Title: Optimal Thresholds for Fracton Codes and Random Spin Models with Subsystem Symmetry

Speakers: Hao Song

Series: Perimeter Institute Quantum Discussions

Date: March 02, 2022 - 11:00 AM

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Abstract: Fracton models provide examples of novel gapped quantum phases of matter that host intrinsically immobile excitations and therefore lie beyond the conventional notion of topological order. Here, we calculate optimal error thresholds for quantum error correcting codes based on fracton models. By mapping the error-correction process for bit-flip and phase-flip noises into novel statistical models with Ising variables and random multi-body couplings, we obtain models that exhibit an unconventional subsystem symmetry instead of a more usual global symmetry. We perform large-scale parallel tempering Monte Carlo simulations to obtain disorder-temperature phase diagrams, which are then used to predict optimal error thresholds for the corresponding fracton code. Remarkably, we found that the X-cube fracton code displays a minimum error threshold (7.5%) that is much higher than 3D topological codes such as the toric code (3.3%), or the color code (1.9%). This result, together with the predicted absence of glass order at the Nishimori line, shows great potential for fracton phases to be used as quantum memory platforms. If time allows, I will also present some of our more recent progress on fractons.

Reference: arXiv:2112.05122.

Zoom Link: https://pitp.zoom.us/j/97053396111?pwd=Ny9tK295dGVacENJMzg0aHRObjZEZz09

# Optimal Thresholds for Fracton Codes and Random Spin Models with Subsystem Symmetry

HS, J Schönmeier-Kromer, K Liu, O Viyuela, L Pollet, MA Martin-Delgado, arXiv:2112.05122.


Hao Song

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March 2, 2022 @ Perimeter Institute


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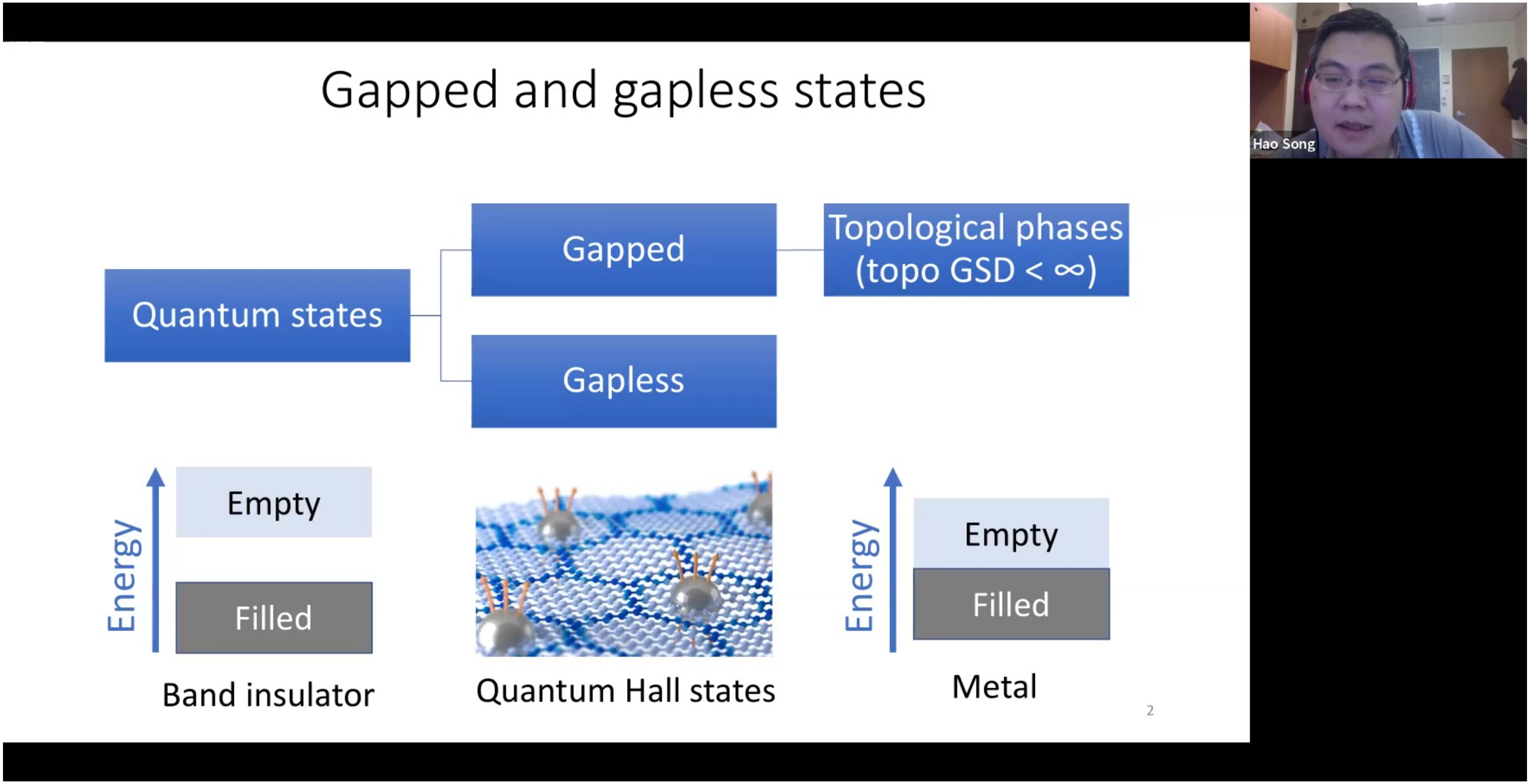
Gapped and gapless states
Quantum states
Gapped
Gapless
Topological phases
(topo GSD < ∞)
Energy
Empty
Filled
Band insulator
Quantum Hall states
Energy
Empty
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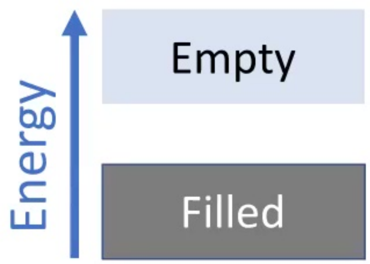
Metal
2
Hao Song

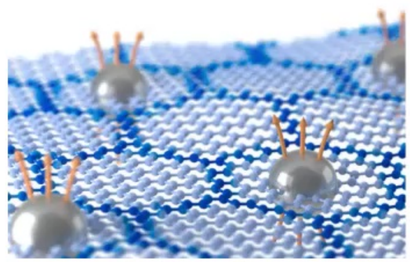
# Gapped and gapless states

GSD = Ground State Degeneracy

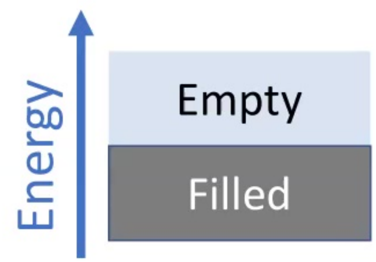
- Quantum states
  - Gapped
    - Topological phases (topo GSD < ∞)
    - Fracton phases (GSD ~ $\#^L$ on $L^3$)
  - Gapless




Band insulator



Quantum Hall states




Metal

2

# What is a fracton?


## Example: X-cube model

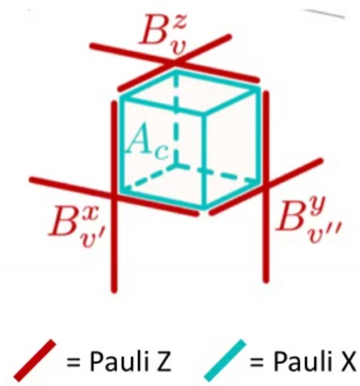
- Exactly solvable stabilizer Hamiltonian on cubic lattice (3D).

$$H_{\mathrm{Xcube}} = -\sum_{c} A_c - \sum_{v} (B_v^x + B_v^y + B_v^z)$$

- Ground states $A_c = B_v^{\mu} = 1$.
- GSD on 3-torus of size L x L x L:
  - $\log_2$ GSD = 6L – 3 (sub-extensive)
  - Gapped
  - Locally indistinguishable
- Similar to topological order but not fully topological



a qubit per edge $\ell$ on cubic lattice




$$A_c = \prod_{\ell \in c} X_\ell \qquad B_v^{\mu} = \prod_{\ell \sim v : \ell \perp \mu} Z_\ell$$

