Title: Compact gauge fields minimally coupled to Causal Dynamical Triangulations

Speakers: Giuseppe Clemente

Series: Quantum Gravity

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Abstract: We discuss the minimal coupling of gauge fields to the dual graph lattice of Causal Dynamical Triangulations (CDT) as a preliminary step towards the investigation of more realistic systems of gravity coupled to bosonic and fermionic matter. We first introduce the CDT approach and how the gravity and gauge fields are discretized, for general dimensions and gauge groups, focusing on some of the algorithmic complications which arise. Next, we discuss some results from 2D simulations of CDT coupled to U(1) and SU(2) gauge groups, where we studied both gravity and gauge-related observables and the ones related to gauge fields, comparing them to analogous simulations in the flat case.

Compact gauge fields minimally coupled to Causal Dynamical Triangulations.

HELMHOLTZ RESEARCH FOR GRAND CHALLENGES

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Perimeter Institute, 17 March 2022

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DESY.

Overview

- Introduction
  - Overview of CDT
  - Regge formalism
  - Phase diagram and some standard results
- Coupling CDT with Yang–Mills gauge fields
  - Minimal coupling gauge fields in the continuum
  - Discretization on the dual lattice
- Algorithm
  - Brief recap on Monte Carlo importance sampling
  - Particular case: 2D CDT with gauge groups U(1) and SU(2)
  - Complexity for higher dimensions and gauge groups
- Numerical Results
  - Gravity-related observables
  - Yang–Mills-related observables
- Current developments
- Conclusions

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The QG problem

Manifest difficulties:

- Standard perturbation theory fails.
  G. 't Hooft and M.J.G. Veltman
  M.H. Goroff and A. Sagnotti
- Gravitational quantum effects unreachable on lab:
  $E_{Pl} = \sqrt{\frac{\hbar c}{G}} c^2 \simeq 10^{19} GeV$

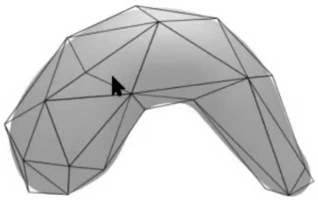
Possible philosophies in QG research

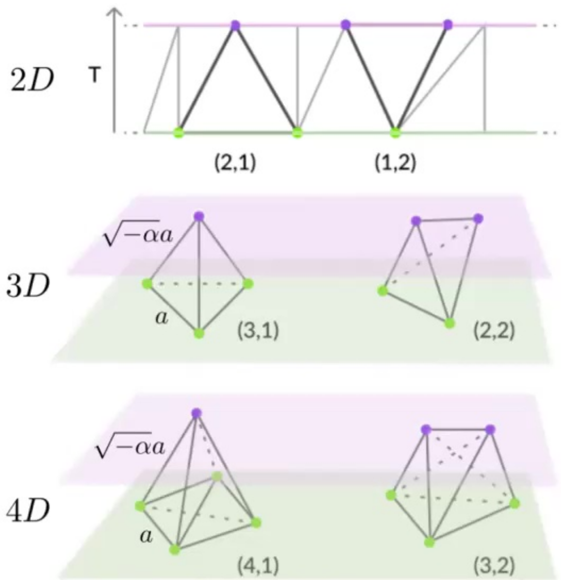
- non-conservative: introduce new short-scale physics (e.g., string theory);
- conservative: keep Einstein theory, but change the framework (e.g., asymptotic safety).

Causal Dynamical Triangulations (CDT): conservative approach of non-perturbative renormalization of the Einstein gravity via Monte-Carlo simulations.

Brief overview of Causal Dynamical Triangulations

> **Simplicial manifolds** approximate smooth ones;





> A **causality condition** is enforced by considering only foliated manifolds (fixed slice topology);

> The action is the **Regge discretization** of the Einstein-Hilbert one;

> Wick rotation of timelike links to the Euclidean. Luckily no sign problem!

> **Continuum limit**: search for a second order critical point in the phase diagram.

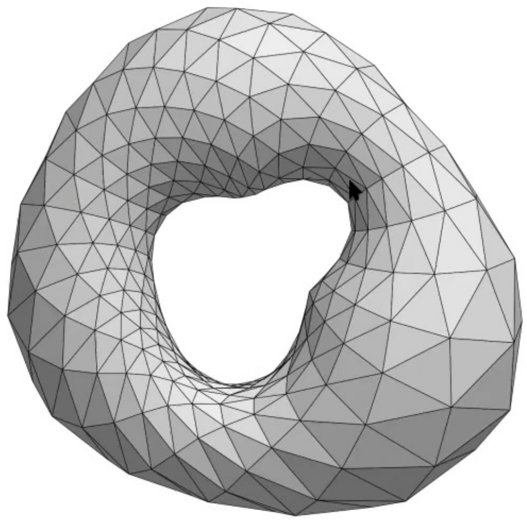
Reviews: J. Ambjorn, A. Goerlich, J. Jurkiewicz, R. Loll, Phys.Rept. 519 (2012)
R. Loll, Classical Quant. Grav. 37 (2019)

