Title: Reducedness of quiver varieties

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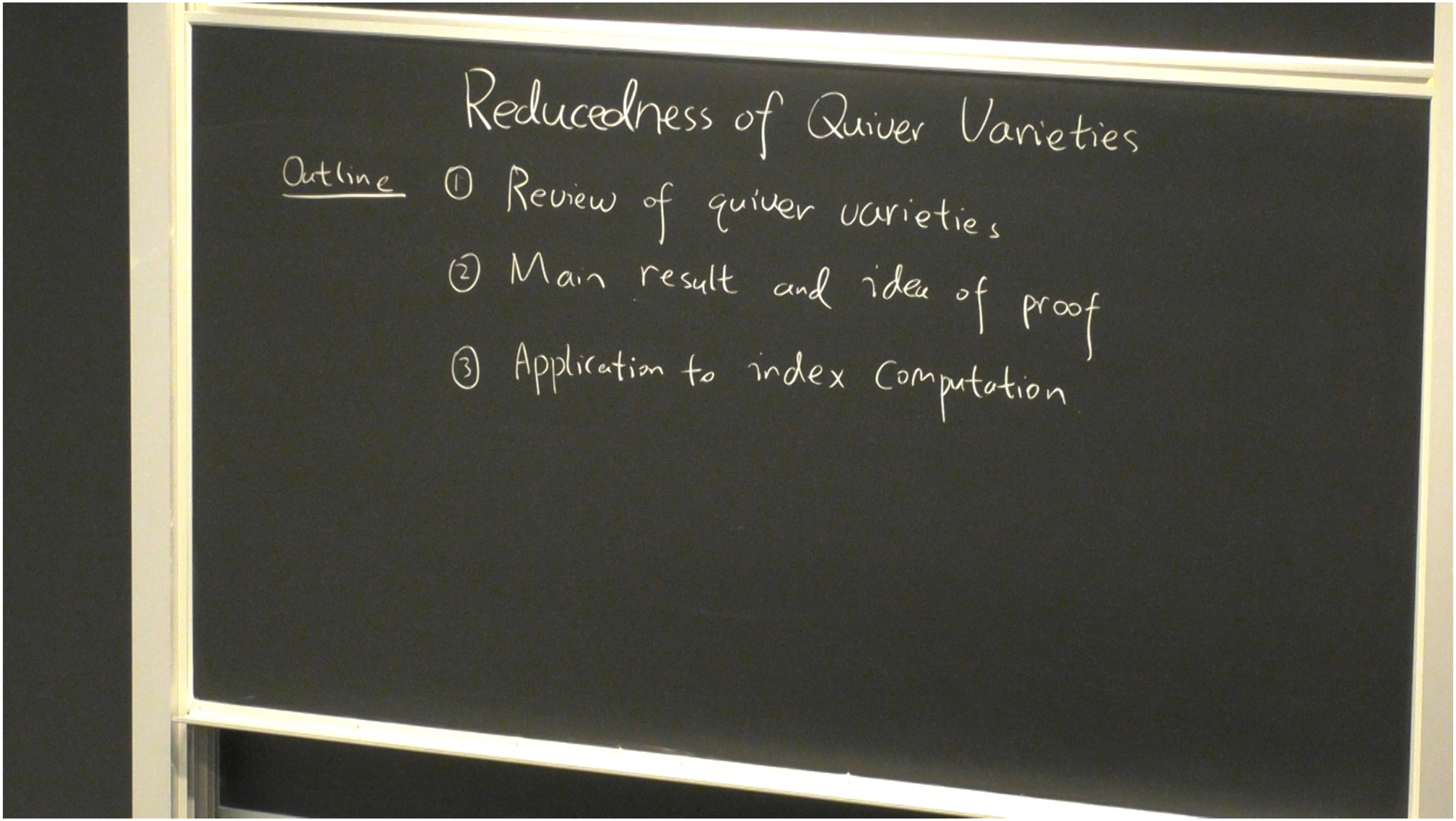
Abstract: Nakajima's quiver varieties play important roles in mathematical physics and representation theory. They are defined as symplectic reduction of the space of representations of the doubled quivers, and they are equipped with natural scheme structures. It is not known in general whether this scheme is reduced or not, and the reducedness issue does show up in certain scenario, for example the integration formula of the K-theoretic Nekrasov's partition function. In this talk I will show that the quiver variety is reduced when the moment map is flat, and I will also give some applications of this result. This talk is based on my work arXiv: 2201.09838.

