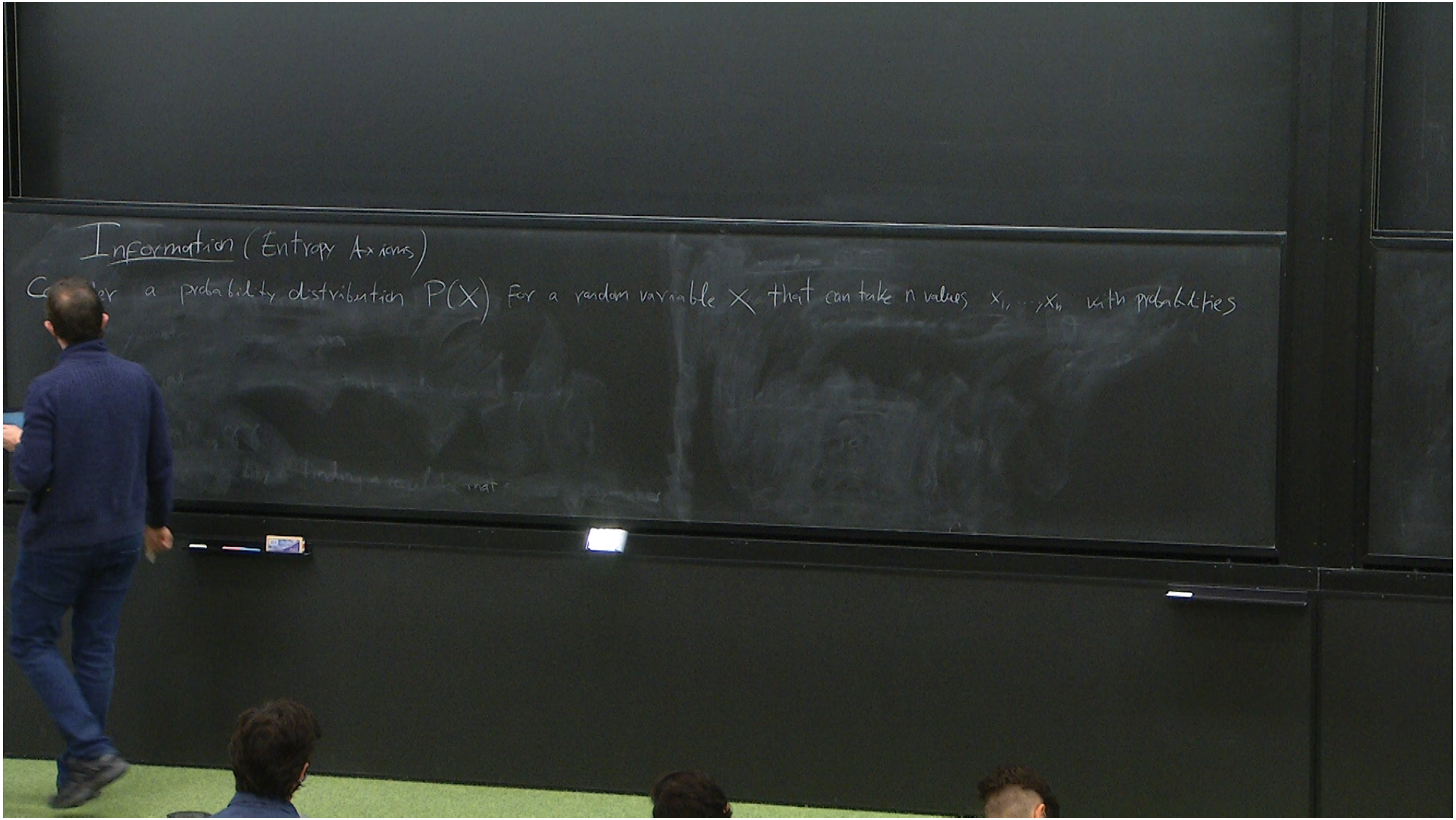
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