Regge Formalism: Features of Triangulations

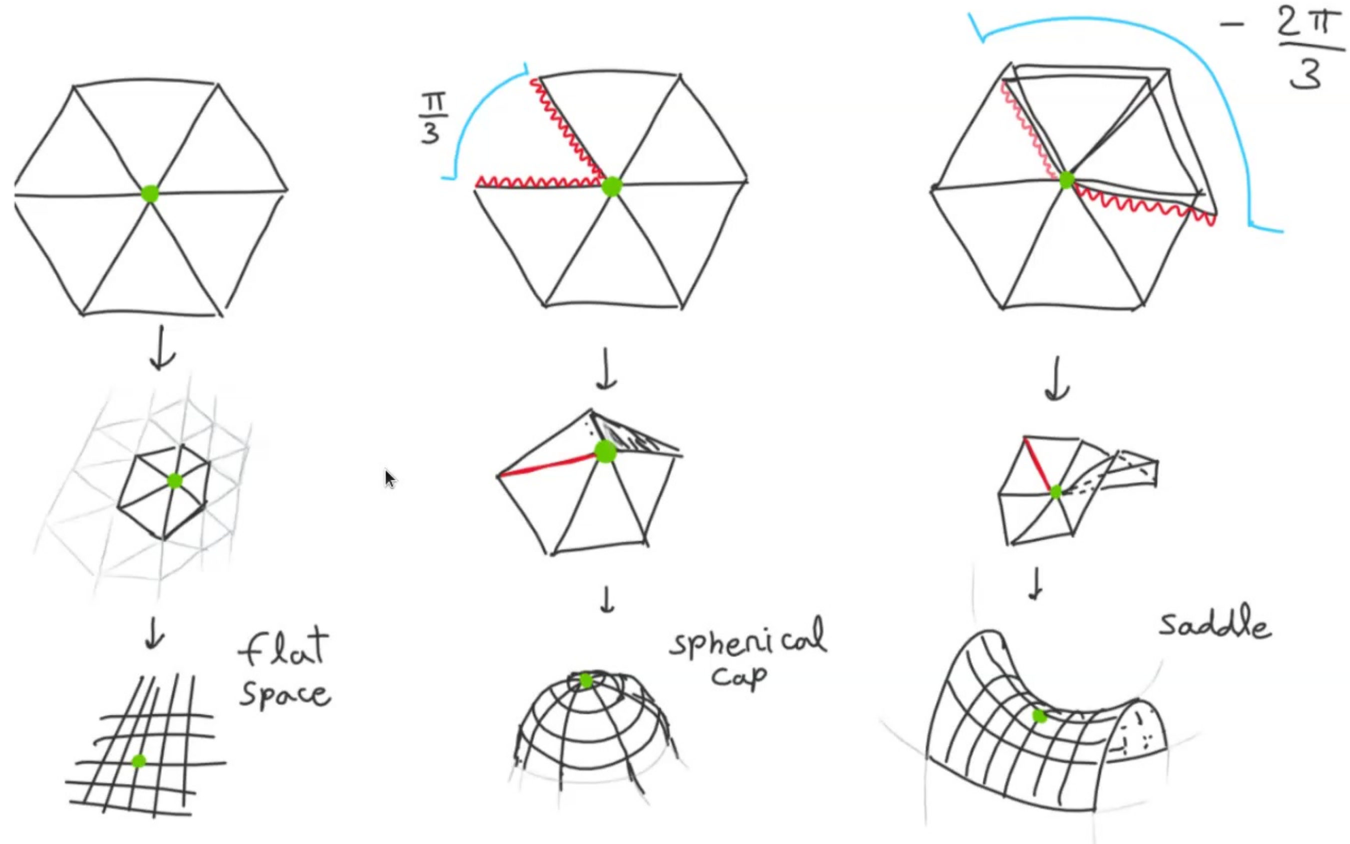
Features of a d-dimensional *triangulation* $\mathcal{T}$:

- k-simplices $\sigma^{(k)}$ have *flat* interior
- metric represented by adjacency relations between d-simplexes
- curvature localized on (d-2)-simplexes $\sigma^{(d-2)}$, twice the **deficit angle** in parallel transport:

$$\varepsilon_{\sigma^{(d-2)}} = 2\pi - \sum_{\sigma^{(d)} \ni \sigma^{(d-2)}} \theta(\sigma^{(d-2)}, \sigma^{(d)})$$



Example: Curvature for Equilateral Triangles (2D)



Regge calculus: a coordinate-free formulation

T. Regge, Nuovo Cimento 19, 568 (1961)

$$\begin{array}{ccc} \text{Metrics} & & \text{Triangulations} \\ (g_{\mu\nu}, \mathcal{M}) & \xrightarrow[\textit{discretization}]{\textit{Regge}} & (\{\sigma^k\}, \mathcal{T}) \\ \text{(coordinate-dependent)} & & \text{(coordinate-free)} \end{array}$$

$$S_{EH}[g_{\mu\nu}] = \frac{1}{16\pi G}\left[\underbrace{\int d^d x \sqrt{|g|}\, R}_{\text{Total curvature}} - 2\Lambda \underbrace{\int d^d x \sqrt{|g|}}_{\text{Total volume}}\right]$$

$\Downarrow$ discretization $\Downarrow$

$$S_{Regge}[\mathcal{T}] = \frac{1}{16\pi G}\left[\sum_{\sigma^{(d-2)}\in\mathcal{T}} 2\varepsilon_{\sigma^{(d-2)}} V_{\sigma^{(d-2)}} - 2\Lambda \sum_{\sigma^{(d)}\in\mathcal{T}} V_{\sigma^{(d)}}\right],$$

$V_{\sigma^{(k)}}$: k-volume of simplex $\sigma^{(k)}$; $\varepsilon_{\sigma^{(d-2)}}$: deficit angle around $\sigma^{(d-2)}$

The action can be computed in an almost combinatorial fashion, greatly simplifying simulation algorithms.
But **triangulations are still manifolds**: even if curvature is discretized, the simplices' flat interior matters.

Wick rotated action in 4D

[J. Ambjorn, A. Goerlich, J. Jurkiewicz, R. Loll, Phys.Rept. 519 (2012)]

After Wick rotation:

$$S_{CDT} = -k_0 N_0 + k_4 N_4 + \Delta(N_4 + N_4^{(4,1)} - 6N_0)$$

- New parameters: (k_0, k_4, Δ), related respectively to G, Λ and α.
- New variables: N_0, N_4 and $N_4^{(4,1)}$, counting the total numbers of vertices, pentachorons and type-$(4,1)/(1,4)$ pentachorons respectively.

It is convenient to "fix" the total spacetime volume $N_4 = V$ by fine-tuning k_4
$\Longrightarrow$ actually free parameters (k_0, Δ, V).

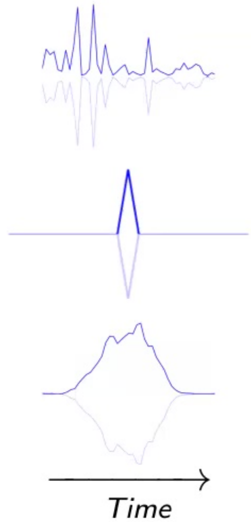
Simulations at different volumes V allow finite-size scaling analysis;

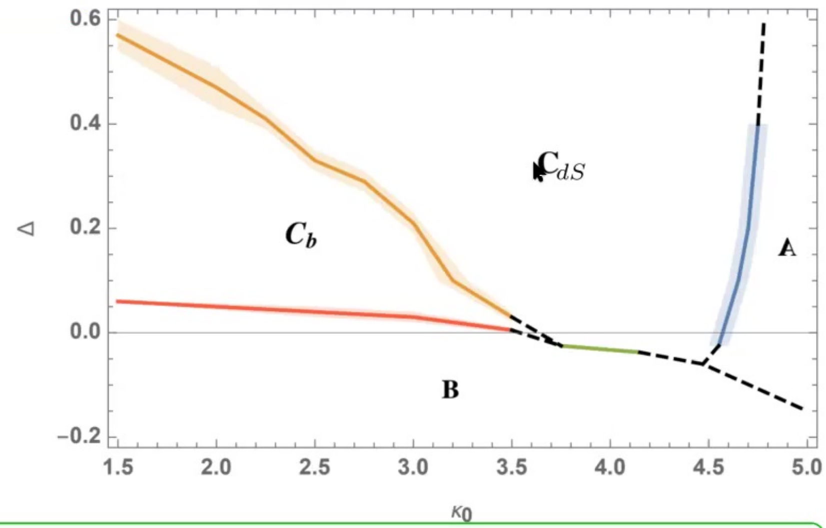
Sum over geometries according to $\mathcal{P}[\mathcal{T}] \equiv \frac{1}{Z} \exp\left(-S_{CDT}[\mathcal{T}]\right)$.