Zoom Link: https://pitp.zoom.us/j/97405405211?pwd=dEtVeHhQVjNrdGN4Vkh0ZlRrbEpVQT09

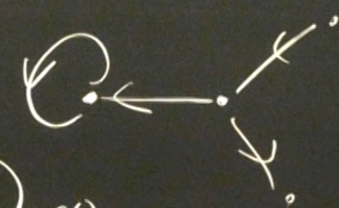

Reducedness of Quiver Varieties
Outline
① Review of quiver varieties
② Main result and idea of proof
③ Application to index computation

§1 Quiver Varieties

Definition: $Q = (Q_0, Q_1)$ be a quiver

Q_0: nodes, Q_1: edges.

Ex



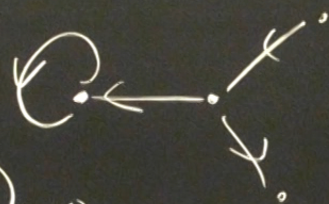
- A rep. of Q of dim. $\underline{V} = (V_i)_{i \in Q_0}$ is $(X_a)_{a \in Q_1}$, $X_a \in \mathrm{Hom}(\mathbb{C}^{V_{t(a)}}, \mathbb{C}^{V_{h(a)}})$
- An isom. between rep's of Q is $(g_i)_{i \in Q_0}$, $g_i \in GL(V_i)$ s.t. $X_a^{(2)} = g_{h(a)} \cdot X_a^{(1)} \cdot g_{t(a)}^{-1}$

...er Varieties

$Q = (Q_0, Q_1)$ be a quiver

...odes $\quad$ edges

Ex



Q of dim. $\underline{V} = (V_i)_{i \in Q_0}$

$)_{a \in Q_1} \quad x_a \in \mathrm{Hom}(\mathbb{C}^{V_{t(a)}}, \mathbb{C}^{V_{h(a)}})$

between rep's of Q. is $(g_i)_{i \in Q_0}$ $g_i \in GL(V_i)$

$^{(2)} = g_{h(a)} \cdot x_a^{(1)} \cdot g_{t(a)}^{-1}$ $\quad$ $GL(\underline{V}) = \prod_{i \in Q_0} GL(V_i)$

$R(Q, \underline{v})$: linear space of reps of dim $\underline{v}$

$\in GL(V_i)$

$= \prod_{i \in Q_0} GL(V_i)$

Remark: $\mathbb{C}^\times_{diag} \hookrightarrow GL(\underline{v})$

acts trivially on reps,

so we define $G(\underline{v}) = GL(\underline{v}) / \mathbb{C}^\times_{diag}$

$$G(\underline{v}) \circlearrowright R(Q, \underline{v}).$$

$$\rightsquigarrow G(\underline{v}) \circlearrowright T^* R(Q, \underline{v}).$$

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$$G(\underline{v}) \circlearrowright R(Q, \underline{v}).$$

$$\rightsquigarrow G(\underline{v}) \circlearrowright T^*R(Q, \underline{v}).$$

Remark: $T^*R(Q, \underline{v}) = R(\overline{Q}, \underline{v})$

$\overline{Q}_0 = Q_0$, $\overline{Q}_1 = Q_1 \sqcup Q_{\bar{1}}$

reverse direction

Natural symplectic form:

$$\omega(x, y) = \sum_{a \in Q_1} \left(Tr(x_a y_{\bar{a}}) - Tr(x_{\bar{a}} y_a) \right)$$