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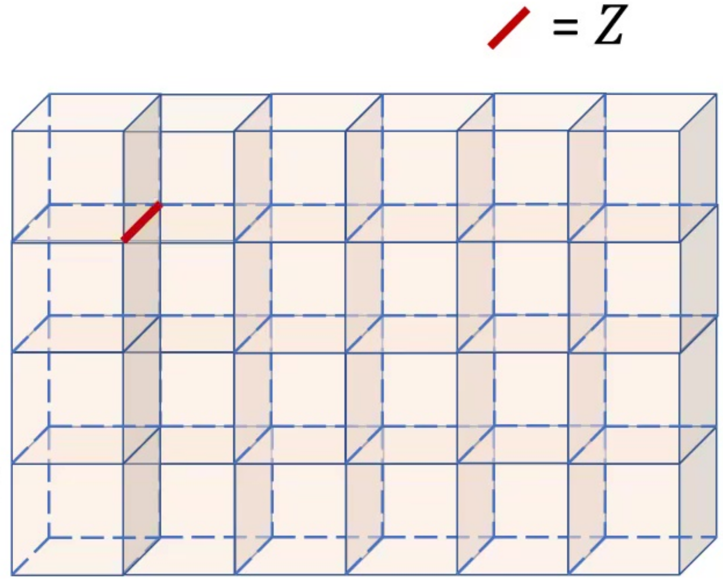
- cyan cube: $A_c = -1$.
- Isolated A-excitations are created at corners of a membrane operator





A **fracton** is an emergent quasiparticle which **fractionalizes into pieces while moving**.
It is **immobile** in the conventional sense!

4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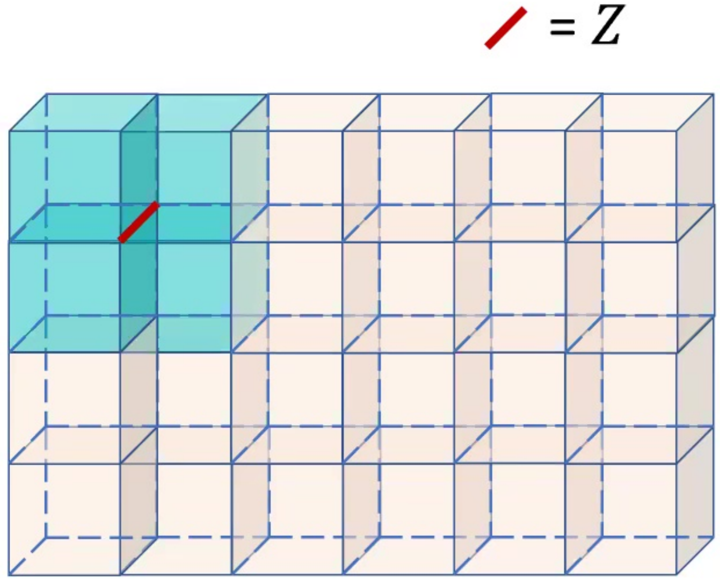


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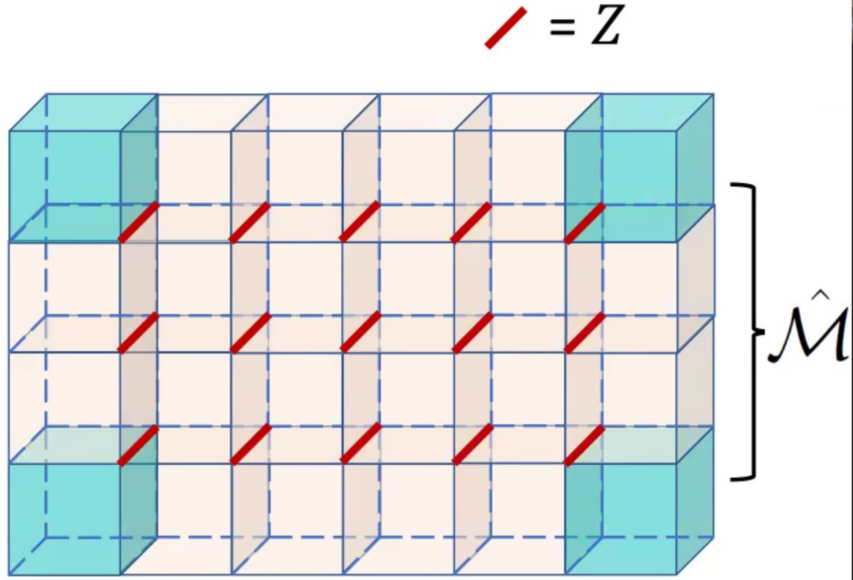



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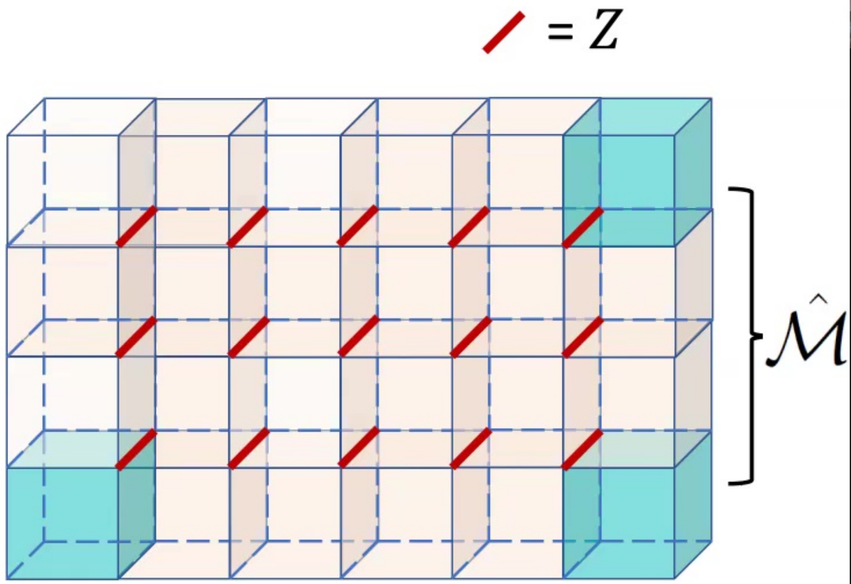



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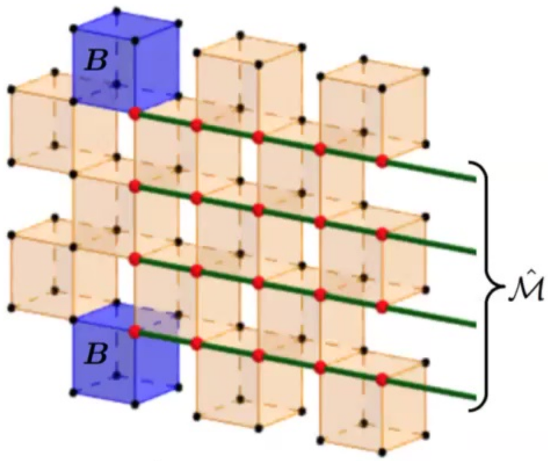
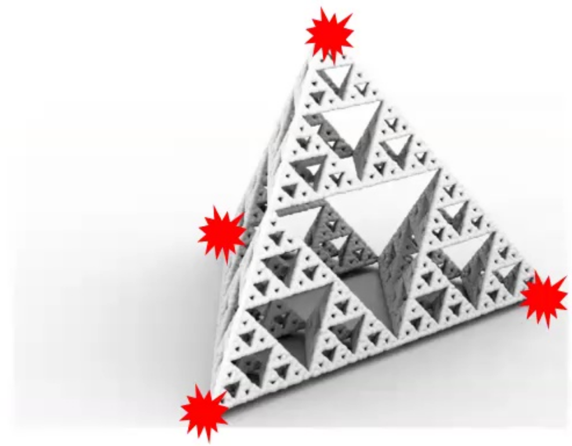
# Fracton order (3+1D)

- Unconventional topological order
  - Interesting gapped quantum phases beyond topological quantum field theories? Yes!
  - Self-correcting quantum memories in d=3? Not fully realized yet!