Sketch of 4d-CDT phase diagram

k_0 and Δ parametrize the theory (related to G and Λ)

phase	spatial volume per slice
A:	
B:	
C_{dS}/C_b:	





Possible 2nd order lines have been found.
C_{dS} spatial volume profiles compatible with *de Sitter* spacetime.

J. Ambjorn, S. Jordan, J. Jurkiewicz, R. Loll PRL 107 (2011)
J. Ambjorn, D. Coumbe, J. Gizbert-Studnicki, A. Goerlich, J. Jurkiewicz PRD 95 (2017)
G.C., M. D'Elia PRD 97 (2018) / G.C., M. D'Elia, A. Ferraro PRD 99 (2019)
J. Ambjorn, J. Gizbert-Studnicki,A. Goerlich, J. Jurkiewicz, D. Nemeth JHEP 07 (2019)
J. Ambjorn, G. Czelusta, J. Gizbert-Studnicki, A. Goerlich, J. Jurkiewicz JHEP 05 (2020)

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Coupling CDT with Yang–Mills gauge fields

Main motivation:
move the first steps in the direction of simulating more realistic systems: gravity coupled to bosonic (and ultimately also fermionic) fields.

To start: minimal coupling Yang–Mills gauge bosons with compact Lie group G.

Continuous action:

$$S_{\mathrm{EH+YM}} = \int_M d^d x \sqrt{|g_{\mu\nu}|} \left[\frac{1}{16\pi G}(R - 2\Lambda) + \frac{1}{2} g^{\mu\alpha} g^{\nu\beta} Tr[F_{\mu\nu} F_{\alpha\beta}] \right],$$

where $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu + ig\,[A_\mu, A_\nu]$ is in Lie(G) and g is the gauge coupling parameter.

In the lattice regularization, the field $A_\mu(x) \in$ Lie(G) is replaced by finite parallel transporters called **link variables** $U_{\vec{n},\mu} \in G$.

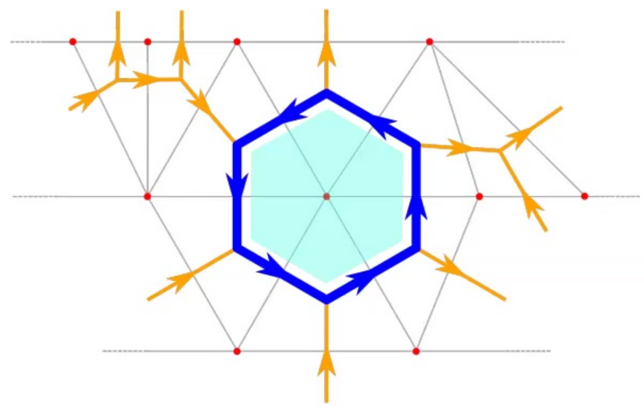
Minimal coupling: the discretized action

∃ analytical studies in 2D on the primal lattice [J. Ambjorn and A. Ipsen (2013)], but *the dual graph accounts for the correct counting for the density of gauge d.o.f.*
[J. Ambjorn, K.N. Anagnostopoulos and J. Jurkiewicz (1999)] [A. Candido, G.C., M. D'Elia and F. Rottoli (2021)]

For a discretized configuration of triangulation and fields $(\mathcal{T}, \Phi_{\mathcal{T}})$:

$$S[\mathcal{T}, \Phi_{\mathcal{T}}] = S_{CDT}[\mathcal{T}] - \beta \sum_{b \in \mathcal{T}^{(d-2)}} \frac{1}{n_b} \left[\frac{1}{N} ReTr\Pi_b[\Phi_{\mathcal{T}}] - 1 \right],$$

where the **plaquette** Π_b is the oriented product of the n_b link variables in loop around a $(d-2)$-simplex b (called **bone**).



A plaquette in the dual graph of a 2D triangulation.

Recovering the continuum action

As in hypercubic lattices, the plaquette term in the YM part of the action contributes with an area term of the type:

$$\Pi_b \simeq \exp\left[igAn_bF_{\mu\nu}\right],$$

where An_b is the area of the plaquette (b dictates the pair μ,ν on a local chart). By expanding for small g (continuum limit) we have

$$\Pi_b \to 1 + igAn_bF^a_{\mu,\nu}T_a - \frac{1}{2}g^2A^2n_b^2\sum_{a,b}F^a_{\mu,\nu}F^b_{\mu,\nu}T_aT_b + O(g^3),$$

with $\{T_a\}_{a=1}^{N_G}$ generators of Lie(G) ($N_{U(1)} = 1$, $N_{SU(N)} = N^2-1$); by inserting this expression in the discretized action only the third term survives:

$$\Longrightarrow \frac{1}{n_b}\left[\frac{1}{N}ReTr\Pi_b[\Phi_{\mathcal{T}}] - 1\right] \to -\frac{1}{4}\left(\frac{g^2A}{N}\right)(An_b)\sum_a F^a_{\mu,\nu}F^a_{\mu,\nu}$$

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Brief recap on Monte Carlo method for path-integration