Then $G(\underline{v}) \circlearrowright T^*R(Q, \underline{v})$ is Hamiltonian with moment map

$$\mu : R(\overline{Q}, \underline{v}) \longrightarrow \mathfrak{g}(\underline{v})^*$$

$$\mathfrak{g}(\underline{v})^* = \{ M_i \in End(\mathbb{C}^{v_i}) \mid$$

Remark: $\mathbb{C}^{\times}_{diag} \hookrightarrow GL(\underline{v})$

acts trivially on reps,

so we define $G(\underline{v}) = GL(\underline{v}) / \mathbb{C}^{\times}_{diag}$

$$G(\underline{v}) \curvearrowright R(Q, \underline{v}).$$

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Natural symplectic form:

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Then $G(\underline{v}) \curvearrowright T^{*}R(Q, \underline{v})$ is

Hamiltonian with moment map

$$\ldots(\overline{Q}, \underline{v}) \longrightarrow \mathfrak{g}(\underline{v})^{*} = \left\{ M_i \in End(\mathbb{C}^{v_i}) \;\middle|\; \sum_{i \in Q_0} Tr(M_i) = 0 \right\}$$

$$\mu(X)_i = \sum_{h(a)=i} X_a X_{\bar{a}} - \sum_{t(a)=i} X_{\bar{a}} X_a .$$

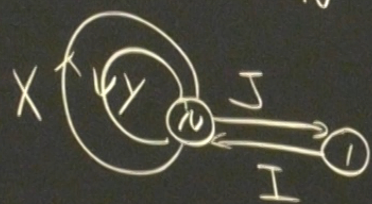
$i \in Q_0$

Example: $Q =$ 

$\underline{v} =$ N 1.

$$\mu(X,Y,I,J) = \left([X,Y] + IJ \;,\; -JI \right)$$

N 1



ADHM quiver.

Stability Condition

Let θ be a charcter of $G(\underline{v})$

i.e. $(g_i) \longmapsto \prod_{i \in Q_0} \det(g_i)^{\theta_i}$ $\theta_i \in \mathbb{Z}$

s.t. $\sum v_i \theta_i = 0$. $(\underline{v} \cdot \theta = 0)$

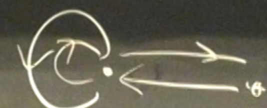
By geometric invariant theory.

we can talk about θ-(semi)stable reps

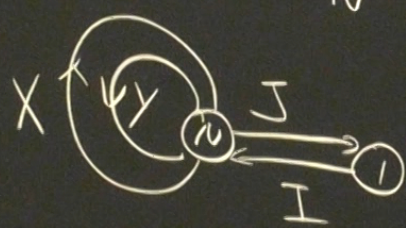
- $X \in R(\bar{Q}, \underline{v})$. is called θ-stable if $\forall$ subrep Y. we have $\theta \cdot \underline{y} < 0$.
- ..... θ-semistable if ..... $\theta \cdot \underline{y} \leq 0$.
- Observe that θ-(semi)stable $\Leftrightarrow$ $k\theta$-(semi)stable $\hookrightarrow k \in \mathbb{Z}_+$

$$\mu(X)_i = \sum_{h(a)=i} X_a X_{\bar{a}} - \sum_{t(a)=i} X_{\bar{a}} X_a , \quad i \in Q_0$$