Type-I: Chamon, Bravyi, Leemhuis, Terhal (2005, 2010)
Vijay, Haah, Fu (2015, 2016)



Type-II: Haah (2011)
Yoshida (2013)

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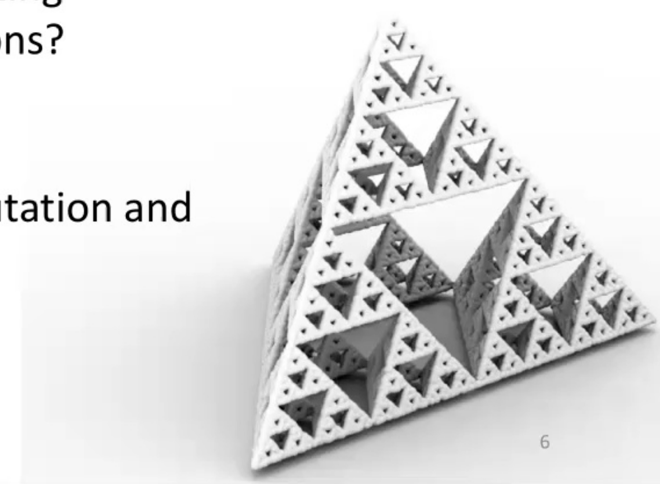
# Open questions on fracton Order (3+1D)

- How to properly define such phases in the continuum, RG fixed point?
- A systematic way to distinguish or characterizing fracton phases, like algebraic theory of fractons?
- Its potential applications for quantum computation and error correction?



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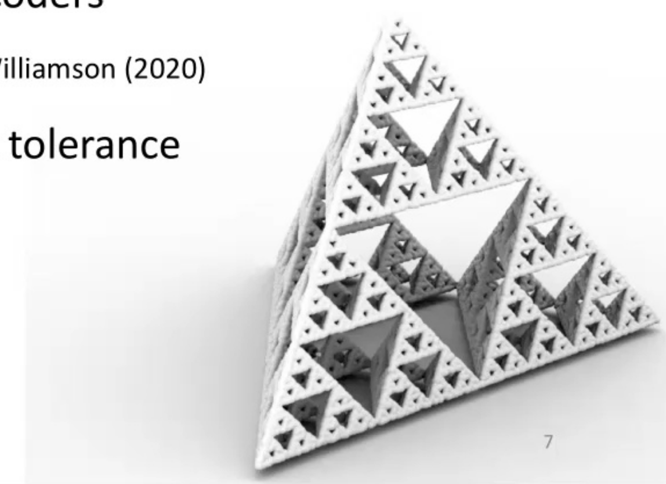
# Self-error correction vs active error correction

- Self-error correction: errors are correctable by thermal bath. “Decoder = thermal bath.”

- Various practical active decoders, e.g., RG decoder.
  - Fracton order may allow more efficient decoders

    Brown and Williamson (2020)

  - Practical decoders may sacrifice some fault tolerance for efficiency



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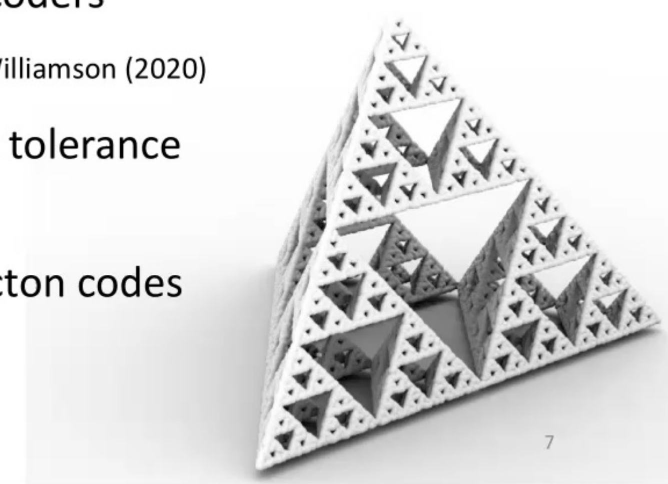
  - Practical decoders may sacrifice some fault tolerance for efficiency
- The theoretical limit of fault tolerance of fracton codes (in dependent of the choice of decoders)?



# Outline

1. Review of quantum error correction in toric code
   [Dennis, Kitaev, Landahl, Preskill, 2001]

2. Optimal Thresholds for the X-cube Code
   [arXiv:2112.05122]

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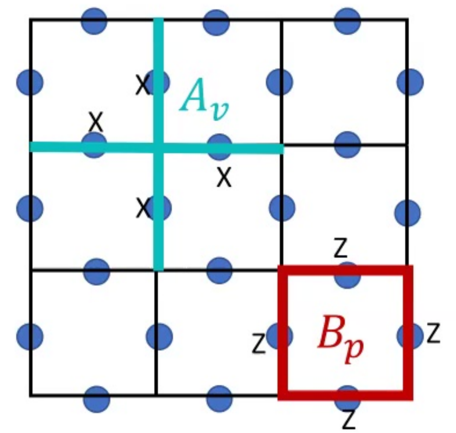
# Toric code as a topological quantum memory


- Toric code is a Calderbank-Shor-Steane (CSS) code allows us to treat Z and X errors separately
- Stabilizer generators

$$A_v = X^{\otimes 4}, \qquad B_p = Z^{\otimes 4}.$$

- Code space $C$ is selected by $A_v = B_p = 1$.
  - $\dim C = 4 = 2^2$ on torus (i.e. periodic boundary condition).




physical qubits (on edges)

= Pauli Z, = = Pauli X.

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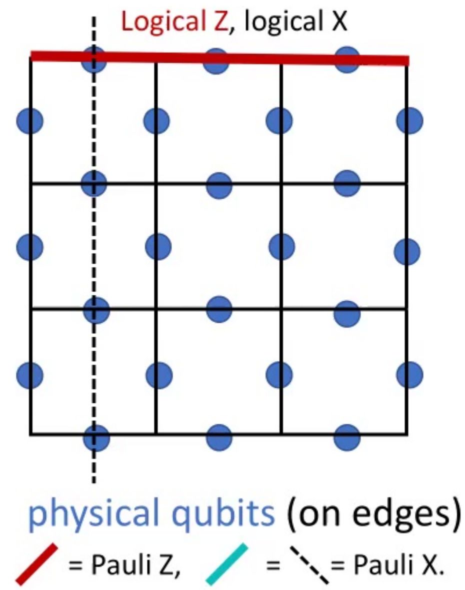



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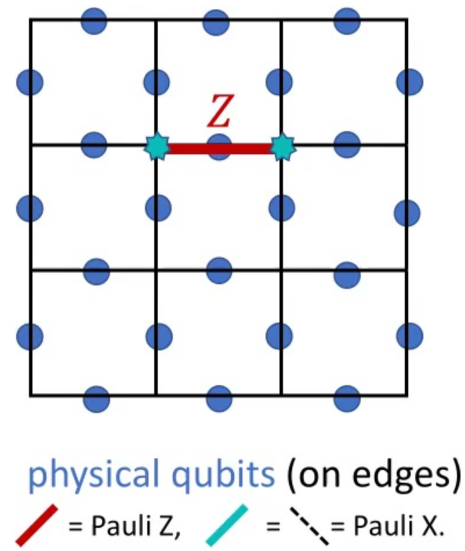
  - An $Z$ error → two flipped $A_v$ ("**A-syndrome**" = the set of flipped A-operators.)

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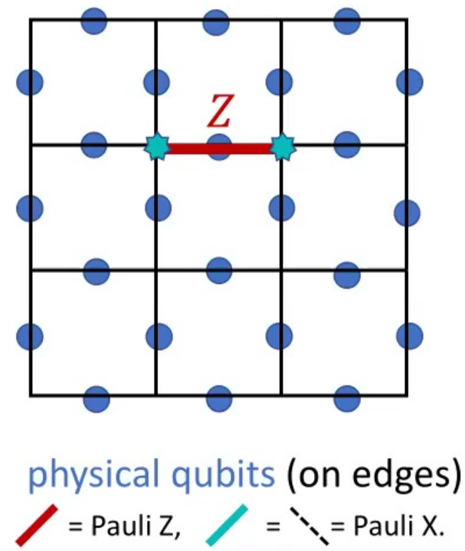
- **Dense** errors → uncorrectable logical errors.




