Title: Harnessing S-duality in N=4 SYM and supergravity as SL(2,Z)-averaged strings

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Abstract: I will describe an approach to extracting the physical consequences of S-duality of four-dimensional N = 4 super Yang-Mills (SYM) and its string theory dual based on SL(2,Z) spectral theory. I will show that processing S-duality in this way leads to strong consequences for the CFT data, both perturbatively and non-perturbatively in all parameters. In large-N limits, I will argue for the existence and scaling of non-perturbative effects, both at large N and at strong 't Hooft coupling. An elegant benchmark for these techniques is a certain integrated stress-tensor multiplet four-point function, whose form I will elucidate. I will explain how the ensemble average of CFT observables over the N = 4 supersymmetric conformal manifold with respect to the Zamolodchikov measure is cleanly isolated by the spectral decomposition, and will show that the large-N limit of the ensemble average is equal to the strong-coupling limit of the observable in the planar theory, which is its value in type IIB supergravity on AdS_5 x S^5. This embeds an emergent averaged holographic duality within the conventional holographic paradigm.

Zoom link: https://pitp.zoom.us/j/95197874062?pwd=QU4vbXNNeFVmS0hNbTVLL24wdDBndz09
Harnessing S-duality in $\mathcal{N} = 4$ SYM &
supergravity as $SL(2,\mathbb{Z})$-averaged strings

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based on [2201.05093] with Eric Perlmutter (CEA Saclay)
Motivating remarks

- $\mathcal{N} = 4$ SYM is a beautiful theory from a multitude of perspectives
  - Maximal SUSY $\Rightarrow$ rigid structure of CFT data
  - Approaches: perturbation theory, bootstrap, localization, integrability
  - Via AdS/CFT, it furnishes our most robust and explicit definition of non-perturbative quantum gravity

- It enjoys S-duality [Montenon, Olive; Witten, Olive; Osborn; Sen]:
  - Local CFT observables $\mathcal{O}(\tau)$ are invariant under $SL(2,\mathbb{Z})$ transformations: $\mathcal{O}(\tau) = \mathcal{O}(\gamma\tau)$
  - An inherently non-perturbative duality; obscured in the 't Hooft limit
  - A symmetry that has not been fully used!

\[
\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{\text{YM}}^2} \quad \mapsto \quad \gamma\tau = \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})
\]
Motivating remarks

- In practice, very little concrete information about the modular structure of CFT observables $\mathcal{O}(\tau)$ is known to complement perturbation theory

  **Idea:** Bake in modular invariance at the outset. Reduce CFT observables to their dynamical content and systematically explore the consequences of S-duality.

- Facilitated by a robust $SL(2,\mathbb{Z})$ spectral theory
  - **Simple calculations** yield a wealth of insights into the structure of perturbation theory and of the instanton expansion

- Structure is especially rigid at large $N$ in the 't Hooft limit with $\lambda \equiv N g_{\text{SYM}}^2$ held fixed
  - $SL(2,\mathbb{Z})$ invariance implies the existence of non-perturbative effects at large $N$ and at strong coupling ($\lambda \gg 1$)
Motivating remarks

- At strong coupling ($\lambda \gg 1$) $\mathcal{N} = 4$ SYM is famously dual to type IIB supergravity on $\text{AdS}_5 \times S^5$, with a prescribed set of stringy $\alpha'$ corrections

- Remarkably, S-duality has something to say about this:

  The large-$N$ limit of ensemble averaged $\mathcal{N} = 4$ SYM is the strong-coupling limit of the planar theory, i.e. $\text{AdS}_5 \times S^5$ supergravity