• **Example:** $Q =$ 



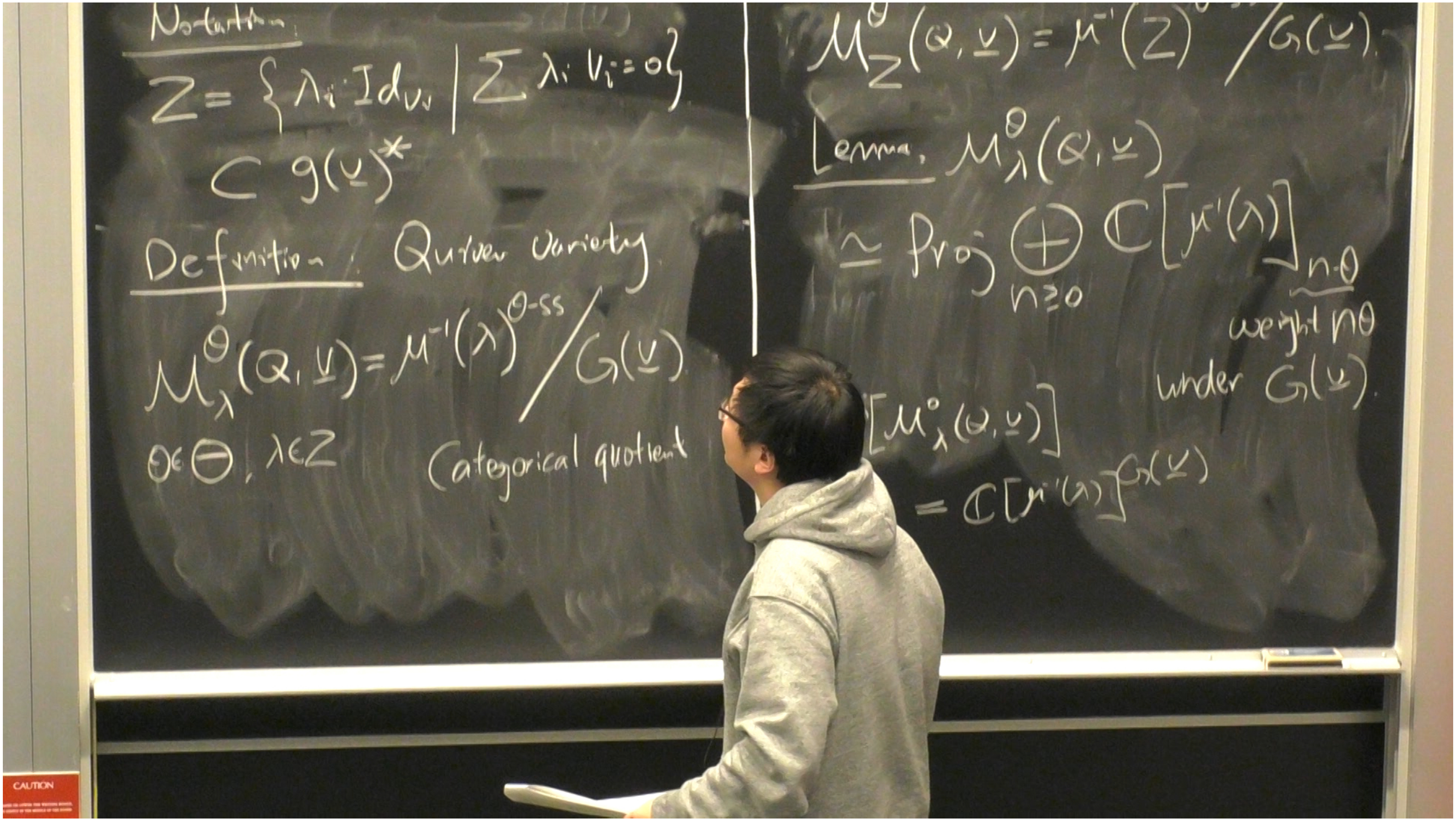
$\underline{v} =$	N	1
θ_+	1	$-N$
θ_-	-1	N



ADHM quiver.

θ_-: $\forall T \subset \mathbb{C}^N$, $X(T) \subset T$, $Y(T) \subset T$, $T \supset \mathrm{Im}(I)$, then $T = \mathbb{C}^N$

θ_+-stable: $\forall S \subset \mathbb{C}^N$ s.t. $X(S) \subset S$, $J(S) = 0$, $Y(S) \subset S$, then $S = 0$.


