ≡ ẏ = Q(x, y)
Let X* = (x₀, y₀) be an eq. pt., so


2-D (Two dimensional) continuous-time (autonom
$\frac{dx}{dt} \equiv \dot{x} = P(x, y)$
$\frac{dy}{dt} \equiv \dot{y} = Q(x, y)$
Let $X^* = (x_0, y_0)$ be an eq. pt., so that $P(x_0, y_0) =$
To determine the stability, we infinitesimally perturb it
$x(t) = x_0 + \xi(t)$
$y(t) = y_0 + \eta(t)$
motion of this pertu

- Use Taylor expansion about the eq. pt.

$$P(x,y) = P(x_0+\xi,\, y_0+\eta)$$
$$= P(x_0,y_0) + \frac{\partial P}{\partial x}\bigg|_{(x_0,y_0)} \cdot \xi + \frac{\partial P}{\partial y}\bigg|_{(x_0,y_0)} \cdot \eta$$

- Use Taylor expansion about the eq. pt.

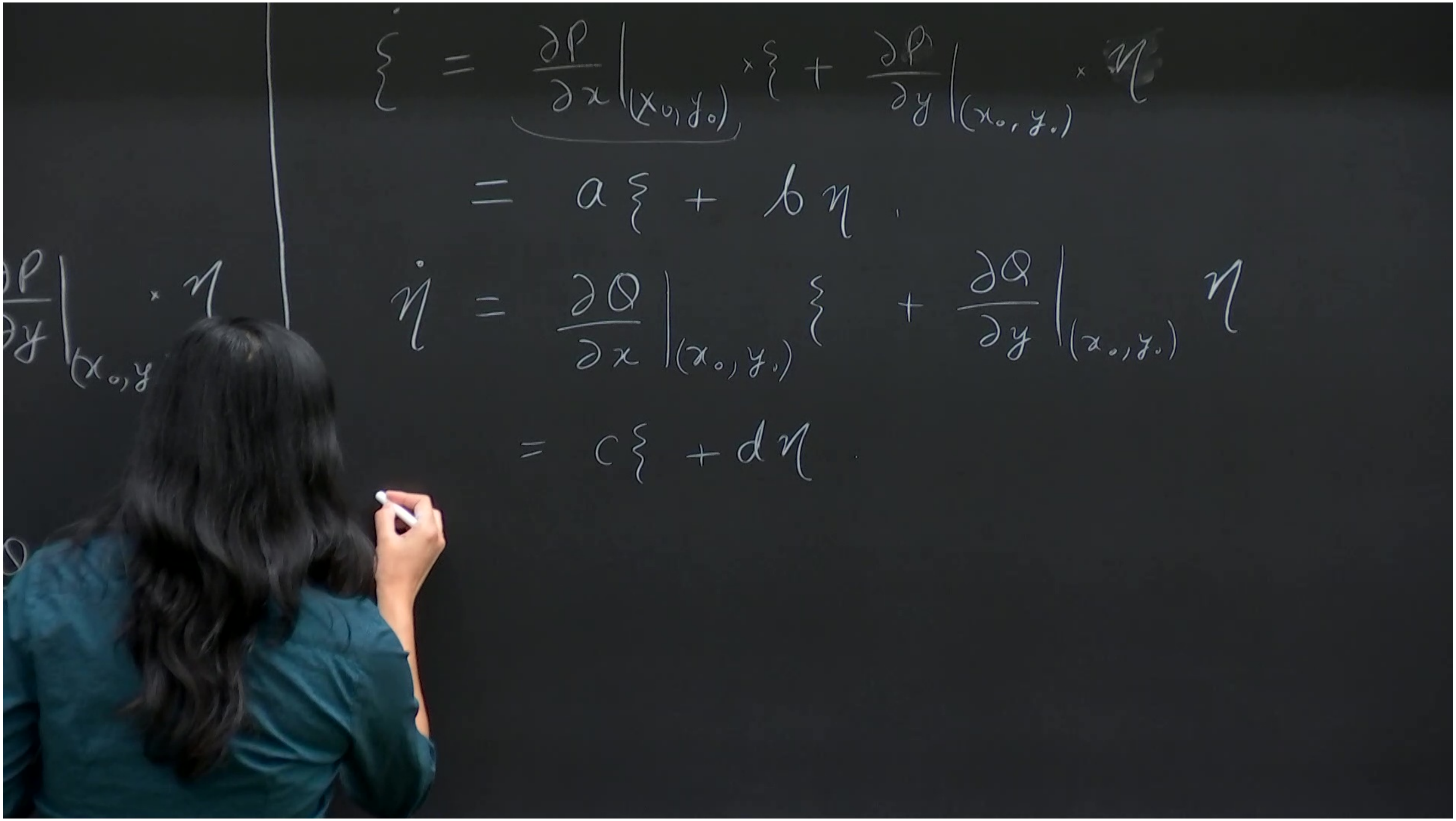
$$P(x,y) = P(x_0+\xi,\, y_0+\eta)$$

$$= \underbrace{P(x_0,y_0)} + \left.\frac{\partial P}{\partial x}\right|_{(x_0,y_0)} \times \xi + \left.\frac{\partial P}{\partial y}\right|_{(x_0,y_0)} \times \eta$$

+ higher order terms.

$$Q(x,y) = \underbrace{Q(x_0,y_0)} + \left.\frac{\partial Q}{\partial x}\right|_{(x_0,y_0)} \times \xi + \left.\frac{\partial Q}{\partial y}\right|_{(x_0,y_0)} \times \eta$$