- An emergent averaged holographic duality within string theory
Outline

1. $SL(2,\mathbb{Z})$ spectral theory for $\mathcal{N} = 4$ SYM observables

2. Example: integrated stress-tensor multiplet four-point function

3. Perturbation theory and the analytic structure of spectral overlaps

4. Large-$\mathcal{N}$ and the ’t Hooft limit

5. Supergravity as an emergent ensemble average

6. Statistics of the “$SL(2,\mathbb{Z})$ ensemble”

7. Remarks on AdS/CFT and wormholes
1. $SL(2,\mathbb{Z})$ spectral theory

The fundamental domain $\mathcal{F}$ of $SL(2,\mathbb{Z})$:

$$\mathcal{F} = \left\{ \tau = x + iy \in \mathbb{H} \mid -\frac{1}{2} \leq x \leq \frac{1}{2}, |\tau| \geq 1 \right\}$$

- Natural metric on moduli space:
  $$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

- Laplacian: $\Delta_\tau = -y^2(\partial_x^2 + \partial_y^2)$
$SL(2,\mathbb{Z})$ spectral theory

- There is a natural **inner product** on this space:

$$ (f, g) = \int_{\mathcal{F}} \frac{dx dy}{y^2} f(\tau) \overline{g(\tau)} $$

- Basic idea: decompose $\mathcal{O}(\tau)$ into eigenfunctions of $\Delta_\tau$ using this inner product
  - This is possible when $\|\mathcal{O}\|^2 = (\mathcal{O}, \mathcal{O}) < \infty$:
  - This is generic for $\mathcal{N}=4$ SYM observables $\mathcal{O}(\tau)$ (cusp $\tau \to i\infty$: free limit)
Spectral resolution of $\Delta_\tau$

$$\Theta(\tau) = \overline{\Theta} + \int_{\text{Res} = \frac{1}{2}} \frac{ds}{4\pi i} (\Theta, E_s) E_s(\tau) + \sum_{n=1}^{\infty} (\Theta, \phi_n) \phi_n(\tau)$$

The complete basis of eigenfunctions of $\Delta_\tau$ splits into three branches:

1. The constant function:

$$\Delta_\tau 1 = 0$$

2. Continuous branch: **real analytic Eisenstein series**

$$\Delta_\tau E_s(\tau) = s(1-s)E_s(\tau), \quad s \in \frac{1}{2} + i\mathbb{R}$$

3. (Infinite) discrete branch: **Maass cusp forms**

$$\Delta_\tau \phi_n(\tau) = \left( \frac{1}{4} + t_1^2 \right) \phi_n(\tau), \quad 0 < t_1 < t_2 < ...$$
Real analytic Eisenstein series

- Defined via a Poincaré series
  \[ E_s(\tau) = \sum_{\gamma \in \text{PSL}(2, \mathbb{Z})/\mathbb{Z}} \text{Im}(\gamma \tau)^s, \quad \text{Re}(s) > 1 \]

- Satisfies the functional equation
  \[ E^*_s(\tau) \equiv \Lambda(s)E_s(\tau) = E^*_{1-s}(\tau) \quad \left( \Lambda(s) \equiv \pi^{-s}\Gamma(s)\zeta(2s) = \Lambda(1/2 - s) \right) \]

- Admits a **meromorphic continuation** to the entire complex \( s \) plane
  \[ E^*_s(\tau) = \Lambda(s)y^s + \Lambda(1-s)y^{1-s} + \sum_{k=1}^{\infty} 2\cos(2\pi k x) \frac{2\sigma_{2s-1}(k)}{k^{s-\frac{1}{2}}} \sqrt{y} K_{s-\frac{1}{2}}(2\pi ky) \]
Maass cusp forms

- The discrete branch is much more **wild & mysterious**

- Related to the energy eigenstates of the quantum mechanics of a particle propagating on $\mathcal{F}$, a **classically chaotic system**

- Functional form is similar to the continuous branch

\[ \phi_n(\tau) = \sum_{k=1}^{\infty} a_k^{(n)} \cos(2\pi k x) \sqrt{y} K_{\nu_k}(2\pi k y) \]