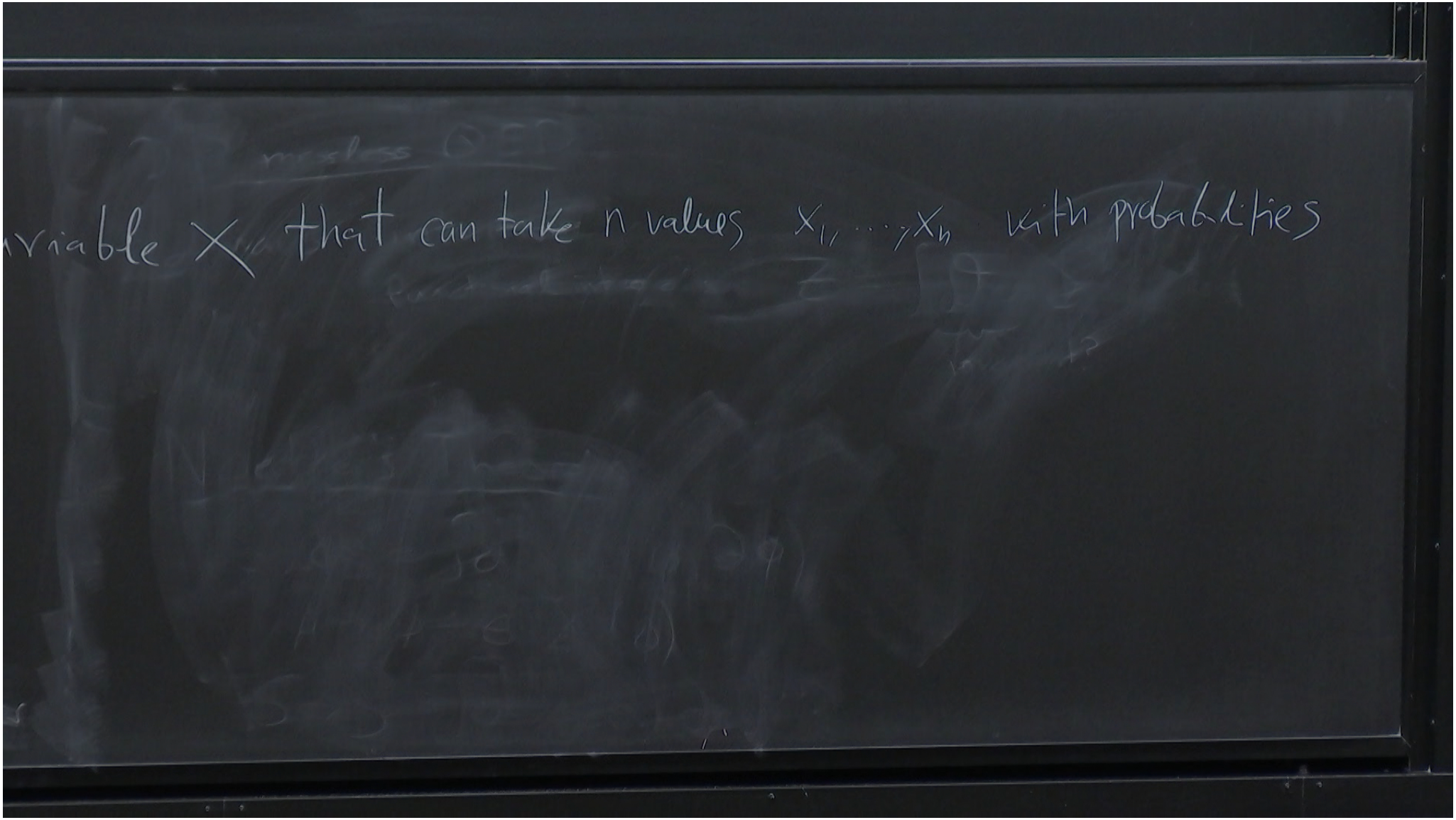
Speakers: Eduardo Martin-Martinez, Philippe Allard Guerin