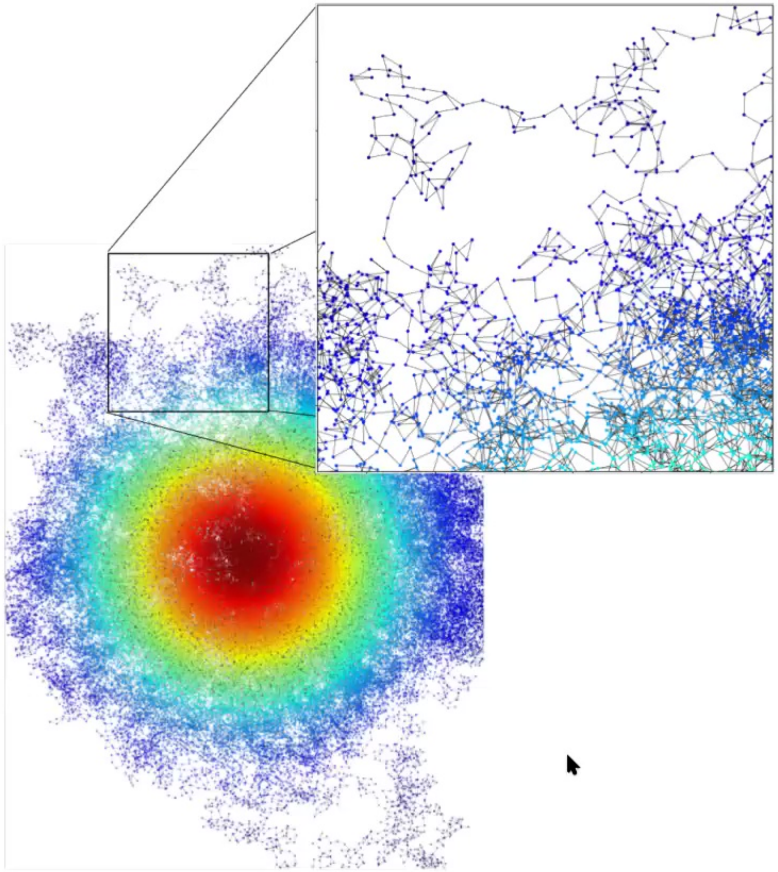
Monte carlo sampling:

- Sample configurations $\{\phi_i\}$ by evolving a quasi-random walk in the configuration space, weighted with probability:

$$\mathcal{P}[\phi] \equiv \frac{1}{Z} \exp\left(-S[\phi]\right)$$

- Average of any observable $\mathcal{O}$ computed as standard average over the sample:

$$\overline{\mathcal{O}} = \frac{1}{m} \sum_{i=1}^{m} \mathcal{O}[\phi_i]$$



Monte Carlo importance sampling and detailed balance

Monte Carlo **importance sampling** involves a Markov process where each step from $i \equiv (\mathcal{T}_i, \Phi^{(i)}_{\mathcal{T}_i})$ to $f \equiv (\mathcal{T}_f, \Phi^{(f)}_{\mathcal{T}_f})$ must satisfy the **detailed balance condition** (DBC):

$$P_i W_{i\to f} = P_f W_{f\to i},$$

where $W_{i\to f}$ and $W_{f\to i}$ are the **transition probabilities** of going from i to f or viceversa respectively; these can be decomposed into **selection** and **acceptance probabilities** (same for $f \leftrightarrow i$):

$$W_{f\to i} \equiv P_{\text{sel.}}(i|f) A_{f\to i},$$

- $P_{\text{sel.}}(f|i)$: probability of selecting f among the many possibile final configurations starting from i;
- $A_{i\to f}$: probability to accept the $i \to f$ move.

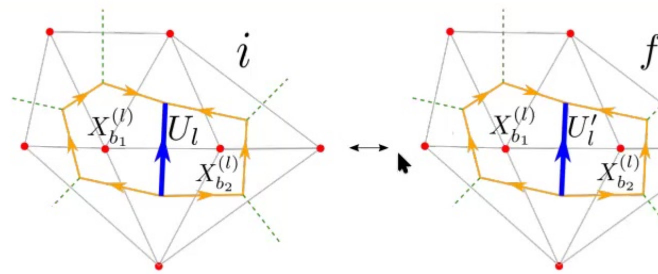
The As can be chosen to both satisfy DBC and maximize acceptance:

$$A_{i\to f} = \min\left(1, \frac{P_{\text{sel.}}(i|f)}{P_{\text{sel.}}(f|i)} \cdot \frac{P_f}{P_i}\right), A_{f\to i} \qquad = \min\left(1, \frac{P_{\text{sel.}}(f|i)}{P_{\text{sel.}}(i|f)} \cdot \frac{P_i}{P_f}\right).$$

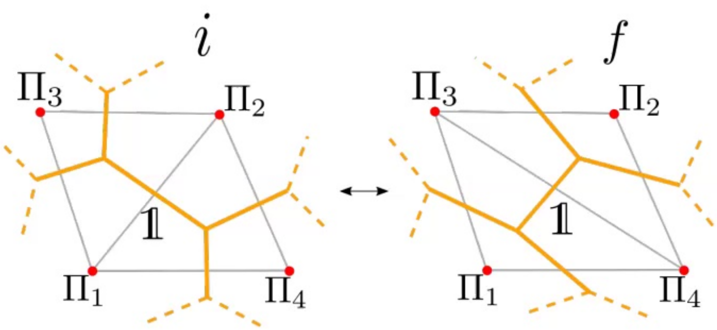
Monte Carlo moves for CDT coupled to Yang–Mills in 2D

The space of all composite configurations is made of pairs $(\mathcal{T}, \Phi_{\mathcal{T}})$, where $\Phi_{\mathcal{T}}$ is a gauge configuration in the fixed triangulation $\mathcal{T}$.

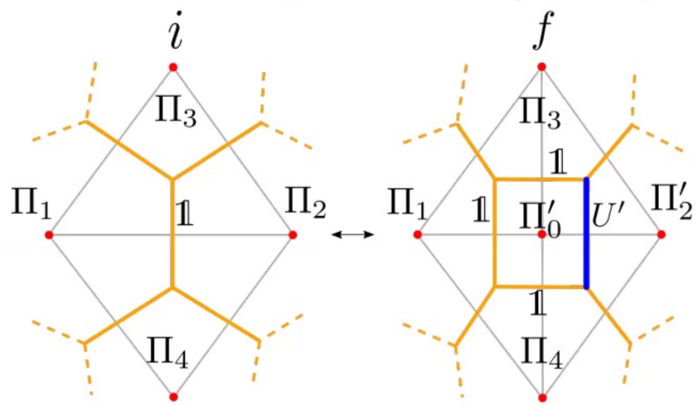
1. The action changes locally at the level of plaquettes involved;
2. Each move is accepted according to the detailed balance conditions;
3. Gauge invariance allows convenient gauge transformations (GT).