+ h.o.t


$\dot{\xi} = \underbrace{\frac{\partial P}{\partial x}\Big|_{(x_0,y_0)}}\times\xi + \frac{\partial P}{\partial y}\Big|_{(x_0,y_0)}\times\eta$
$= a\xi + b\eta$.
$\dot{\eta} = \frac{\partial Q}{\partial x}\Big|_{(x_0,y_0)}\xi + \frac{\partial Q}{\partial y}\Big|_{(x_0,y_0)}\eta$
$= c\xi + d\eta$.

$$\dot{\xi} = \underbrace{\frac{\partial P}{\partial x}\bigg|_{(x_0, y_0)}}\,\xi + \frac{\partial P}{\partial y}\bigg|_{(x_0, y_0)}\,\eta$$

$$= a\xi + b\eta \Rightarrow \eta = \frac{1}{b}\left(\dot{\xi} - a\xi\right)$$

$$\dot{\eta} = \frac{\partial Q}{\partial x}\bigg|_{(x_0, y_0)}\xi + \frac{\partial Q}{\partial y}\bigg|_{(x_0, y_0)}\eta$$

$$= c\xi + d\eta$$

$$\ddot{\xi} = a\dot{\xi} + b\dot{\eta} = a\dot{\xi} + b(c\xi + d\eta) = a\dot{\xi} + bc\xi + bd\left(\frac{\dot{\xi} - a\xi}{b}\right)$$

$\Rightarrow$

$$\dot{\xi} = \underbrace{\frac{\partial P}{\partial x}\Big|_{(x_0, y_0)}}\xi + \frac{\partial P}{\partial y}\Big|_{(x_0, y_0)}\eta$$