- The eigenvalues $\{t_n\}$ and Fourier coefficients $\{a_k^{(n)}\}$ are sporadic real numbers with very interesting statistics ("**arithmetic quantum chaos**" [Sarnak])
Eisenstein series vs. Maass cusp forms

\[ (\phi^2_1)_0(y) \]
\[ (\phi^2_2)_0(y) \]
\[ (\phi^2_3)_0(y) \]
\[ \frac{1}{4}(|E_{\frac{1}{2}+i\sqrt{2}}|^2)_0(y) \]
Spectral decomposition of CFT observables

CFT observables admit a **unique decomposition** in this basis:

\[
\mathcal{O}(\tau) = \overline{\mathcal{O}} + \int_{\text{Res}=\frac{1}{2}} \frac{ds}{4\pi i} (\mathcal{O}, E_s) E_s(\tau) + \sum_{n=1}^{\infty} (\mathcal{O}, \phi_n) \phi_n(\tau)
\]

where:

\[
\overline{\mathcal{O}} = \text{vol}(\mathcal{F})^{-1} \int_\mathcal{F} \frac{dx dy}{y^2} \mathcal{O}(\tau) = \text{Res}_{s=1}(\mathcal{O}, E_s) \text{ is the modular average}
\]

\[
(\mathcal{O}, E_s) = \int_\mathcal{F} \frac{dx dy}{y^2} \mathcal{O}(\tau) E_s(\tau) = \int_0^\infty \frac{dy}{y^2} y^s \mathcal{O}_0(y) \text{ is a Mellin transform of the zeromode}
\]
Spectral decomposition of CFT observables

Consequences:

1. The overlap \( \langle \mathcal{O}, E_s \rangle \) satisfies a \textbf{functional equation}

\[
\langle \mathcal{O}, E_s \rangle \equiv \frac{\langle \mathcal{O}, E_s \rangle}{\Lambda(s)} = \langle \mathcal{O}, E_{1-s} \rangle
\]

2. The constant term \( \overline{\mathcal{O}} \) is the \textbf{ensemble average} over the conformal manifold (wrt the Zamolodchikov measure)

\[
\langle \mathcal{O} \rangle \equiv \text{vol}(\mathcal{M})^{-1} \int_{\mathcal{M}} d\mu_{\mathcal{M}} \mathcal{O}(\tau) = \overline{\mathcal{O}} \quad \text{by virtue of maximal SUSY}
\]

3. "\textbf{Instantons are redundant:}" \( \mathcal{O}(\tau) = \mathcal{O}_0(\nu) + \sum_{k=1}^{\infty} 2 \cos(2\pi k x) \mathcal{O}_k(\nu) \)
Instantons are redundant

\[ \mathcal{O}_0(y) \]

\[ \{ \mathcal{O}, E_s \} \]

\[ \mathcal{O}_k(y) = \int_{\text{Re } s = \frac{1}{2}} \frac{ds}{4\pi i} \{ \mathcal{O}, E_s \} E_{s,k}^*(y) \]
2. Integrated correlator

- “CFT observables:” non-perturbatively well-defined

  e.g.: correlators of protected operators, spectrum of the dilatation operator,
  structure constants
  **not**, e.g.: “anomalous dimension of the Konishi”

- A rare and beautiful example: integrated $O_{20}$' four-point function of [Binder
  Chester Pufu Wang 2019]

  \[ \mathcal{G}_N(\tau) = \int du dv \rho(u, v) T_N(u, v) \]

  SUSY-preserving measure

  \[ = -\frac{1}{4} \Delta_{\tau} \partial^2_m \log Z^S_N(m, \tau) \bigg|_{m=0} \]

  (Dynamical part of) 20` 4-pt function

  Mass-deformed sphere free energy (localization)
Integrated correlator

- Remarkably, $\mathcal{G}_N(\tau)$ is conjecturally known for all $N, \tau$ (!) [Dorigoni Green Wen 2021]

$$\mathcal{G}_N(\tau) = \frac{1}{2} \sum_{(m,n) \in \mathbb{Z}^2} \int_0^\infty d\xi \, B_N(\xi) \exp \left( -\frac{\pi \xi}{y} |m\tau + n|^2 \right)$$