Yang–Mills move (fixed $\mathcal{T}$)

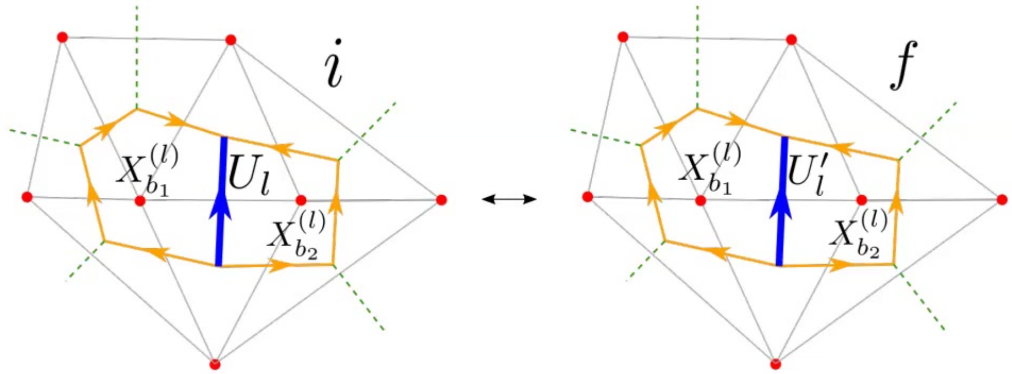


Flip of timelike link



Vertex creation/destruction

Detailed balance for the YM moves



$$\Delta S^{(\text{gauge})} = -\frac{\beta}{N} ReTr\left[\left(U'_l - U_l\right)F_l^\dagger\right], \text{ with } F_l \equiv \sum_{b \ni l} \frac{X_b^{(l)}}{n_b} \text{ (force)},$$

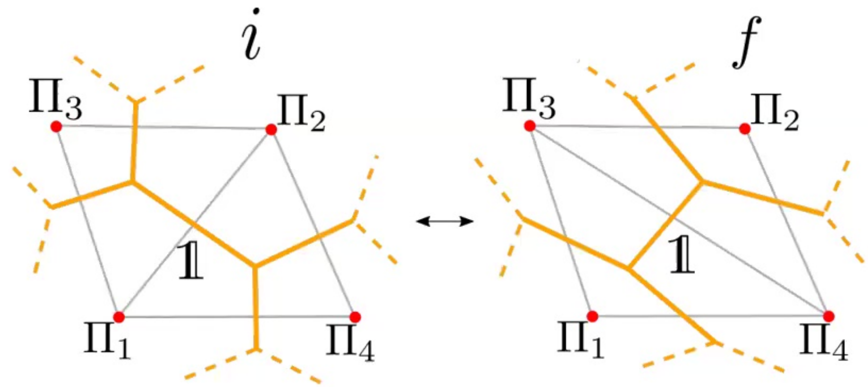
with $X_b^{(l)}$ being the so called *staple* (the two oriented yellow paths) Extracting U' can be done either by **Metropolis-Hastings** or by a **heat-bath** procedure:

$$\rho_{\text{sel.}}^{\widetilde{U}}(\Phi) = \frac{1}{Z_G(F_l)} \exp\left\{\frac{\beta}{N} ReTr\left[\widetilde{U} F_l^\dagger\right]\right\}, \text{ with } \widetilde{U} \text{ either } U_l \text{ or } U'_l,$$

Normalization $Z_G(F)$ is computed as a Haar integral on G:

$$Z_G(F) \equiv \int_G dU'' \exp\left\{\frac{\beta}{N} ReTr\left[U'' F^\dagger\right]\right\}.$$

Detailed balance for the $(2,2)$ move



We can use **gauge transformations** on a vertex to fix the central link variable to $\mathbb{1}$.

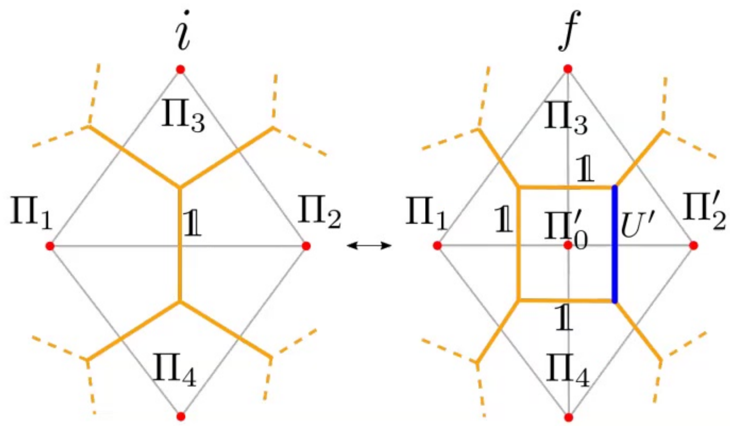
$$\Delta S^{(2,2)} \equiv S^{(2,2)}[f] - S^{(2,2)}[i] = \beta\left[\frac{\widetilde{\Pi}_1}{n_1(n_1+1)} + \frac{\widetilde{\Pi}_2}{n_2(n_2+1)} - \frac{\widetilde{\Pi}_3}{n_3(n_3-1)} - \frac{\widetilde{\Pi}_4}{n_4(n_4-1)}\right],$$

where we defined $\widetilde{\Pi}_b \equiv \left[\frac{1}{N} Re Tr \Pi_b - 1\right]$.

Selection probability is trivial: $P_{\text{sel.,YM}}[i] = P_{\text{sel.,YM}}[f] = 1$, but acceptance is not (because of the change in the n_bs):

$$A^{(2,2)} = \min\left(1, e^{-\Delta S^{(2,2)}} \cdot \frac{P_{\text{sel.,CDT}}[\mathcal{T}]}{P_{\text{sel.,CDT}}[\mathcal{T}']}\right).$$

Detailed balance for the (2, 4)–(4, 2) moves



On i we can use gauge transformations to fix the central link to $\mathbb{1}$, but only 3 out of 4 links in f (plaquettes are gauge invariants!!).

$$\Delta S_{\mathrm{YM}}^{(2,4)} = \beta\Big\{\frac{\widetilde{\Pi}_3}{n_3(n_3+1)} + \frac{\widetilde{\Pi}_4}{n_4(n_4+1)} + \frac{\widetilde{\Pi}_2}{n_2} + \frac{1}{4} - \frac{1}{n_2} - \frac{1}{N} Re Tr\Big[U'\Big(\frac{\mathbb{1}}{4} + \frac{X_2^\dagger}{n_2}\Big)\Big]\Big\},$$

where we can isolate the U'-dependent part by defining:

$$\widetilde{\Delta S}_{\mathrm{YM}}^{(2,4)}(U', \{\Pi_b\}) = -\frac{\beta}{N} Re Tr\Big[U'\mathcal{F}^\dagger\Big],$$

$$\widehat{\Delta S}_{\mathrm{YM}}^{(2,4)}(\{\Pi_b\}) = +\beta\Big[\frac{\widetilde{\Pi}_3}{n_3(n_3+1)} + \frac{\widetilde{\Pi}_4}{n_4(n_4+1)} + \frac{\widetilde{\Pi}_2}{n_2} + \frac{1}{4} - \frac{1}{n_2}\Big],$$