$$= a\xi + b\eta \Rightarrow \eta = \frac{1}{b}\left(\dot{\xi} - a\xi\right)$$

Solution

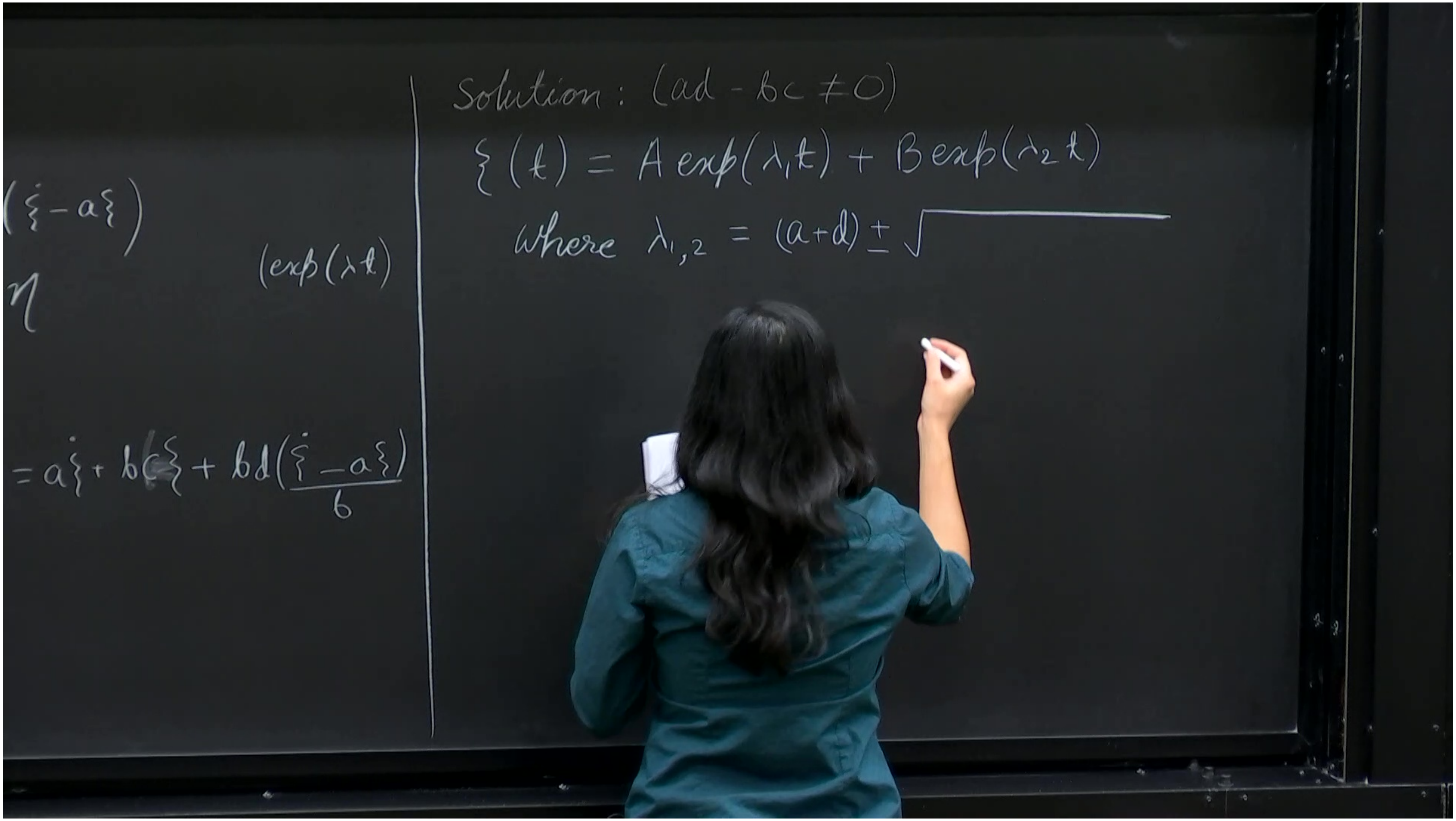
$(\exp(\lambda t))$

$$\dot{\eta} = \frac{\partial Q}{\partial x}\Big|_{(x_0, y_0)}\xi + \frac{\partial Q}{\partial y}\Big|_{(x_0, y_0)}\eta$$

$$= c\xi + d\eta$$

$$\ddot{\xi} = a\dot{\xi} + b\dot{\eta} = a\dot{\xi} + b(c\xi + d\eta) = a\dot{\xi} + b(c\xi \; [illegible] \; a\xi)$$

$$\Rightarrow \ddot{\xi} - (a+d)\dot{\xi} + (ad - bc)\xi = 0$$


Solution : (ad - bc ≠ 0)
ξ(t) = A exp(λ₁t) + B exp(λ₂t)
where λ₁,₂ = (a+d) ± √
(ξ̇ - aξ)
(exp(λt)
η
= aξ̇ + bcξ + bd((ξ̇ - aξ)/b)