- The kernel $B_N(\xi)$ is determined by recursion from $N = 2$ via a Laplace difference equation satisfied by $\mathcal{G}_N(\tau)$

- Another useful representation

$$\mathcal{G}_N(\tau) = \frac{N(N-1)}{8} - \frac{1}{2} \sum_{s=2}^\infty (-)^s c_s^{(N)} E_s^*(\tau)$$

  e.g. for $SU(2)$, $c_s^{(2)} = s(1-s)(2s-1)^2$
Integrated correlator

- We claim that this integrated correlator is given exactly by the following spectral decomposition

\[
\mathcal{G}_N(\tau) = \frac{N(N - 1)}{4} + \int_{\text{Res} = \frac{1}{2}} \frac{ds}{4\pi i} \frac{\pi}{\sin \pi s} C_s^{(N)} E_s^{*}(\tau)
\]

\[
\langle \mathcal{G}_N \rangle \quad \{ \mathcal{G}_N, E_s \}
\]

- In particular: \((\mathcal{G}_N, \phi_n) = 0\)
3. Perturbation theory and the analytic structure of spectral overlaps

- Consistency of the weak-coupling ($y \gg 1$) expansion constrains ($\mathcal{O}, E_s$) (recall: $y = 4\pi/g_{YM}^2$)

$$\mathcal{O}_0(y) = \langle \mathcal{O} \rangle + \int_{\text{Re } s = \frac{1}{2}} \frac{ds}{4\pi i} \{ \mathcal{O}, E_s \} (\Lambda(s)y^s + \Lambda(1 - s)y^{1-s})$$

- Perturbation theory: upon contour deformation, **only non-negative integer powers of** $y$, **no logarithms**

- Large- $|s|$ asymptotics of $\{ \mathcal{O}, E_s \}$ encodes asymptotics of perturbation theory
Perturbation theory and the analytic structure of spectral overlaps

- Claim: the most general \( \{ \mathcal{O}, E_s \} \) consistent with perturbation theory takes the form

\[
\{ \mathcal{O}, E_s \} = \frac{\pi}{\sin \pi s} s(1-s)f_p(s) + \frac{f_{np}(s)}{\text{perturbative } \sim y^{-n}} \quad \text{non-perturbative } \sim (q\bar{q})^n
\]

where both \( f_p(s), f_{np}(s) \) are:
- symmetric under reflection \( s \to 1-s \)
- regular for \( s \in \mathbb{C} \) away from \( s = 0,1 \)
- such that \( \{ \mathcal{O}, E_s \} = 0 \) for \( s = 1/2 \)
The optimal simplicity of $\mathcal{G}_2(\tau)$

Recall the $SU(2)$ integrated correlator $\mathcal{G}_2(\tau)$:

\[
\begin{align*}
\langle \mathcal{G}_2, \phi_n \rangle &= 0 \\
f_{np}(s) &= 0 \\
f_p(s) &= (2s - 1)^2
\end{align*}
\]

An **optimally simple observable** consistent with $SL(2, \mathbb{Z})$-invariance and perturbation theory!
The $SL(2, \mathbb{Z})$ Borel transform

- It is often the case the $\mathcal{N} = 4$ SYM observables have **Borel-summable perturbative series**:

$$\sigma_0(y) \approx \sum_{n=1}^{\infty} (-)^n c_n y^{-n}, \quad (c_n = -n(n+1)\Lambda(n + 1/2)f_p(n + 1))$$

e.g. suppose $c_n \sim (\pi R)^{-n} n!$

- Convenient to define the “$SL(2, \mathbb{Z})$ Borel transform”:

$$B[\xi] = \sum_{n=0}^{\infty} \frac{(-)^n c_n}{\Lambda(n + 1/2)} \xi^{n+1}$$

(radius of convergence $R$)