Suppose Z (or X) errors occur at each qubit independently with probability $p$.

sparser $p_c^{Z,X}$ denser

correctable → uncorrectable

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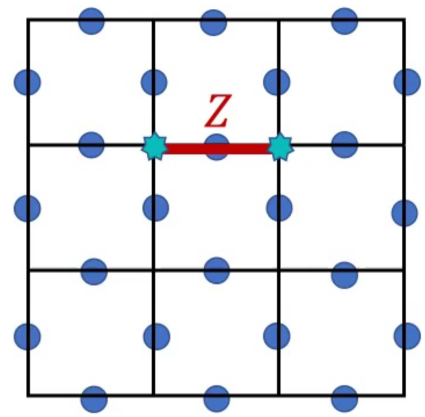


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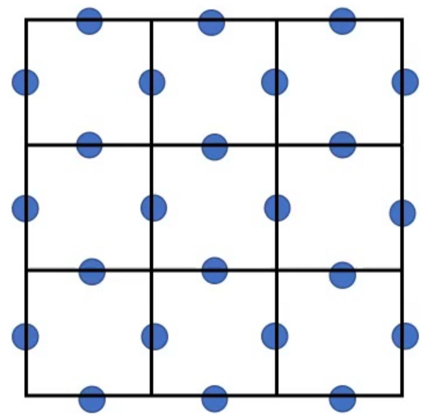
# Toric code in chain complex terminology

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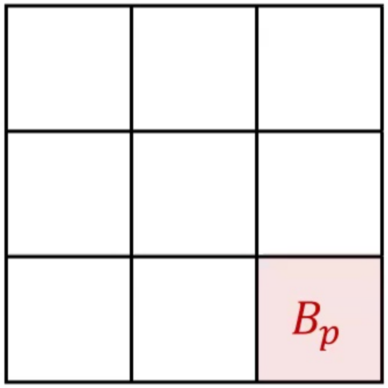


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- Let $\mathbb{Z}_2^{\mathcal{S}} = \mathrm{Fun}(\mathcal{S}, \mathbb{Z}_2 = \{0,1\})$ be the set of $\mathbb{Z}_2$-valued functions (i.e., **indicator functions**) on set $\mathcal{S}$.
- Relation between qubits and stabilizers

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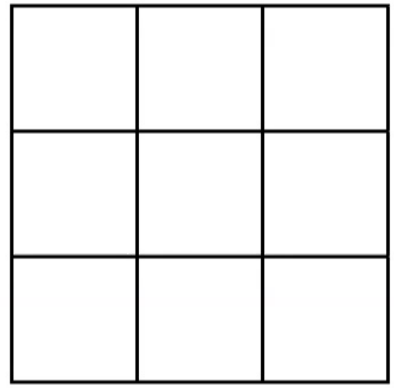

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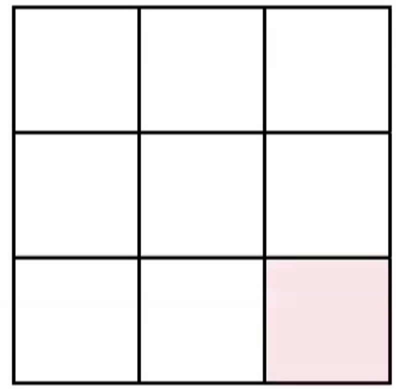

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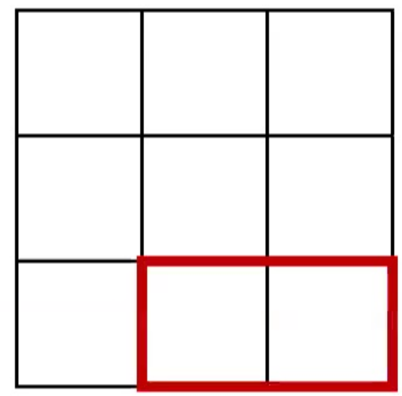

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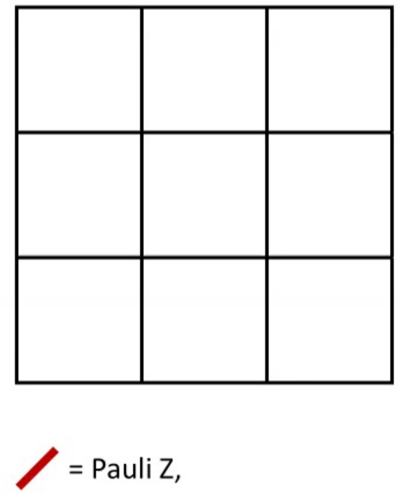


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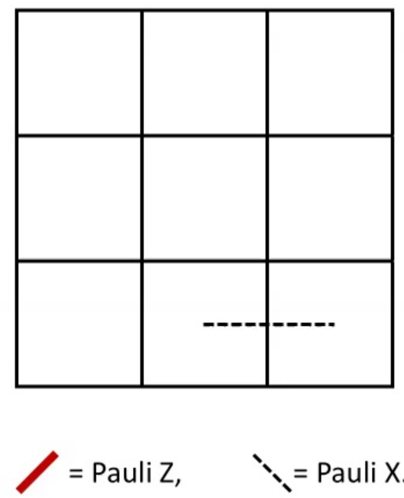


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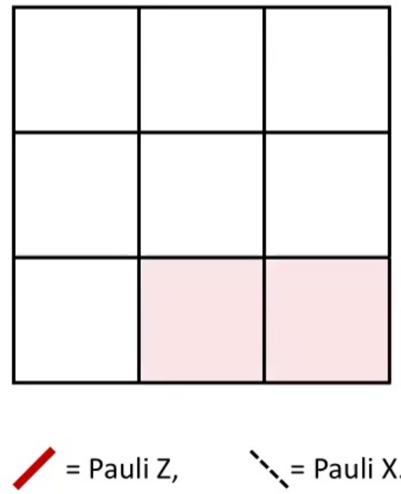


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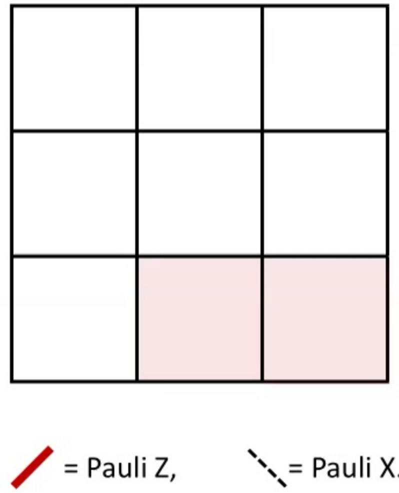


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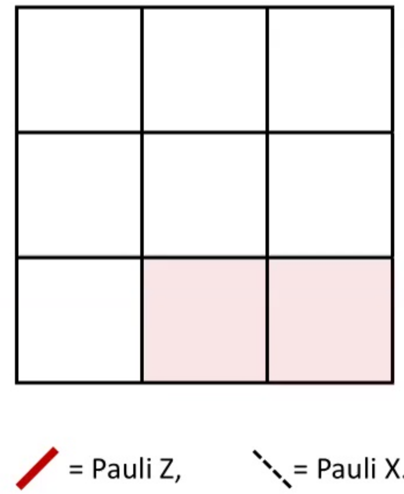
  - $\partial_A^\dagger \circ \partial_B = 0 = \partial_B^\dagger \circ \partial_A$.
  - Logical Z-operators $\in \ker \partial_A^\dagger$, but $\notin \mathrm{im}\, \partial_B$.