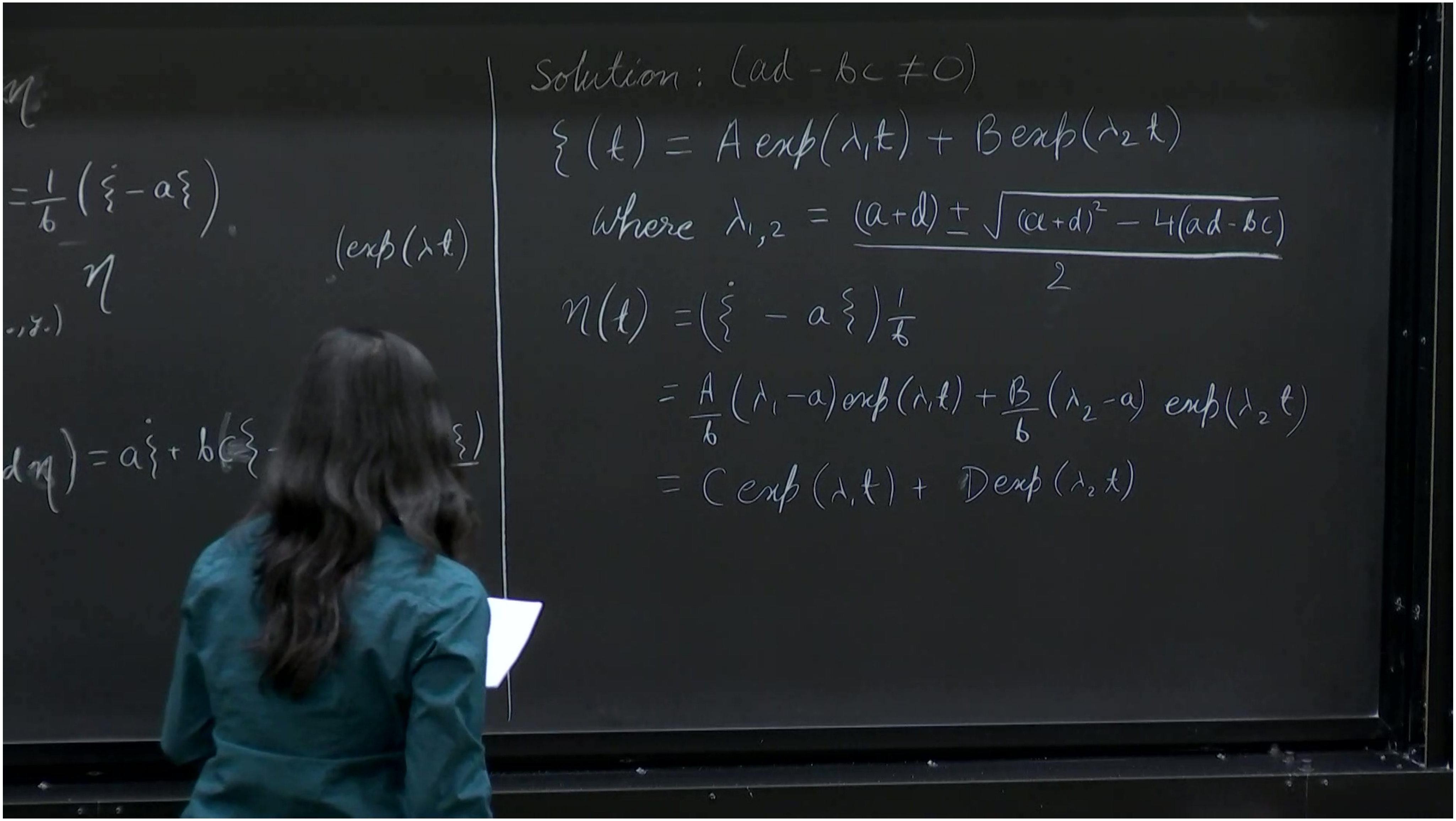
$(\exp(\lambda t))$

$- bd\left(\frac{\dot{\xi} - a\xi}{b}\right)$

Solution: $(ad - bc \neq 0)$

$$\xi(t) = A\exp(\lambda_1 t) + B\exp(\lambda_2 t)$$

where $\lambda_{1,2} = \dfrac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$


Solution : $(ad - bc \neq 0)$
$\xi(t) = A\exp(\lambda_1 t) + B\exp(\lambda_2 t)$
Where $\lambda_{1,2} = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$
$\eta(t) = (\dot{\xi} - a\xi)\frac{1}{b}$
$= \frac{A}{b}(\lambda_1 - a)\exp(\lambda_1 t) + \frac{B}{b}(\lambda_2 - a)\exp(\lambda_2 t)$
$= C\exp(\lambda_1 t) + D\exp(\lambda_2 t)$
$= \frac{1}{b}(\dot{\xi} - a\xi)$
$(\exp(\lambda t)$

η

$= \frac{1}{b}(\dot{\xi} - a\xi)$.