- The **resummation** of the perturbative series is neatly obtained by inverting this transform:

$$\sigma_0(y) = y^{1/2} \int_0^{\xi} \frac{d\xi}{\xi^{3/2}} \left( \frac{\theta_3(y\xi) - 1}{2} \right) B[\xi]$$
The $SL(2,\mathbb{Z})$ Borel transform

- When $(\Theta, \phi_n) = 0$, can reconstruct $\Theta(\tau)$ from $\Theta_0(y)$ in an elegant way

- Sum the resummation of the perturbative series over $SL(2,\mathbb{Z})$ images:

$$\Theta(\tau) = \frac{1}{2} \sum_{\gamma \in PSL(2,\mathbb{Z})/\mathbb{Z}} \Theta_0(\text{Im}(\gamma \tau))$$

$$= \frac{1}{4} \sum_{(m,n) \in \mathbb{Z}^2} \int_0^\infty \frac{d\xi}{\xi} B[\xi] \exp \left( -\frac{\pi \xi}{y} |m\tau + n|^2 \right)$$

- Thus there is a lattice integral representation for rather general CFT observables

- The kernel is the $SL(2,\mathbb{Z})$ Borel transform of the zeromode!
The redundancy of instantons

- If the Borel transform of the perturbative series of $O_0(y)$ has radius of convergence $R$, then that of the $k > 0$ instanton sectors is also Borel summable with radius of convergence:

$$R_k = R \left(1 + \frac{k}{R}\right)^2$$

- $SL(2,\mathbb{Z})$ relates all instanton sectors in a simple way

- Violation of this can be taken as a sharp signal that $(\Theta, \phi_n) \neq 0$
4. Large-$N$ and the 't Hooft limit

- We will mostly be concerned with the 't Hooft limit

$$N \to \infty, \text{ with } \lambda = N g_{YM}^2 = \frac{4\pi N}{y} \text{ held fixed}$$

- The perturbative $1/N$ expansion organizes into a genus expansion

$$\mathcal{O}_0(y) = \sum_{g=0}^{\infty} N^{2-2g} \mathcal{O}^{(g)}_0(\lambda) \quad (k\text{-instantons are non-perturbatively suppressed } \sim e^{-8\pi^2 k N/\lambda})$$

- We then consider the $1/N$ expansion of the spectral overlaps (set $f_{np}(s) = 0$ for now):

$$\mathcal{O}_0(y) = \langle \mathcal{O} \rangle + \sum_{g=0}^{\infty} N^{2-2g} \int \frac{ds}{2\pi i} \frac{\pi}{\sin \pi s} s(1-s) (\Lambda(s) \lambda^{-s} + \Lambda(1-s) N^{1-2s} \lambda^{s-1}) f_p^{(g)}(1-s)$$
Large-$N$ and the ’t Hooft limit

- At the end of the day, one finds the following for the general form of the weak-coupling ($\lambda \ll 1$) expansion:

$$\phi_0(y) \approx \sum_{g=0}^{\infty} N^{2-2g} \sum_{m=1}^{\infty} R_m^{(g)} \lambda^m$$

$$\text{where } R_m^{(g)} = \text{Res}_{s=\frac{1}{2}+m} \left( \frac{\pi}{\sin \pi s} s(1-s) \Lambda(s) f_p^{(g)}(1-s) \right)$$

- While at strong coupling ($\lambda \gg 1$) we have:

$$\phi_0(y) \approx C(N) - \frac{1}{2} \sum_{g=0}^{\infty} N^{2-2g} \sum_{m=1}^{\infty} R_m^{(g)} \left( \lambda^{-m-\frac{1}{2}} + \frac{\Lambda(\frac{1}{2} - m)}{\Lambda(\frac{1}{2} + m)} N^{-2m} \lambda^{m-\frac{1}{2}} \right)$$

where $C(N) = \langle \phi \rangle - \frac{1}{2} \sum_{g=0}^{\infty} N^{1-2g} f_p^{(g)}(0)$

Needed for renormalization of $1/N$ expansion at strong-coupling (stringy regularization of loop divergences in sugra)
Non-perturbative effects implied by $SL(2,\mathbb{Z})$

- A surprising consequence of modular invariance:
  - Convergence of the weak-coupling expansion of $O(\tau)$ implies non-perturbative effects both at $N \gg 1$ and at strong coupling (at fixed orders in the genus expansion).
  - The non-perturbative scale is set by the radius of convergence $|\lambda| < \lambda_s$ of the weak-coupling expansion.