Collection: Quantum Information 2021/2022

Date: February 28, 2022 - 10:15 AM

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Information (Entropy Axioms)
Consider a probability distribution P(X) for a random variable X that can take n values x1, ..., xn with probabilities


variable X that can take n values $x_1, \ldots, x_n$ with probabilities

Information (Entropy Axioms)

Consider a probability distribution $P(X)$ / For a random

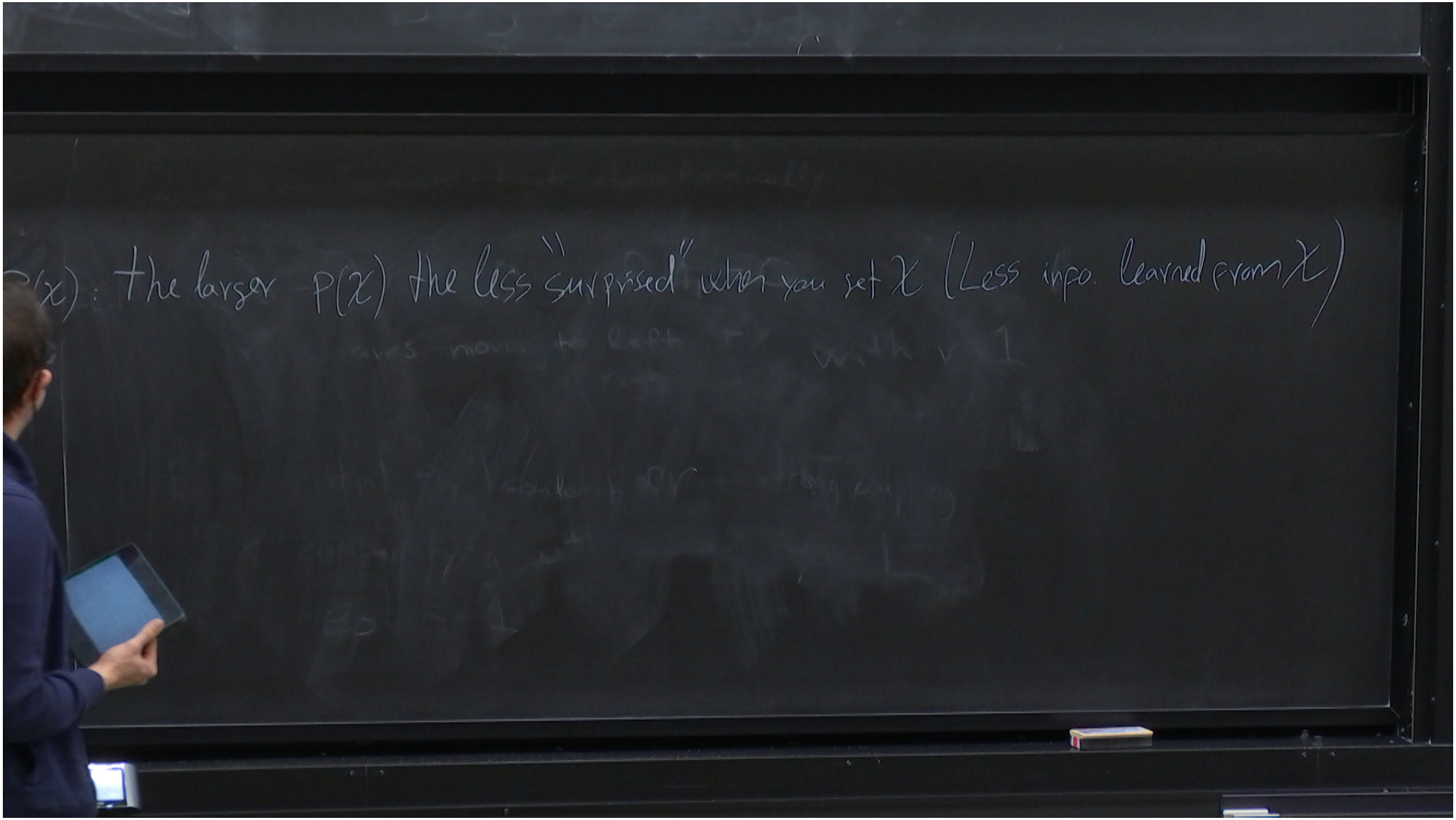
$P_1, \ldots, P_n$ $P_i := P(x_i)$ / $\sum_i P_i = 1$