Notation
$Z = \{ \lambda_i \cdot Id_{V_i} \mid \sum \lambda_i v_i = 0 \}$
$\subset \mathfrak{g}(\underline{v})^*$
Definition Quiver variety
$\mathcal{M}_\lambda^\theta(Q, \underline{v}) = \mu^{-1}(\lambda)^{\theta\text{-}ss} / G(\underline{v})$
$\theta \in \Theta,\ \lambda \in Z$
Categorical quotient
$\mathcal{M}_Z^\theta(Q, \underline{v}) = \mu^{-1}(Z) / G(\underline{v})$
Lemma. $\mathcal{M}_\lambda^\theta(Q, \underline{v})$
$\simeq \mathrm{Proj} \bigoplus_{n \geq 0} \mathbb{C}[\mu^{-1}(\lambda)]_{n\cdot\theta}$
weight $n\theta$
under $G(\underline{v})$
$[\mathcal{M}_\lambda^0(Q, \underline{v})]$
$= \mathbb{C}[\mu^{-1}(\lambda)]^{G(\underline{v})}$

$x \in R(\bar{Q}, \underline{v})$ is called

θ- stable if $\forall$ subrep

y we have $\theta \cdot \underline{y} < 0$

.... θ-semistable if

..... $\theta \cdot \underline{y} \leq 0$.

- Observe that θ-(semi)stable
  $\iff k\theta$-(semi)stable
  $\hookrightarrow k \in \mathbb{Z}_+$

In particular, $\exists$ projective map

$$\mathcal{M}^{\theta}_{\lambda}(Q, \underline{v}) \longrightarrow \mathcal{M}^{0}_{\lambda}(Q, \underline{v})$$

Remark: ① Points in $\mathcal{M}_\lambda^\theta(Q,\underline{v}) \xleftrightarrow{1:1}$ closed $G(\underline{v})$-orbits

in $\mu(\lambda)^{\theta\text{-ss}} \xleftrightarrow{1:1}$ θ-polystable reps

$$X = X^{(1)} \oplus \cdots \oplus X^{(n)}$$

$X^{(i)}$ are θ-stable.

② Framed quivers $(Q, \underline{v}, \underline{d})$.

Remark: ① Points in $\mathcal{M}_\lambda(\ \ )$ $\longrightarrow$ closed $G(\underline{v})$-orbits

in $\mu(\lambda)^{\theta\text{-ss}}$ $\longleftrightarrow$ θ-polystable reps

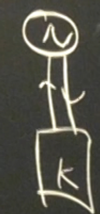
$$X = X^{(1)} \oplus \cdots \oplus X^{(r)}$$

$X^{(i)}$ are θ-stable.

② Framed quivers $(Q, \underline{v}, \underline{d})$

$\Longleftrightarrow$ Unframed quiver $(Q^{\underline{d}}, \underline{v}^{\underline{d}})$

Crawley-Boevey trick.



$X_a X_{\bar{a}} - \sum X_{-} X \qquad \theta_- : \forall I \subset \mathbb{C}^N$

Remark: ① Points in $\mathcal{M}_\lambda(\ldots)$ $\longleftrightarrow$ closed $G(\underline{v})$-orbits
in $\mu(\lambda)^{\theta\text{-ss}}$ $\overset{1:1}{\longleftrightarrow}$ θ-polystable reps

$$X = X^{(1)} \oplus \cdots \oplus X^{(r)}$$

$X^{(i)}$ are θ-stable.

② Framed quivers $(Q, \underline{v}, \underline{d})$
$\Longleftrightarrow$ Unframed quiver $(Q^{\underline{d}}, \underline{v}^{\underline{d}})$
Crawley-Boevey trick.