- In $\mathcal{N} = 4$ SYM, $\lambda_s = \sqrt{\pi}$ is generic.
  This implies the non-perturbative scales $\Lambda^2_\lambda$ and $\Lambda^2_{\lambda_s}$ (for $\lambda \gg 1$ and $\lambda_s \gg 1$ respectively):

$$
\begin{align*}
\Lambda^2_\lambda &= e^{-\sqrt{\lambda}} = e^{-2\pi T_{\text{ir1}}} \\
\Lambda^2_{\lambda_s} &= e^{-\sqrt{\lambda_s}} = e^{-2\pi T_{\text{di1}}},
\end{align*}
$$

where $\lambda_s = \frac{(4\pi N)^2}{\lambda}$ is an S-dual 't Hooft coupling.

- Worldsheet instanton effects.
Non-perturbative effects implied by $SL(2,\mathbb{Z})$

- A surprising consequence of modular invariance:
  - Convergence of the weak-coupling expansion of $\mathcal{O}(\tau)$ implies non-perturbative effects both at $N \gg 1$ and at strong coupling (at fixed orders in the genus expansion)
  - The non-perturbative scale is set by the radius of convergence $|\lambda| < \lambda_s$ of the weak-coupling expansion

- In $\mathcal{N} = 4$ SYM, $\lambda_s = \pi^2$ is generic. This implies the non-perturbative scales $\Lambda_\lambda^2$ and $\Lambda_{\lambda_s}^2$ (for $\lambda \gg 1$ and $\lambda_s \gg 1$ respectively):

\[
\begin{align*}
\Lambda_\lambda &= e^{-\sqrt{\lambda}} = e^{-2\pi T_{i1}}, \\
\Lambda_{\lambda_s} &= e^{-\sqrt{\lambda_s}} = e^{-2\pi T_{d1}},
\end{align*}
\]

where $\lambda_s = \frac{(4\pi N)^2}{\lambda}$ is an S-dual 't Hooft coupling

- Worldsheet instanton effects

[Bargheer Coronado Vieira 2019]
Allowing for instanton-anti-instanton effects

- Essentially the exact same structure persists for $f_{np}(s) \neq 0$.
  - one lesson: at strong-coupling, **integer powers of** $1/\lambda \leftrightarrow f_{np}(s) \neq 0$

- At the end of the day, the strong-coupling expansion is slightly modified

\[
\Theta_0(y) \approx C(N) - \sum_{g=0}^{\infty} N^{2-2g} \sum_{m=0}^{\infty} \left( a_m^{(g)} \lambda^{-m+3 \over 2} + b_m^{(g)} N^{-2-2m} \lambda^m \right)
\]

where $a_m^{(g)}$ and $b_m^{(g)}$ are computable residues.
5. Supergravity as an emergent ensemble average

• The general expression for the strong-coupling expansion has striking consequences

• Given the large-$N$ expansion of the average (turning $f_{np}(s)$ off for clarity):

\[
\langle \Theta \rangle = \frac{1}{2} \sum_{g=0}^{\infty} N^{2-2g} \left( \frac{f_p^{(g)}(1)}{\langle \langle \Theta^{(g)} \rangle \rangle} + N^{-1} f_p^{(g)}(0) \right)
\]