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$\mathcal{B}$ = label set of B-terms (plaquettes)

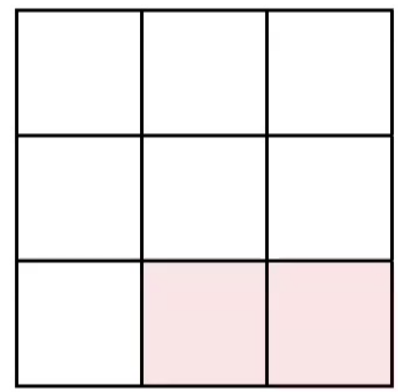



# Toric code in chain complex terminology

- Let $\mathbb{Z}_2^{\mathcal{S}} = \mathrm{Fun}(\mathcal{S}, \mathbb{Z}_2 = \{0,1\})$ be the set of $\mathbb{Z}_2$-valued functions (i.e., **indicator functions**) on set $\mathcal{S}$.
- Relation between qubits and stabilizers

$$\mathbb{Z}_2^{\mathcal{A}} \underset{\partial_A^\dagger}{\overset{\partial_A}{\rightleftharpoons}} \mathbb{Z}_2^{\mathcal{Q}} \underset{\partial_B}{\overset{\partial_B^\dagger}{\rightleftharpoons}} \mathbb{Z}_2^{\mathcal{B}}$$

  - $\partial_B$ maps a collection of B-terms to the Z-error configuration generated by them.
  - The transpose $\partial_B^\dagger$ maps an X-error configuration to the collection of B-terms flipped by them.
  - $\partial_A^\dagger \circ \partial_B = 0 = \partial_B^\dagger \circ \partial_A$.
  - Logical Z-operators $\in \ker\partial_A^\dagger$, but $\notin \mathrm{im}\partial_B$.
  - Logical X-operators $\in \ker\partial_B^\dagger$, but $\notin \mathrm{im}\partial_A$.



= Pauli Z, = Pauli X.

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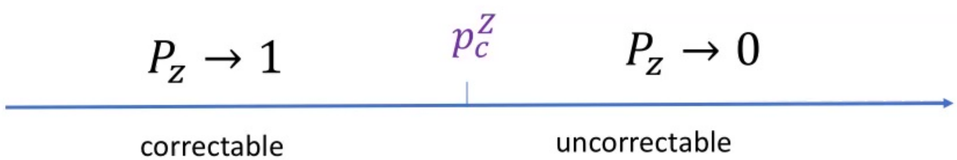
# Maximal success probability in Z-error correction

- All Z-error configurations $\eta, \eta'$ acts trivially on the code space, iff $\eta - \eta' \in \mathrm{im}\partial_B$.

$$\text{Z-error equivalence class } [\eta] = \eta + \mathrm{im}\partial_B \in \mathbb{Z}_2^{\mathcal{Q}}/\mathrm{im}\partial_B.$$

- For each possible A-syndrome $\sigma \in \mathbb{Z}_2^{\mathcal{A}}$, there are four compatible Z-error equivalence classes $[\eta_\sigma]$, $[\eta_\sigma] + \mathfrak{z}_1$, $[\eta_\sigma] + \mathfrak{z}_2$, and $[\eta_\sigma] + \mathfrak{z}_1 + \mathfrak{z}_2$.
- Choose $\eta_\sigma$ such that $[\eta_\sigma]$ has the largest probability among the four.
- maximal success primality in Z-error correction $P_z = \sum_\sigma [\eta_\sigma]$.
- Suppose Z errors occur at each qubit independently with probability *p*. By analogy to statistical-mechanical models, as system size → ∞,




# Mapping to statistical-mechanical models

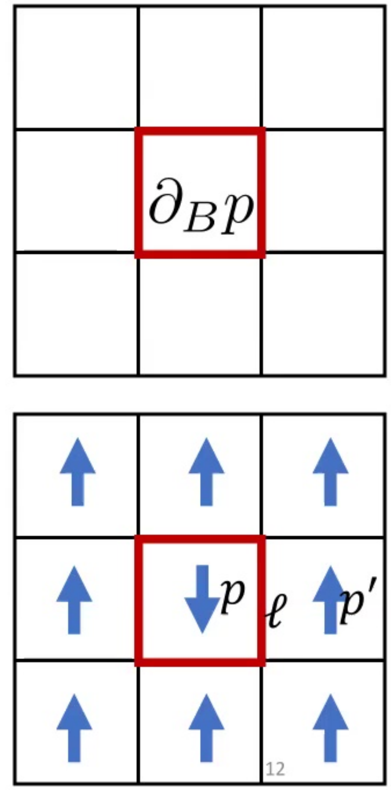
- For Z-error configuration $\eta$ with $\eta(\ell) = 0$ or 1 on edges with or without an error,

$$\mathrm{pr}\,(\eta; p) = \prod_{\ell \in \mathcal{Q}} p^{\eta(\ell)} (1-p)^{1-\eta(\ell)} \propto \left(\frac{p}{1-p}\right)^{\sum_\ell \eta(\ell)}$$

- Introduce auxiliary temperature $e^{-\frac{2}{T}} = \frac{p}{1-p}$
- The relation is called the **Nishimori line** in the p-T plane.
- The total probability of a Z-error equivalence class $[\eta] = \eta + \mathrm{im}\partial_B$ → partition function of random Ising model at T

$$H_\eta^{\mathcal{B}} = -\sum_{\ell \in \mathcal{Q}} (-1)^{\eta(\ell)} \prod_{p \in \partial_B^\dagger \ell} S_p = -\sum_{\langle pp' \rangle} J_{pp'} S_p S_{p'}$$

$J_{pp'} = \pm 1$ with probability $1-p$ and $p$.



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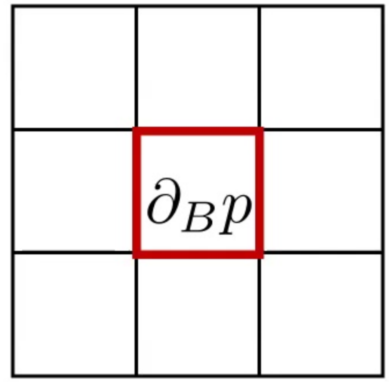
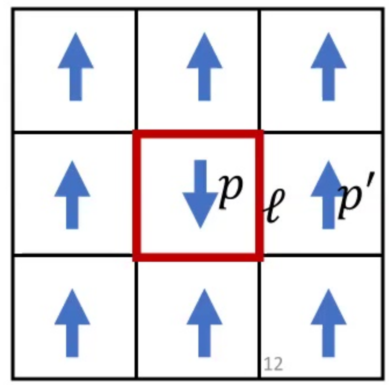
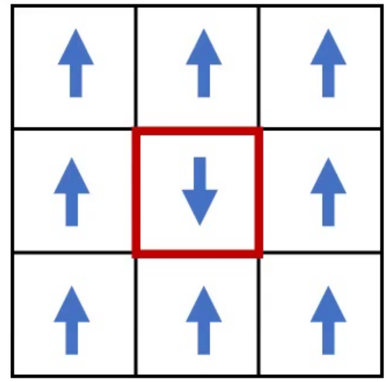


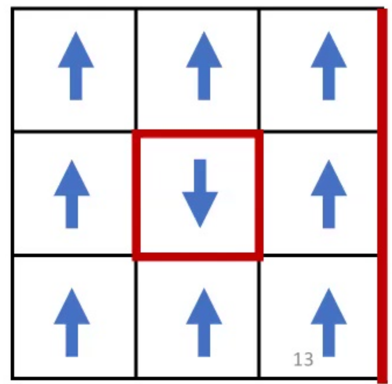





# Wisdom from statistical-mechanical models

- Probabilities for the four classes $[\eta]$, $[\eta] + \mathfrak{z}_1$, $[\eta] + \mathfrak{z}_2$, and $[\eta] + \mathfrak{z}_1 + \mathfrak{z}_2$ with the same syndrome differs by a contractible domain wall (on random Ising model side).
  - In the ordered phase, one dominates → correctable
  - In the disordered phase, they are comparable → uncorrectable