Shannon: Measure of ignorance $H(X) := \sum_i P(x_i) S_u(P(x_i)) = \langle S_u$

χ is a set of outcomes out of evaluating a random variable

What is $S_u(P(x))$. Axioms

1. $S_u(x) = S_u(P(x))$ is monotonically decreasing with $P(x)$

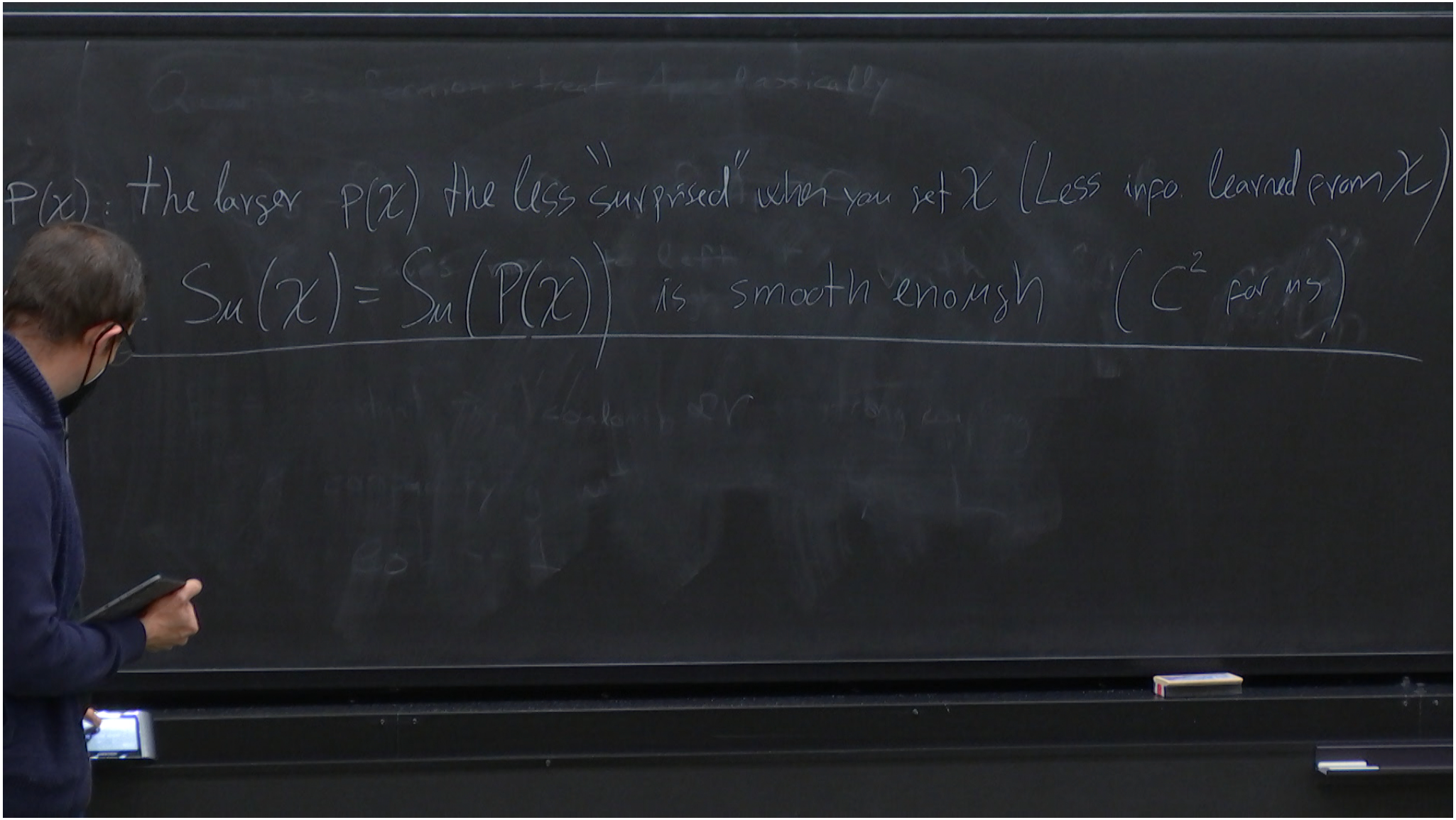

(x): the larger P(x) the less "surprised" when you see x (Less info. learned from x)

What is $S_u(P(\chi))$. Axioms

1. $S_u(\chi) = S_u(P(\chi))$ is monotonically decreasing with $P(\chi)$:
2. $S_u(\chi) \geq 0$
3. $S_u(\chi) = 0 \Leftrightarrow P(\chi) = 1$
4. $S_u(\{x_1, x_2\}) = S_u(P(\{x_1, x_2\})) = S_u(P_1 P_2) \stackrel{\downarrow}{=}$

What is $S_u(P(x))$. Axioms

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4. $S_u(\{x_1, x_2\}) = S_u(P(\{x_1, x_2\})) = S_u(P_1 P_2) \stackrel{!}{=} S_u(P_1) + S_u(P_2)$


P(x): the larger P(x) the less "surprised" when you see x (Less info. learned from x)
S_M(x) = S_M(P(x)) is smooth enough (C^2 for us)

$S'_u(P_1P_2) = S'_u(P_1) \xrightarrow{\partial_{P_2}} S'_u(\overset{P}{\widetilde{P_1P_2}}) + \overset{P}{\widetilde{P_1}}\,\overset{P}{\widetilde{P_2}}\, S''_u(\overset{P}{\widetilde{P_1P_2}}) = 0 \longrightarrow (*)$