$$Q_0^d = Q_0 \sqcup \infty$$

$$Q_1^d = Q_1 \sqcup \left\{ \underbrace{\infty \rightrightarrows i}_{\# = w_i} \right\}$$

$$V_i^d = V_i \qquad V_\infty^d = 1$$

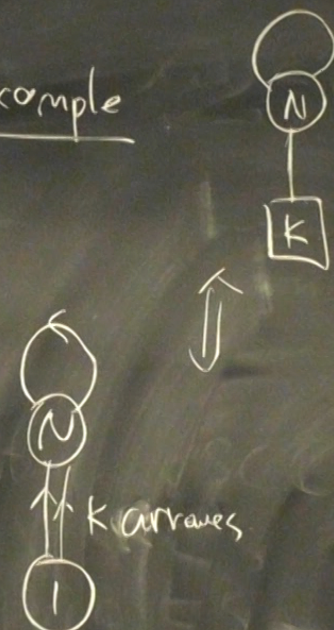
$i \in Q_0$

$$\theta \mapsto \theta^d \qquad \theta_i^d = \theta_i \qquad \theta_\infty = -\theta \cdot \underline{v}$$

$$= -\sum \theta_i v_i$$

$\lambda \mapsto \dots\dots$

Example



$Q_0^{\underline{d}} = Q_0 \sqcup \infty$

$Q_1^{\underline{d}} = Q_1 \sqcup \{\infty \Rightarrow i\}$ $\quad \# = d_i$

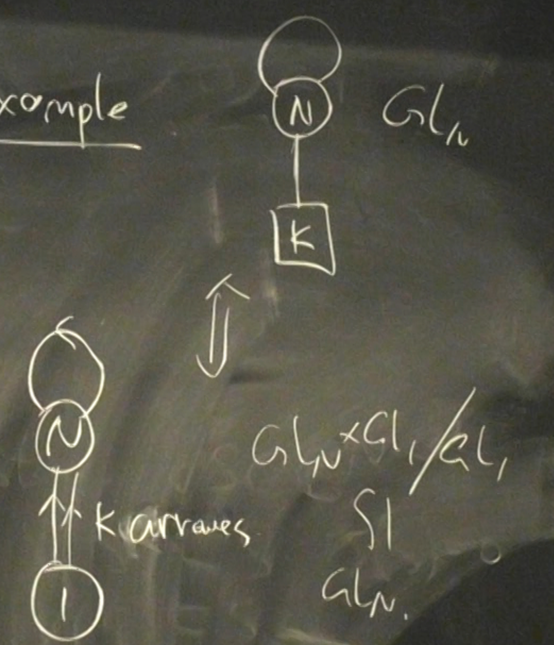
$V_i^{\underline{d}} = V_i \quad V_\infty^{\underline{d}} = 1$

$i \in Q_0 \qquad \underline{d} = (d_i)_{i \in Q_0}$

$\theta \mapsto \theta^{\underline{d}} \quad \theta_i^{\underline{d}} = \theta_i \quad \theta_\infty = -\theta \cdot \underline{v}$

$\lambda \mapsto \dots \qquad = -\sum \theta_i \cdot v_i$

Example



- $\mathcal{M}^{\theta}_{\lambda}(Q, \underline{v})$ are highly singular,
- but $\mathcal{M}^{\theta}_{\lambda}(Q, \underline{v})_{\underline{red}}$ has symplectic singularities

(Bellamy - Schedler) $\hookrightarrow$ reduced scheme structure.

ω on $T^*R(Q, \underline{v})$

$\rightsquigarrow$ Poisson str on $\mathcal{M}^{\theta}_{\lambda}$

$\exists\, \widehat{\mathcal{M}}^{\theta}_{\lambda} \xrightarrow{f} \mathcal{M}^{\theta}_{\lambda}(Q, \underline{v})_{red}$. resolution.

s.t. $f^*(\omega_{reg})$ extends to global 2-form on $\widehat{\mathcal{M}}^{\theta}_{\lambda}$.

- [BS] : Conjectures that $\mathcal{M}^{\theta}_{\lambda}$ is actually reduced

§ 2. Main result

Thm: $\mathcal{M}_\lambda^\theta$ is reduced if $\mu: R(\overline{Q}, \underline{v}) \to \mathfrak{g}(\underline{v})^*$ is flat

proved by induction on $|\underline{v}| = \sum v_i$ using reduction to Codim 1 argument.