Detailed balance for the (2, 4)–(4, 2) moves (cont'd)

We choose to select a random value for the new link variable $U' \in G$ with a **heat-bath** sampling:

$$P_{\text{sel.,YM}}[f] = \frac{\exp\left[-\Delta S_{\text{YM}}^{(2,4)}(U')\right]}{\int_G dU'' \exp\left[-\Delta S_{\text{YM}}^{(2,4)}(U'')\right]} = \frac{1}{Z_G(\mathcal{F})} \exp\left(\frac{\beta}{N} ReTr\left[U' \mathcal{F}^\dagger\right]\right),$$

where $\mathcal{F} \equiv \mathbb{1}/4 + X_2/n_2$ is the sum of the staples' contributions. Using heat-bath selection, the final Metropolis acceptance probability simplifies:

$$A^{(2,4)} = \min\left\{1, \frac{P_{\text{sel.,CDT}}(\mathcal{T}_i)}{P_{\text{sel.,CDT}}(\mathcal{T}_f)} e^{-2\lambda - \widehat{\Delta S}_{\mathbf{YM}}^{(\mathbf{2,4})}} Z_G(\mathcal{F})\right\},$$

$$A^{(4,2)} = \min\left\{1, \frac{P_{\text{sel.,CDT}}(\mathcal{T}_f)}{P_{\text{sel.,CDT}}(\mathcal{T}_i)} e^{+2\lambda + \widehat{\Delta S}_{\mathbf{YM}}^{(\mathbf{2,4})}} / Z_G(\mathcal{F})\right\}.$$

No dependence on the selected U', but the integral $Z_G(\mathcal{F})$ must be computed for each value of the force!

Closed form for the group integrals in 2D

For $G = U(1)$ and $G = SU(2)$ in 2D, the group integrals have closed forms:
[R. Brower, P. Rossi and C. I. Tan (1981)]

$$Z_{U(1)}(\mathcal{F}) = I_0\Big(2\,det(J)\Big); \tag{2}$$

$$Z_{SU(2)}(\mathcal{F}) = \frac{1}{\sqrt{K(J)}} I_1\Big(2\sqrt{K(J)}\Big), \tag{3}$$

where $J \equiv \frac{\beta}{2N}\mathcal{F}$, I_k are the modified Bessel functions of the first kind, while $K(J) \equiv Tr\left(J^\dagger J\right) + det(J) + det(J^\dagger)$.

For $U(N)$ and $SU(N)$ groups with $N > 2$ a numerical evaluation of the Haar integral has to performed.

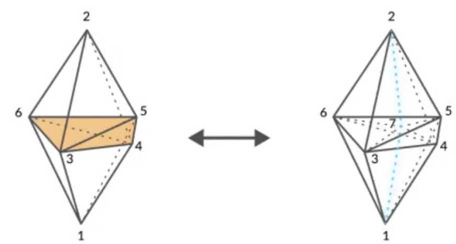
Complexity for higher dimensions

The same ideas sketched on 2D can be applied for higher dimensions and for generic Lie groups, but the the group integrals become more expensive! About higher dimensions, gauge transformations can fix only part of the involved link variables.

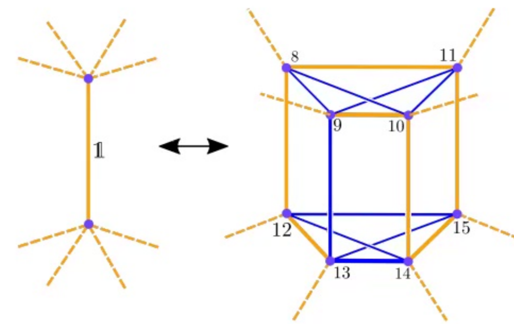
E.g. the final configuration of move $(2, 6)$ used 3D CDT has dual links structured into a triangular prism: only 5 out of 9 links can be fixed to $\mathbb{1}$.

All unfixed links are degrees of freedom involved in the detail balance selection: Metropolis-Hasting has very poor acceptance, while heat-bath requires integrations on many group variables.

Worst case example: group integral for $(2,8)$ move in 4D



Move $(2,8)$ on the primal lattice



Move $(2,8)$ on the dual lattice

9 unfixable link variables $\implies$ sampling from 9 copies of the gauge group (!!)

$$
\begin{aligned}
-\frac{\widetilde{\Delta S}^{(G)}}{\beta} =& + \frac{|S_{0,1}U'_{11,10}|}{n_{0,1}+1} + \frac{|S_{0,2}U'_{11,9}|}{n_{0,2}+1} + \frac{|S_{1,3}U'_{10,8}|}{n_{1,3}+1} + \frac{|S_{2,3}U'_{9,8}|}{n_{2,3}+1} + \frac{|S_{2,6}U'_{9,13}|}{n_{2,6}} + \frac{|S_{4,6}U'_{15,13}|}{n_{4,6}+1} + \frac{|S_{4,7}U'_{15,12}|}{n_{4,7}+1} \\
&+ \frac{|S_{5,6}U'_{14,13}|}{n_{5,6}+1} + \frac{|S_{5,7}U'_{14,12}|}{n_{5,7}+1} + \frac{|U'_{8,9}U'_{10,8}|}{3} + \frac{|U'_{8,9}U'_{9,11}|}{3} + \frac{|U'_{8,10}U'_{10,11}|}{3} + \frac{|U'_{10,11}U'_{11,9}|}{3} + \frac{|U'_{13,14}U'_{14,12}|}{3} \\
&+ \frac{|U'_{13,15}U'_{15,12}|}{3} + \frac{|U'_{12,14}U'_{15,12}|}{3} + \frac{|U'_{13,14}U'_{15,13}|}{3} + \frac{|U'_{8,9}U'_{9,13}|}{4} + \frac{|U'_{8,10}U'_{14,12}|}{4} \\
&+ \frac{|U'_{15,12}|}{4} + \frac{|U'_{14,13}U'_{13,9}|}{4} + \frac{|U'_{9,11}U'_{15,13}U'_{13,9}|}{4} + \frac{|U'_{10,11}|}{4}
\end{aligned}
$$

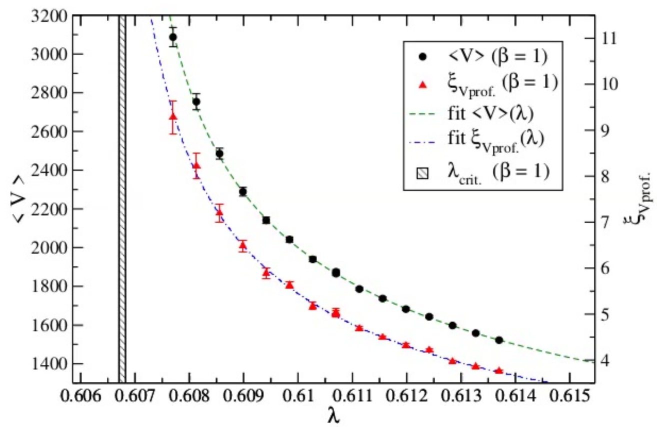
Maybe the only viable solution is a Monte Carlo inside the Monte Carlo (ideas?).