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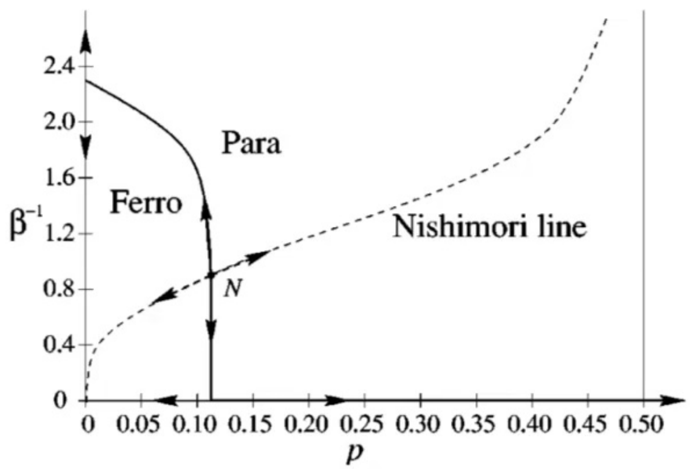


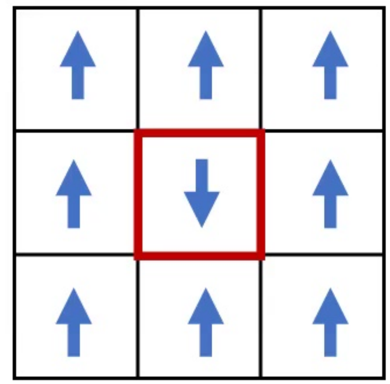

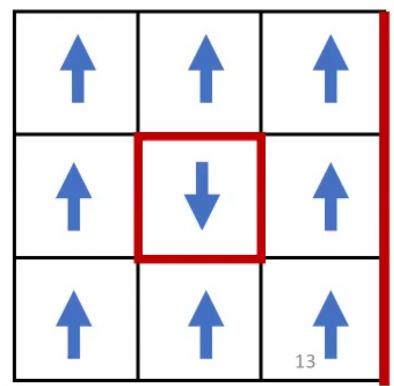

# Outline

1. Review of quantum error correction in toric code

   [Dennis, Kitaev, Landahl, Preskill, 2001]

2. Optimal Thresholds for the X-cube Code

   [arXiv:2112.05122]

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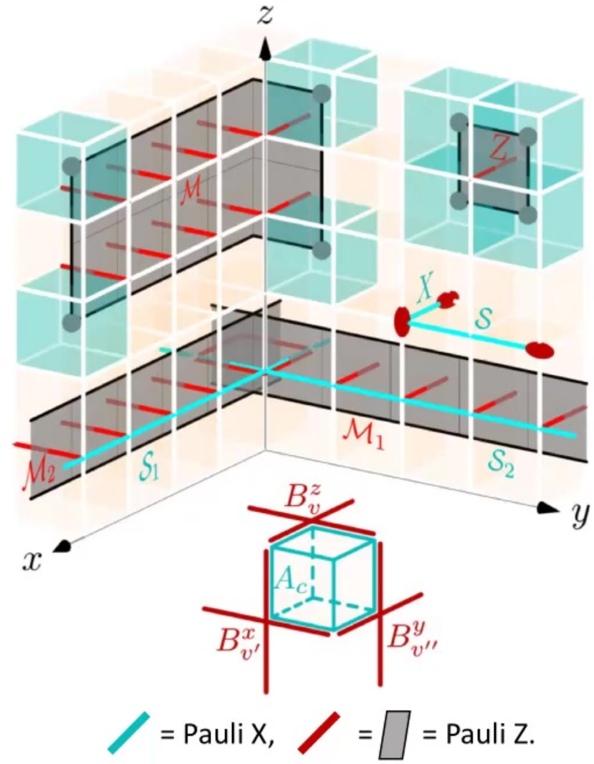
# Logical operators in X-cube code

- Toric code is a Calderbank-Shor-Steane (CSS) code allows us to treat Z and X errors separately
- Stabilizer generators

$$A_v = X^{\otimes 12}, \qquad B_p = Z^{\otimes 4}.$$

- Code space $C$ is selected by $A_v = B_p = 1$.
  - $\dim C = 2^{6L-3}$ on $L \times L \times L$ torus (i.e. periodic boundary condition).
  - 6L-3 logical qubits
  - Logical Z operators: $\mathcal{M}_1, \mathcal{M}_2, \ldots$
  - Logical X operators: $\mathcal{S}_1, \mathcal{S}_2, \ldots$

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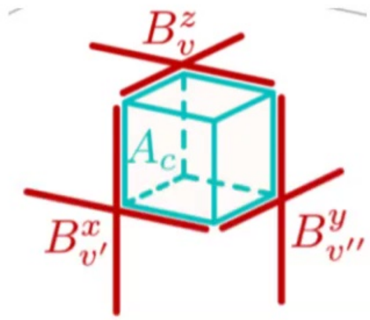
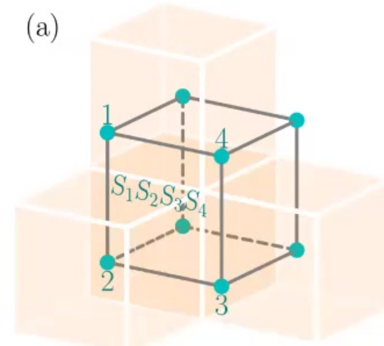
# Mapping to statistical-mechanical models




X-cube fracton code
(on a cubic lattice with a qubit per edge)

Suppose X (and Z) errors occur independently at each qubit with probability $p$.

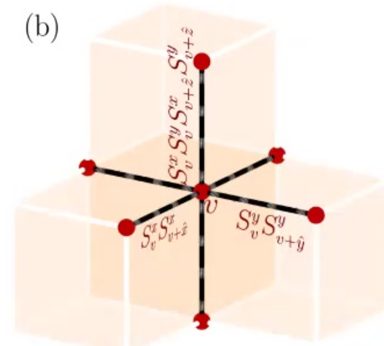
$$\mathrm{pr}(\eta; p) = \prod_{\ell \in Q} p^{\eta(\ell)} (1-p)^{1-\eta(\ell)} \propto \left(\frac{p}{1-p}\right)^{\sum_\ell \eta(\ell)}$$




**R**andom **P**laquette **I**sing model

$$H_\eta^{\mathcal{A}} = -\sum_{\ell \in Q} (-1)^{\eta(\ell)} \prod_{c \in \partial_A^\dagger \ell} S_c$$

For X-error equivalence class
$[\eta]_X = \eta + \mathrm{im}\partial_A$




**R**andom **A**nisotropically
**C**oupled **A**shkin-**T**eller model

$$H_\eta^{\mathcal{B}}(\{S_v^x, S_v^y\}_v) = -\sum_v \sum_{\mu=x,y,z} J_v^\mu S_v^\mu S_{v+\hat{\mu}}^\mu,$$

$$S_v^z \equiv S_v^x S_v^y, \qquad J_v^\mu \equiv (-1)^{\eta(\langle v, v+\hat{\mu}\rangle)}$$

For Z-error equivalence class
$[\eta]_Z = \eta + \mathrm{im}\partial_B$

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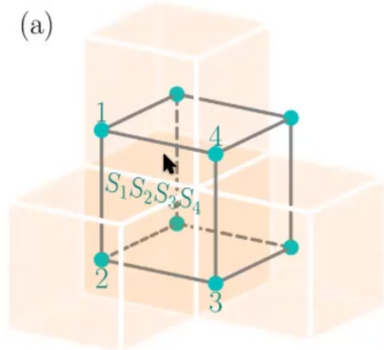
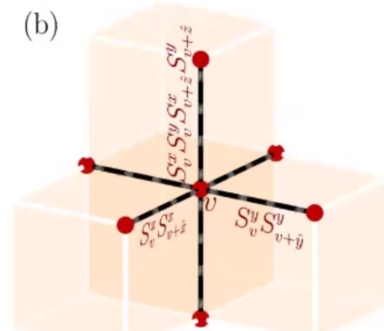
# Planar subsystem symmetry and order paramters