Reduction ①. Lemma. $\forall x \in \mathcal{M}^0_\lambda(Q, \underline{v})$.

$\exists$ a new quiver Q', $\underline{v}'$ s.t.

$$\mathcal{M}^0_\lambda(Q, \underline{v})^\wedge_x \cong \mathcal{M}^0_0(Q', \underline{v}')^\wedge_0$$

Moreover, μ' is flat if μ is flat.

Consider $\mathcal{M}_\lambda^\theta(Q, \underline{v})$ s.t. θ is generic.

then it's not hard to prove the thm in this case

<u>Rmk:</u> If θ is generic and $\underline{v} \neq k \cdot \underline{w}$ $k \in \mathbb{Z}_{\geq 2}$

$\Rightarrow \mathcal{M}_\lambda^\theta$ is smooth.

Consider: $p : \mathcal{M}_Z^\theta \longrightarrow \mathcal{M}_Z^0$

Can show that birational when θ is generic.

$\mathcal{M}_Z^\theta$ is normal.

Lemma: If $\mathcal{M}_Z^0$ is normal, then $\mathcal{M}_0^0$ is reduced.

then $Rp_* \mathcal{O}_{\mathcal{M}_Z^\theta} \simeq \mathcal{O}_{\mathcal{M}_Z^0} \Rightarrow \mathcal{O}_{\mathcal{M}_0^0} = \Gamma(\mathcal{M}_0^\theta, \mathcal{O})$

reduced

§2 Main result

Thm: M^{Q}_{λ} is reduced if $\mu : R(\bar{Q}, \underline{v}) \to \mathfrak{g}(\underline{v})$ is flat

proved by induction on $|\underline{v}| = \sum v_i$ using reduction to Codim 1 argument.

- Normality $\iff$ Regular at codim 1
  Serre's (S2)

§3 Index Computation

$p_0 : \mathcal{M}_0^\theta \to \mathcal{M}_0^0$ is a birational map

$$\mathbb{C}[\mathcal{M}_0^0] = \Gamma(\mathcal{M}_0^\theta, \mathcal{O})$$

$\Big\Vert$ $\mathcal{M}_0^0 = \mu^{-1}(0)/G(\underline{v})$ $\hookrightarrow$ Computable by localization

$$\left(\mathbb{C}[R(Q,\underline{v})] \overset{L}{\underset{\mathbb{C}[\mathfrak{g}(\underline{v})^*]}{\otimes}} \mathbb{C} \right)^{G(\underline{v})} \longleftarrow$$

Koszul resolve $\mathbb{C}$

+

Molien Weyl integration.

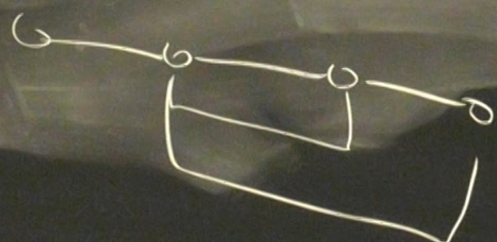
Q is (affine) type A. $(Q, \underline{v}, \underline{d})$ s.t. $v_i > 0$

$\underline{d} \neq 0$.

then flatness of μ $\Longleftrightarrow$

$$e_I \cdot (\underline{d} - C_Q \underline{v}) \geq -1$$

for all $I \subset Q_0$ connected by edges



$$e_I = \sum_{i \in I} e_i$$

Ex $\hat{A}_0$

is always flat

Q is (affine) type A. $(Q, \underline{v}, \underline{d})$ s.t. $v_i > 0$, $\underline{d} \neq 0$.

then flatness of $\mu \iff$

$$e_I \cdot (\underline{d} - C_Q \underline{v}) \geq -1$$

for all $I \subset Q_0$ connected by edges

$$e_I = \sum_{i \in I} e_i$$

Ex ① $\hat{A}_0$ $\boxed{k}$ — N is always flat

② A_1, $N \rightleftarrows \boxed{k}$

$K - 2N \geq -1$.