Outline

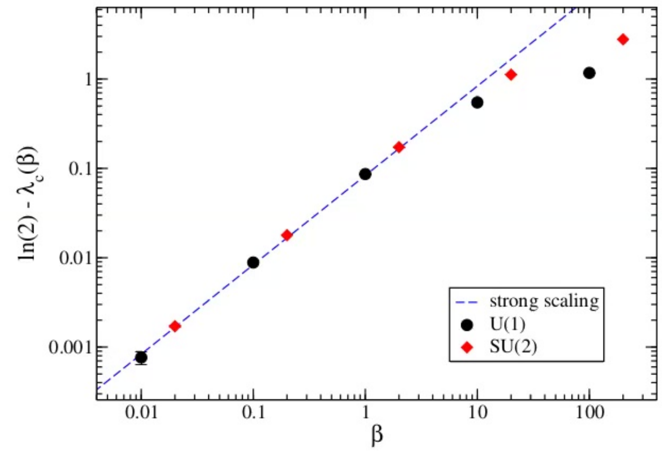
Gravity-related observables

A. Candido, G.C., M. D'Elia and F. Rottoli [JHEP **04** (2021), 184]

We focused on the numerical study of the gauge groups $U(1)$ and $SU(2)$ coupled to CDT in 2D: $S_{CDT,2D} = \lambda N_2[\mathcal{T}]$,
$\lambda \leq \lambda_c$: volumes diverge, $\lambda \gtrsim \lambda_c$: shifted power law $A(\lambda - \lambda_c)^{-\mu}$.



$\langle V \rangle$ and 2-point correlation length of volume profiles $\xi_{\text{Vprof.}}$ as function of λ ($\beta = 1$ fixed, $U(1)$).

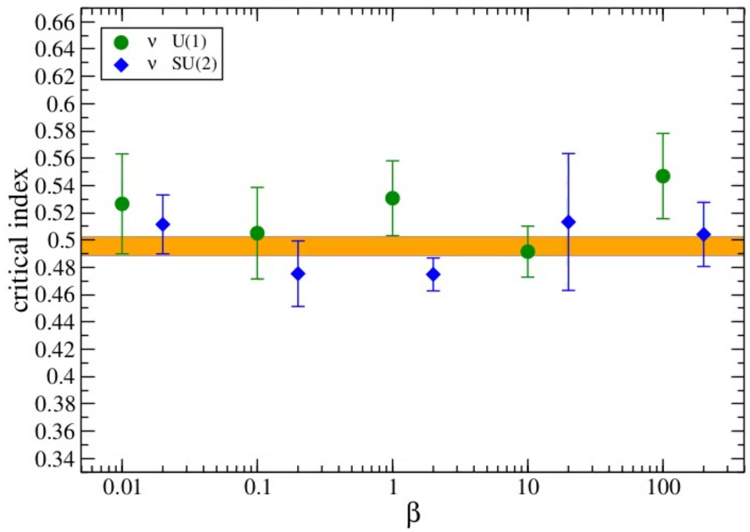


Critical points $\lambda_c(\beta)$ for both $U(1)$ and $SU(2)$ groups. Dashed slope: expected strong coupling behavior.

In the **strong coupling limit** ($\beta \to 0$) $\lambda_c \to \log 2$ (no fields).

Gravity-related observables are independent from YM fields

Besides an unobservable shift in the critical value λ_c, we observed no β-dependence on gravity-related observables.



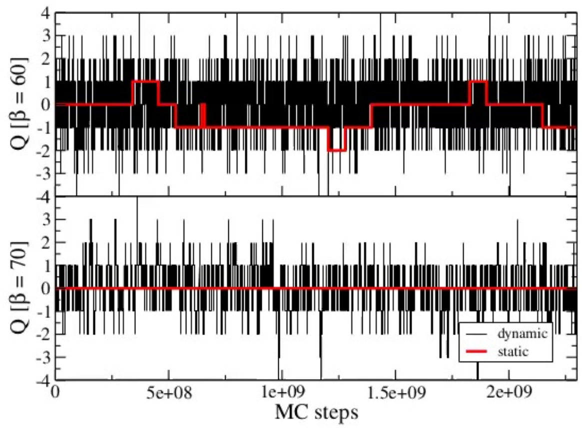
Critical indices for the total volume $\langle V \rangle$ and the correlation length of volume profiles $\xi_{\mathrm{Vprof.}}$ from the shifted scaling law.

Numerical Results: topology for the $U(1)$ gauge group

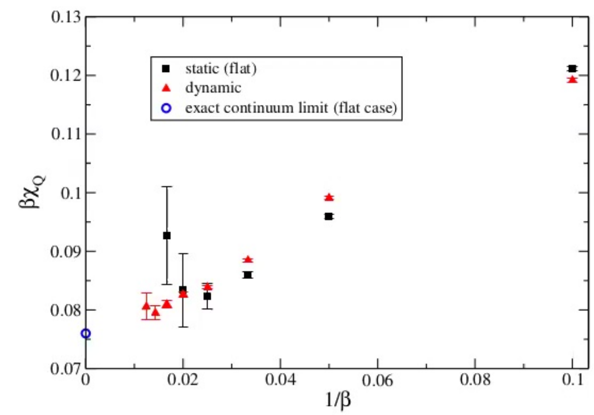
A. Candido, G.C., M. D'Elia and F. Rottoli [JHEP **04** (2021), 184]

$U(1)$ gauge configurations in a toroidal space has non-trivial homotopy, with interesting **topological charge** and **susceptibility**:

$$Q \equiv \frac{1}{2\pi} \sum_{b \in \mathcal{T}^{(d-2)}} \arg[Tr(\Pi_b)] \qquad \chi_Q \equiv \langle Q^2 \rangle / V .$$



Comparison of topological charge histories for a static (flat) and dynamic simulations at same $\langle V \rangle$.