η

$(\exp(\lambda t)$

$(.., y..)$

$d\eta) = a\dot{\xi} + bc\xi + bd\left(\frac{\dot{\xi} - a\xi}{b}\right)$

Solution: $(ad - bc \neq 0)$

$$\xi(t) = A\exp(\lambda_1 t) + B\exp(\lambda_2 t)$$

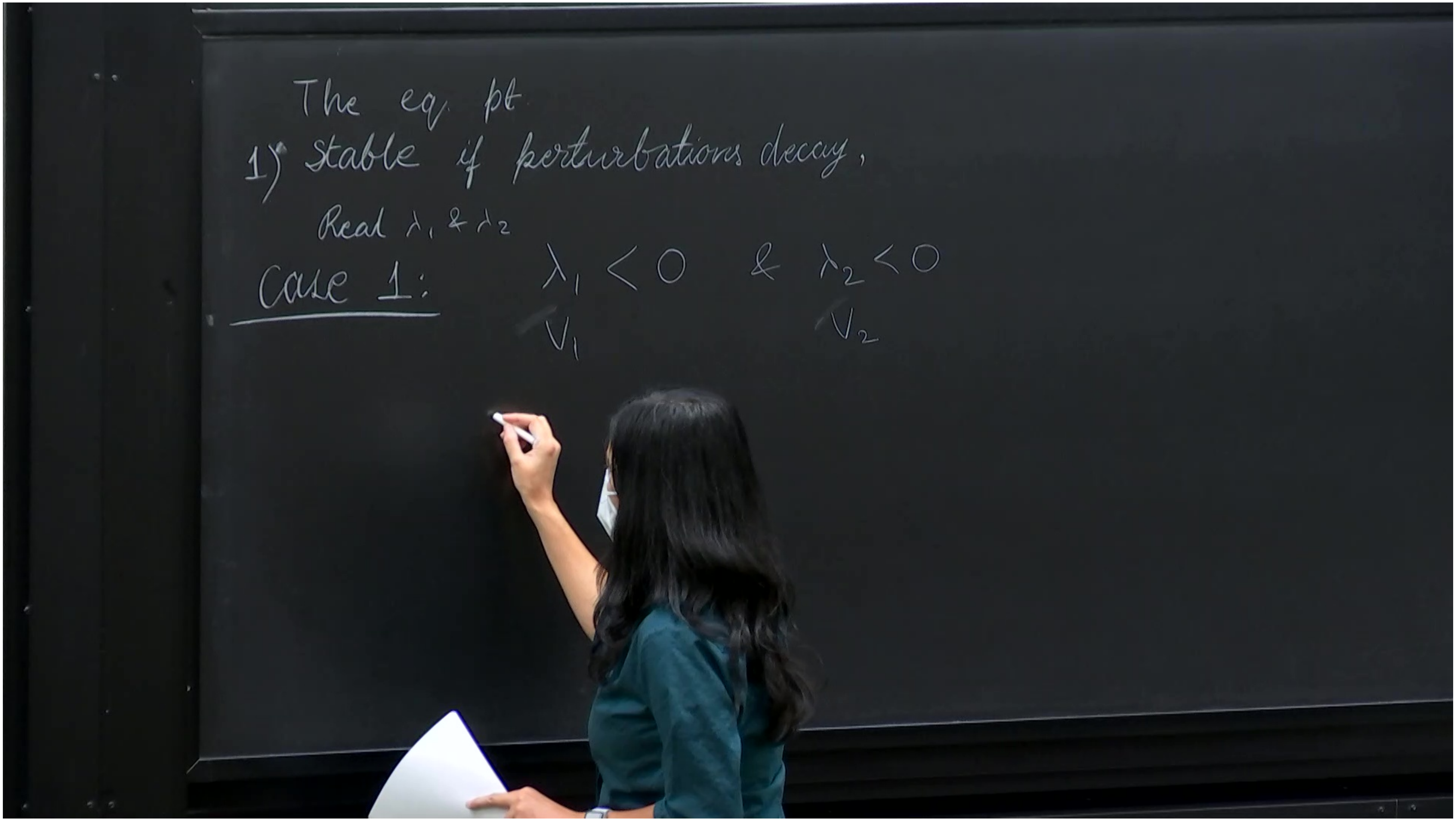
where $\lambda_{1,2} = \dfrac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$