• Then the large-$N$ limit of the average is the only term that survives the strong-coupling limit:

\[
\Theta_0(y) \approx \sum_{g=0}^{\infty} N^{2-2g} \left[ \langle \langle \Theta^{(g)} \rangle \rangle - \sum_{m=0}^{\infty} \left( a_m^{(g)} \lambda^{-m-\frac{3}{2}} + b_m^{(g)} N^{-2-2m} \lambda^{m+\frac{1}{2}} \right) \right]
\]
Supergravity as an emergent ensemble average

• In other words, we have arrived at a large-$N$ equivalence between strong coupling and ensemble averaging in $\mathcal{N} = 4$ SYM:

$$\langle \mathcal{O} \rangle = \mathcal{O}(\lambda \rightarrow \infty) = \mathcal{O}_{\text{sugra}}$$

• This extends to all loop orders:

$$\mathcal{O}_0^{(g)}(\lambda \rightarrow \infty) = \langle \langle \mathcal{O}^{(g)} \rangle \rangle$$
Supergravity as an emergent ensemble average

- Obviously, the traditional holographic paradigm is left entirely intact

- Only the strongly-coupled theory at large $N$, dual to bulk supergravity, emerges as an ensemble average
  - As in low-dimension dualities, averaging generates a simple (extremal) bulk dual

- Perhaps there are scenarios where the average can be studied more easily than the strongly-coupled observable
  (cf. averaged spectral gap at large central charge in Narain ensemble average [Afkhami-Jeddi Cohn Hartman Tajdini 2020])

- Immediate corollary:
  - Any observable that diverges as $\lambda \to \infty$ cannot be modular invariant (e.g. Konishi anomalous dimension)
6. Statistics of the $SL(2,\mathbb{Z})$ ensemble

- We are thus motivated to study the **statistics** of CFT observables in the $SL(2,\mathbb{Z})$ ensemble.

- We have seen that the spectral decomposition cleanly picks out the ensemble average:

\[
\mathcal{O}(\tau) = \langle \mathcal{O} \rangle + \mathcal{O}_{\text{spec}}(\tau)
\]

\[\langle \mathcal{O}_{\text{spec}} \rangle = 0\]

- E.g. the **variance** is straightforwardly computed in terms of the spectral overlaps:

\[
\mathcal{V}(\mathcal{O}) = \langle \mathcal{O}^2 \rangle - \langle \mathcal{O} \rangle^2
\]

\[
= \langle \mathcal{O}^2_{\text{spec}} \rangle
\]

\[
= \text{vol}(\mathcal{F})^{-1} \left[ \int_{\text{Re} s = \frac{1}{2}} ds \frac{ds}{4\pi i} \left| \langle \mathcal{O}, E_s \rangle \right|^2 + \sum_{n=1}^{\infty} \left( \langle \mathcal{O}, \phi_n \rangle \right)^2 \right]
\]

"Second moment in spectral space"
The variance at large $N$

- At large $N$, the variance is **parametrically suppressed** compared to the mean-squared (but still puzzlingly large):

$$\frac{\mathcal{V}(\mathcal{O})}{\langle \mathcal{O} \rangle^2} \sim \frac{1}{N}$$

- Moreover, the $1/N$ expansion is **quadruple-factorially divergent**, leading to non-perturbative effects in powers of $\Lambda_N^2$, where

$$\Lambda_N \equiv e^{-4\sqrt{\pi N}}$$
7. Wormholes in moduli space & emergent averaged holography

- Our results are especially interesting in light of recent developments in low-dimensional quantum gravity
  - 2d: JT gravity ↔ double-scaled random matrix integral [Saad Shenker Stanford 2019]
  - 3d: exotic “$U(1)$ gravity” ↔ averaged Narain lattice CFT [Maloney Witten; Afkhami-Jeddi Cohn Hartman Tajdini 2020]

- **Spacetime wormholes** play an important role in these dualities

- Leads to some natural questions in conventional holographic dualities:
  - What (other) bulk ingredients are needed for factorization of multi-boundary observables? [Saad Shenker Stanford Yao; Blommaert Kruthoff; Blommaert iliesiu Kruthoff 2021]
  - In a product of observables in two decoupled CFTs, why are there contributions that appear to correlate them? (When and how do they appear?) Where are wormholes in the bootstrap?
Wormholes in moduli space

- The large-$N$ equivalence $\langle \mathcal{O} \rangle = \mathcal{O}_{\text{sugra}}$ suggests a gravitational manifestation of $SL(2,\mathbb{Z})$ ensemble statistics
  - How much does semiclassical $\text{AdS}_5 \times S^5$ string theory "know" about them?