**R**andom **P**laquette **I**sing model




**R**andom **A**nisotropically
**C**oupled **A**shkin-**T**eller model

Symmetry: $S_c \to -S_c$ for $c$ on any *xy*-, *yz*-, or *zx*-plane

Symmetry: $S^\mu_v \to -S^\mu_v,\ S^\nu_v \to -S^\nu_v$ for $v$ on any $\mu\,\nu$-plane

$$G^{\mathcal{A}}(\mathbf{r}) = [\langle S_c S_{c+\hat{z}} S_{c+\mathbf{r}} S_{c+\mathbf{r}+\hat{z}} \rangle]$$

$$G^{\mathcal{B}}(r) = [\langle S^x_v S^x_{c+r\hat{x}} \rangle]$$

$$Q^{\mathcal{A}} := \frac{1}{L^3} \sum_{z=0}^{L-1} \left[ \left\langle \left| \sum_{x,y=0}^{L-1} S_{c(x,y,z)} S_{c(x,y,z+1)} \right| \right\rangle \right]$$

planewise dipole moment

$$Q^{\mathcal{B}} := \frac{1}{L^3} \sum_{x,y=0}^{L-1} \left[ \left\langle \left| \sum_{z=0}^{L-1} S^z_{v(x,y,z)} \right| \right\rangle \right]$$

linewise magnetization

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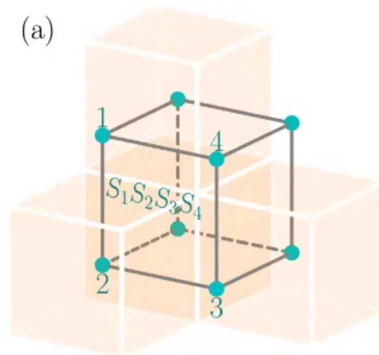
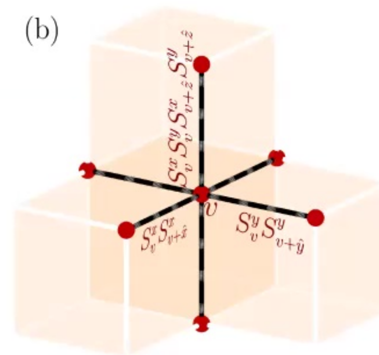
# Phase diagrams (by Monte Carlo simulations)

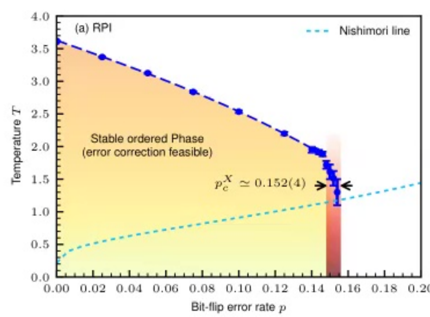
- First-order phase transition for both models at small *p*. (revealed by energy histogram)
- For larger p, the phase transitions are softened to continuous ones.

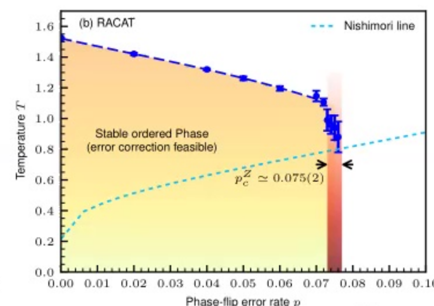



Random Plaquette Ising model




**R**andom **A**nisotropically **C**oupled **A**shkin-**T**eller model



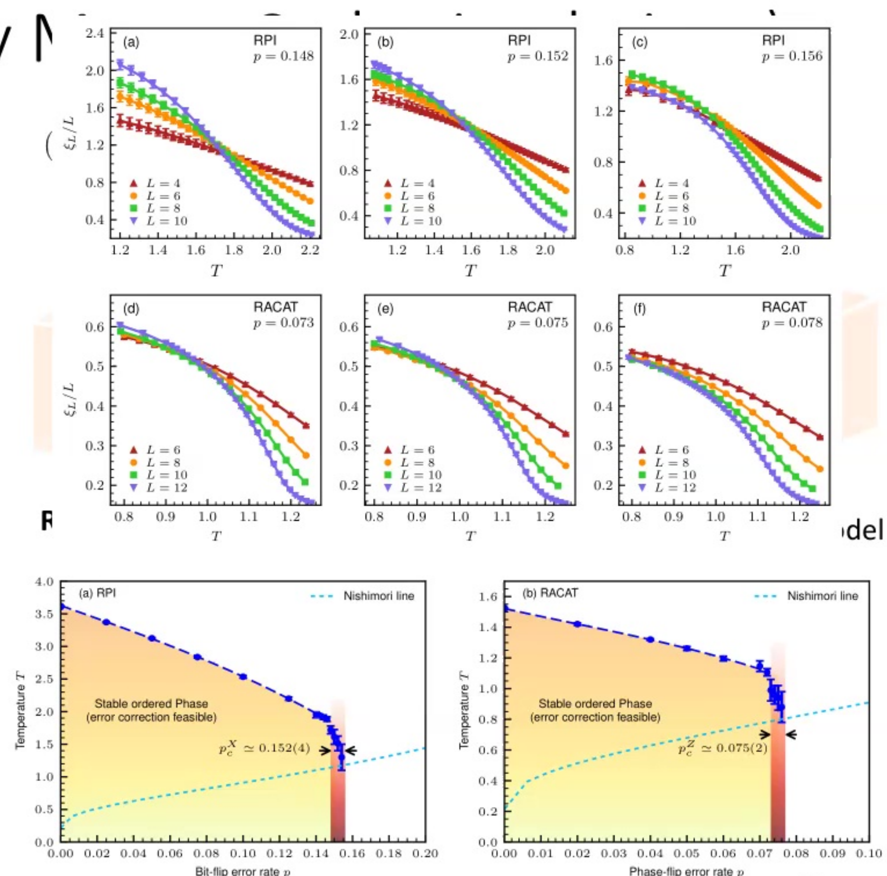
# Phase diagrams (by

- First-order phase transition for both models at small *p.* (revealed by energy histogram)
- For larger p, the phase transitions are softened to continuous ones.
  - To precise identify the location of the transition, we study the second-moment correlation length

$$\xi_L := \frac{1}{2\sin\left(|\mathbf{k}_{\min}|/2\right)}\left(\frac{\tilde{G}(\mathbf{0})}{\tilde{G}\left(\mathbf{k}_{\min}\right)}-1\right)^{1/2}$$

  - Scaling $\frac{\xi_L}{L} = g\left(L^{\frac{1}{\nu}}\left(T-T_c\right)\right)$

→ crossing at $(T_c, g(0))$ if there is a continuous phase transition.







# Optimal thresholds of CSS code

- X-cube code (3D)
  - $p_c^X = 15.2\%$, $p_c^Z = 7.5\%$
- 2D toric code & color code
  - $p_c^X = p_c^X = 10.9\%$

Honecker, Picco, Pujol, 2001

Katzgraber, Bombin, Martin-Delgado, 2009

- 3D toric code
  - $p_c^X = 23.5\%$, $p_c^Z = 3.3\%$

Ozeki and Ito, 1998

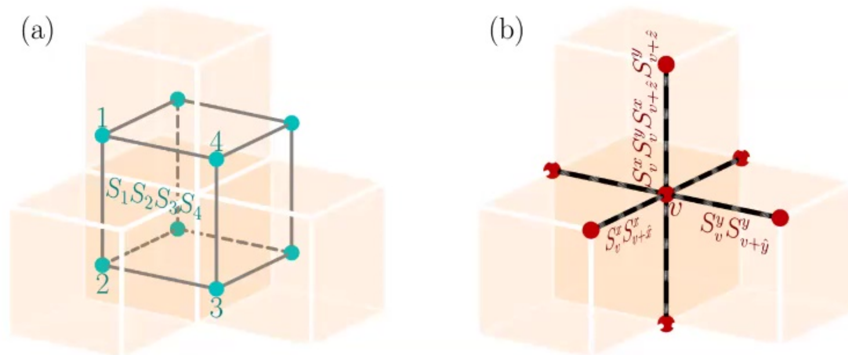
Ohno, Arakawa, Ichinose, Matsui, 2004

- 3D color code
  - $p_c^X = 27.6\%$, $p_c^Z = 1.9\%$

Kubica, Beverland, Brandão, Preskill, Svore,2018

- Approximate duality (Nishimori 2007) works for the X-cube code:

$H(p_c^X) + H(p_c^Z) \approx 1$, where $H(p) = -p\log_2(p) - (1-p)\log_2(1-p)$ is the Shannon entropy.