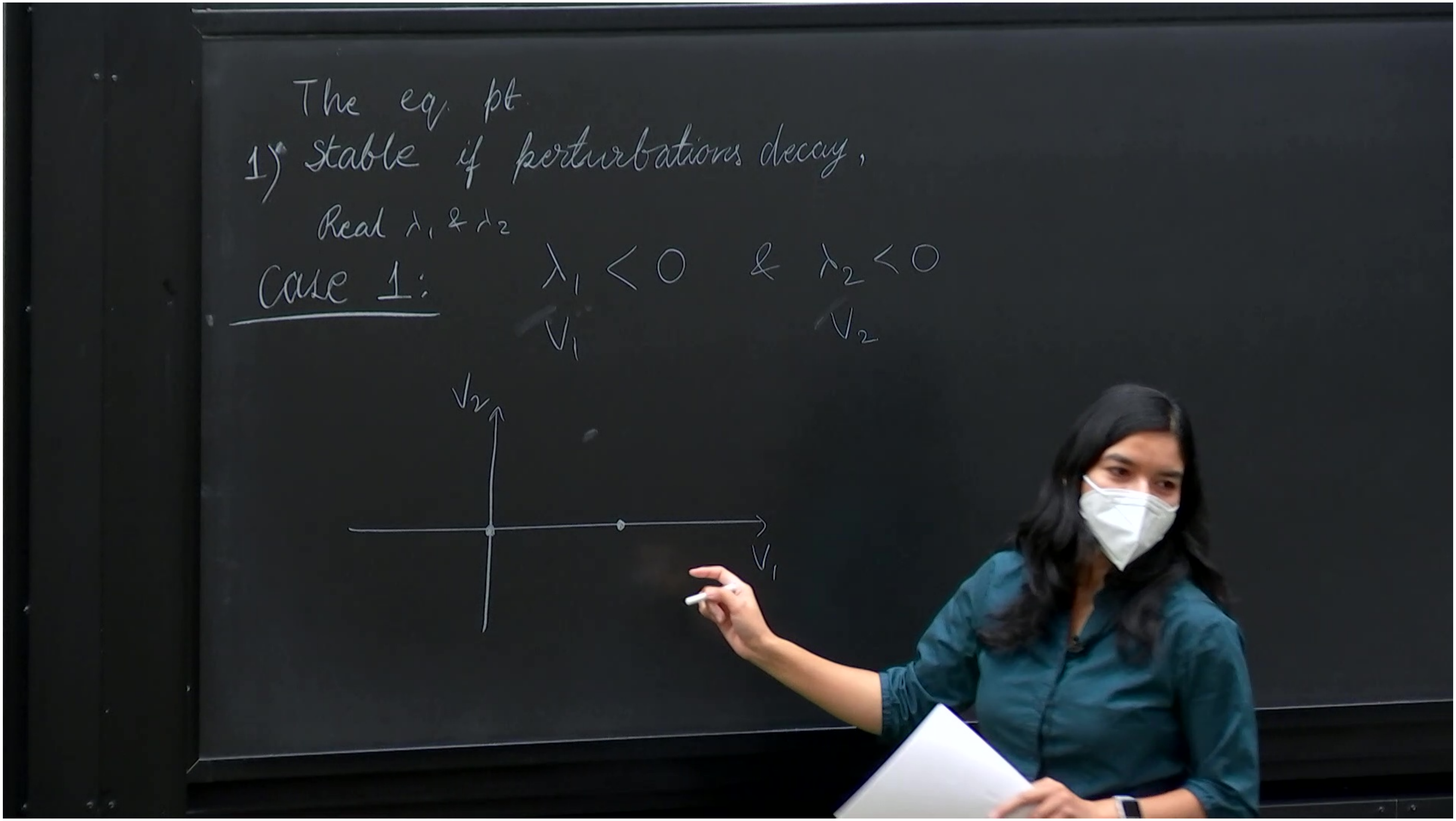
$$\eta(t) = (\dot{\xi} - a\xi)\frac{1}{b}$$

$$= \frac{A}{b}(\lambda_1 - a)\exp(\lambda_1 t) + \frac{B}{b}(\lambda_2 - a)\exp(\lambda_2 t)$$

$$= C\exp(\lambda_1 t) + D\exp(\lambda_2 t)$$

$$\begin{bmatrix} \dot{\xi} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix} = M\begin{bmatrix} \xi \\ \eta \end{bmatrix}$$


The eq. pt
1) Stable if perturbations decay,
Real λ_1 & λ_2
Case 1: $\lambda_1 < 0$ & $\lambda_2 < 0$
V_1 V_2


The eq. pt.
1) Stable if perturbations decay,
Real λ1 & λ2
Case 1:
λ1 < 0 & λ2 < 0
V1
V2
V2
V1