- Note: we are not trying to argue that wormholes dominate multi-boundary observables in semiclassical $\text{AdS}_5 \times S^5$ (they don't, in general)
  - But rather, attempting to address whether and why wormholes may appear as part of the (unaveraged) semiclassical bulk theory (and how they do so consistently with UV completeness)
Wormholes in moduli space

- Our results resonate with recent work on factorization in low-dimensional models of quantum gravity [Saad Shenker Stanford Yao; Blommaert Kruthoff; Mukhametzhanov; Blommaert Iliesiu Kruthoff 2021]

$$\mathcal{O}(\tau) = \langle \mathcal{O} \rangle + \int \frac{ds}{4\pi i} (\mathcal{O}, E_s) E_s(\tau) + \sum_{n=1}^{\infty} (\mathcal{O}, \phi_n) \phi_n(\tau)$$

$$\sim \mathcal{O}_{\text{sugra}}$$

$$\mathcal{O}_{\text{spec}}(\tau): \text{"half-wormhole"}$$

- To probe ensemble statistics, consider higher powers of $\mathcal{O}_{\text{spec}}(\tau)$:

$$\mathcal{O}_{\text{spec}}^2(\tau) = \text{vol}(\mathcal{F})^{-1} \left[ \int \frac{ds}{4\pi i} |(\mathcal{O}, E_s)|^2 + \sum_{n=1}^{\infty} (\mathcal{O}, \phi_n)^2 \right] + \int \frac{ds}{4\pi i} (\mathcal{O}_{\text{spec}}, E_s) E_s(\tau) + \sum_{n=1}^{\infty} (\mathcal{O}_{\text{spec}}, \phi_n) \phi_n(\tau)$$

$$\mathcal{F}(\mathcal{O}): \text{"wormhole"}$$

coupling-dependent fluctuations around $\mathcal{F}(\mathcal{O})$
Wormholes in moduli space

- We have **not** identified the bulk configurations/stringy degrees of freedom that give the noisy contributions responsible for factorization for any particular observable

\[ \text{Diagram} \]

- Rather, we have shown concretely how ensemble statistics (and correspondingly connected bulk configurations) can play a role even in a fixed instance of $\mathcal{N} = 4$ SYM
  - Meaningful holographically because $\langle \mathcal{O} \rangle = \mathcal{O}_{\text{sugra}}$
  - Does the variance have a spacetime wormhole interpretation in semiclassical gravity? Does the answer depend on the type of observable? [Schlenker Witten 2022]
Future directions

- Obvious, urgent question: to what extent does this story generalize to other string/M-theory vacua?
  - Automorphic averages over U-duality groups? Ensemble averages over conformal manifolds? Over all directions in moduli space, or just the “gravity direction”?

- Other observables (and other S-duality groups):
  - Other integrated correlators, e.g. $\langle \overline{22} pp \rangle$
  - Ambitiously: unintegrated correlators
  - Observables particularly sensitive to statistics of high-energy states (thermal observables?)
  - Extremal correlators $\langle \overline{O}_p O_p \rangle$ in 4d $\mathcal{N} = 2$ SQCD

- Better understand worldsheet and spacetime perspectives on non-Borel summability of AdS$_5 \times S^5$ string theory and the semiclassical bulk meaning of the variance

- “Bootstrapping in spectral space”

- Statistics of CFT data over conformal manifolds (cf. [Afkhami-Jeddi Ashmore Córdova 2021])

- Cusp forms and **arithmetic chaos** in $\mathcal{N} = 4$ SYM