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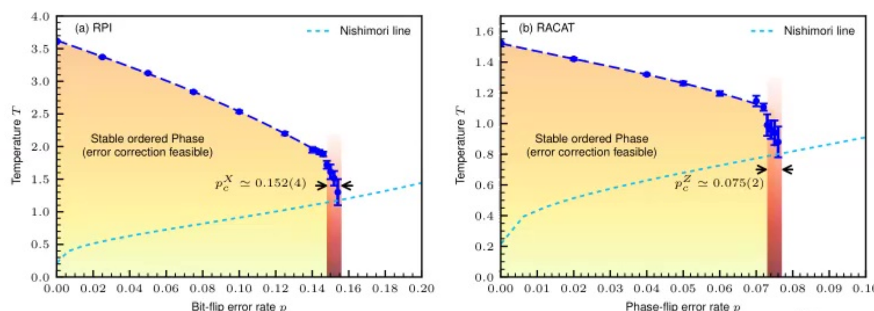


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# Absence of glass order along the Nishimori line

- Normal and spin glass order correlation functions

$$G^{\mathcal{A}}(\mathbf{r}) = [\langle S_c S_{c+\hat{z}} S_{c+\mathbf{r}} S_{c+\mathbf{r}+\hat{z}} \rangle]$$

$$G^{\mathcal{A}}_{\mathrm{SG}}(\mathbf{r}) = \left[ \langle S_c S_{c+\hat{z}} S_{c+\mathbf{r}} S_{c+\mathbf{r}+\hat{z}} \rangle^2 \right]$$

$$G^{\mathcal{B}}(r) = \left[ \left\langle S^x_v S^x_{c+r\hat{x}} \right\rangle \right]$$

$$G^{\mathcal{B}}_{\mathrm{SG}}(r) = \left[ \left\langle S^x_v S^x_{c+r\hat{x}} \right\rangle^2 \right]$$

- Glass order

$$\lim_{r\to\infty} G(r) = 0, \lim_{r\to\infty} G_{\mathrm{SG}}(r) \neq 0$$

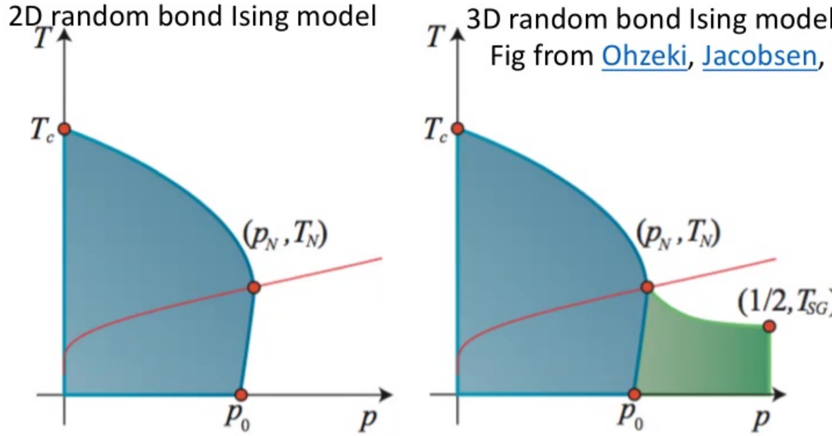
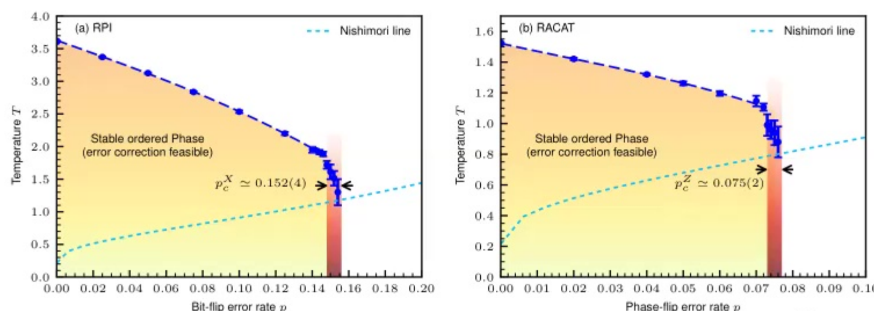



Fig from Ohzeki, Jacobsen, 2014

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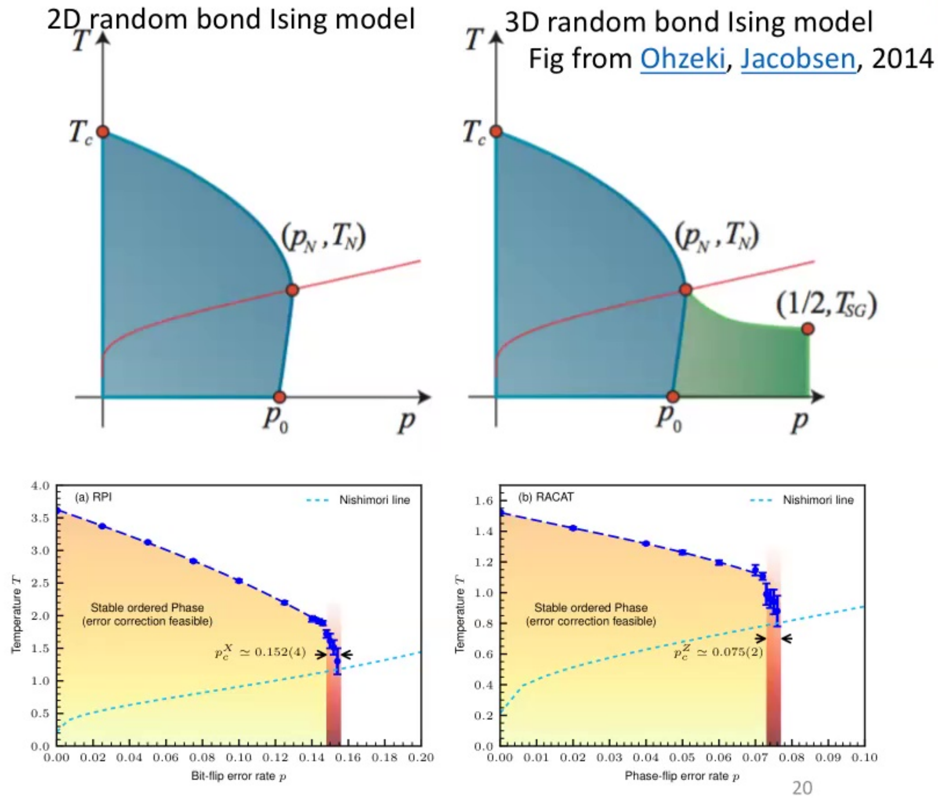
- Glass order

$$\lim_{r\to\infty} G(r) = 0, \lim_{r\to\infty} G_{\mathrm{SG}}(r) \neq 0$$

- Along the Nishimori line → no glass order
  - We double-checked Nishimori's argument and showed in general

$$G(r) = G_{\mathrm{SG}}(r)$$

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# Summary

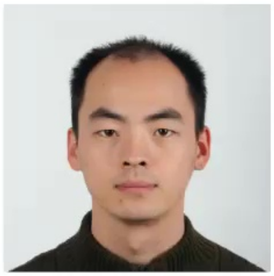
- We make a first study of the optimal thresholds of fracton codes
- For the X-cube model, $p_c^X = 15.2\%$ and $p_c^Z = 7.5\%$---higher minimum error threshold (7.5%) than 3D topological codes.
- Random spin models with subsystem symmetry.
- Analytically show no glass order along Nishimori line.
- Numeric suggests no glass order (even below the Nishimori line).
- Approximate duality relation between $p_c^X$ and $p_c^Z$.

# Outlook

- Haah's code and checkerboard model → $p_c^X = p_c^Z$ → 11%?
- Measurement errors in fracton codes & high-rank tensor gauge theory in 4D?
- Factonic states for universal quantum computing?


# Acknowledgements

Janik Schönmeier-Kromer
(Munich)



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**HS, J Schönmeier-Kromer, K Liu, O Viyuela, L Pollet, MA Martin-Delgado, arXiv:2112.05122.**

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# Thank you for your attention!

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