The extrapolated topological susceptibility $\beta\chi_Q = 0.0758(14)$ ($\chi^2/dof = 1.6/3$) agrees with the exact (flat) limit $3/(4\pi^2)$.

Mitigation of topological freezing

Observation: the estimates of susceptibility for both flat static and curved dynamical are comparable, but the first suffers freezing while the second doesn't. But **why** the dynamical case seems to mitigate the freezing problem?

Hypotheses:

- (H_0) Because of geometric fluctuations in Monte Carlo time;
- (H_1) Because of metric inhomogeneities in the geometry.

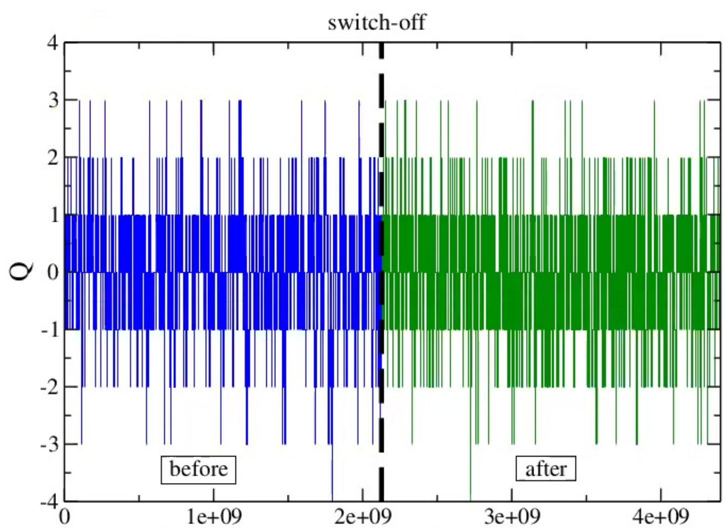
As a test, we started a dynamical simulation where we switched off all geometry updates at a certain Monte Carlo step after which the (locally non-flat) geometry is fixed and only the YM part changes.

This tests both hypotheses, because we already know that there is freezing in the flat static case.

Mitigation of topological freezing: an unexpected result

Unexpectedly, mitigation seems induced by metric inhomogeneities in the geometry, and fluctuations in Monte Carlo time don't seem to have a role! ($\neg H_0$, $\underline{H_1}$ holds)

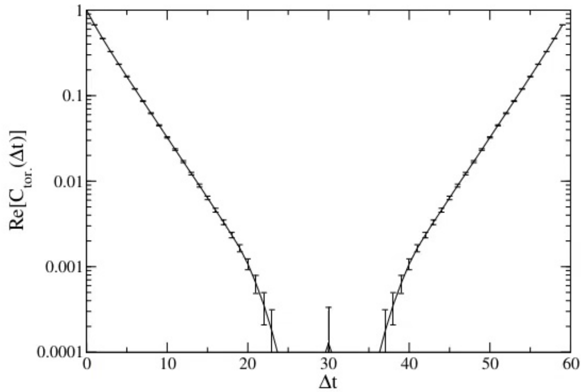
Intuitively, vertices with a large negative curvature (large n_b) represent large holes in the lattice where fluctuations can happen more easily leading to tunneling events between topological sectors.



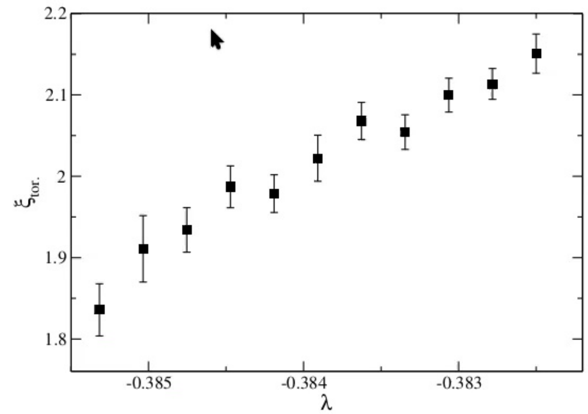
Since topological freezing is just an algorithmic problem, this observation has potentially very fruitful applications to improve algorithms also in other contexts (e.g. QCD).

Torelons

Another YM-related observable is the **torelonic profile** $\Pi(t)$: for a 2D torus, it is the oriented product of link variables on the loop of dual-links that wraps around spatially at (slab) time t.



Typical torelon correlation function ($\lambda = -0.44244$ and $\beta = 90$).



Dependence on λ, at fixed $\beta = 70$

$$C_{\text{tor}}(\Delta t) \equiv \frac{\langle Tr\Pi(t)\, Tr\Pi^\dagger(t+\Delta t)\rangle - |\langle Tr\Pi(t)\rangle|^2}{\langle |Tr\Pi(t)|^2\rangle - |\langle Tr\Pi(t)\rangle|^2}\,.$$

From this we can extract a toreonic 2-pt correlation length $\xi_{tor.}$.

Outline

A recipe for the continuum limit

Our two length scales for our system, $\xi_{\text{Vprof.}}$ and $\xi_{tor.}$. are related to the physical ones $\xi_{X,r}$ by means of the unobservable lattice spacing a, i.e., $\xi_{X,r} = a\,\xi_X$.
The correct procedure for building a sequence of simulations corresponding to a continuum limit is to keep their ratio $\rho \equiv \xi_{\text{tor.},r}/\xi_{\text{Vprof.},r} = \xi_{\text{tor.}}/\xi_{\text{Vprof.}}$ invariant, while the critical line is approached for $\beta \to \infty$ ($a \to 0$).

Physically, keeping ρ invariant means that the relation between Yang–Mills and gravitational ranges is kept the same, while $a \to 0$ since $\xi_X \to \infty$ in lattice spacing units.

Then, we can forget about the original bare couplings λ and β, since each value of ρ identifies a distinct continuum limit physics, while a correlation length (let's say $\xi_{\text{tor.}}$), used as parameter dictating the common scale for every phenomena of the system.

Quenched 4D case

First step towards the 4D case: **quenched simulation**.
G.C., M. D'Elia and D. Nemeth (published soon)

With quenched here we mean:

- geometry configurations are sampled independently;
- gauge simulations are performed on fixed background from the ensamble build in the previous step.

The algorithm is **much simpler** than the dynamical case, just the usual heat-bath on single link variables, but it has not backreaction on gravity.

Conclusions

- Coupling CDT with compact gauge (and eventually fermionic) fields brings us closer to more realistic systems;
- Algorithm manageable in low dimensions and for low order groups (small N);
- Besides a shift in the bare parameter λ, gravity-related observables don't show particular β-dependence.
- The YM topology behaves similarly to flat case, but with algorithmically mitigated freezing.