$\frac{k}{P} \Rightarrow S_u(p) = k \log p + C$

Choice ↓

$= -\log_a P$ But I like to measure info in bits so $S_u(P) = -\log_2 P$

Postulates of QM

Postulate 2: Physical observables are represented by (possibly unbounded) self-adjoint operators. The result of a measurement* of any observable (represented by) $\hat{O}$ will always be an eigenvalue of $\hat{O}$

Postulate 3 (Born's rule): the probability of classically measure* the outcome a_i when measuring $|v\rangle$ is

$$P(A=a_i) = |\langle a_i|v\rangle|^2 \Rightarrow \langle \hat{A} \rangle_{|v\rangle} = \langle v|\hat{A}|v\rangle$$

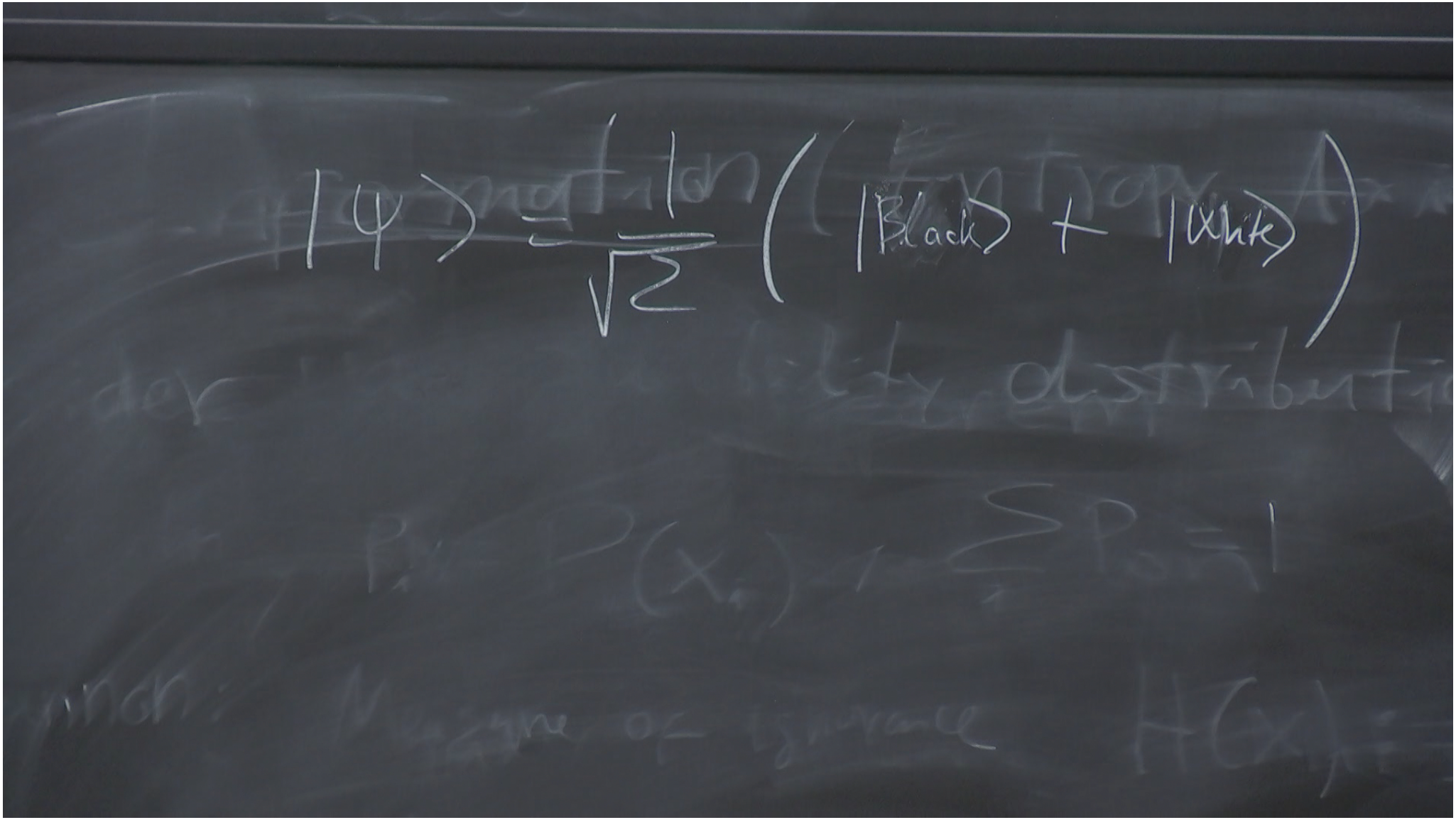
e represented by (possibly unbounded) self-adjoint operators on a Hilbert sp

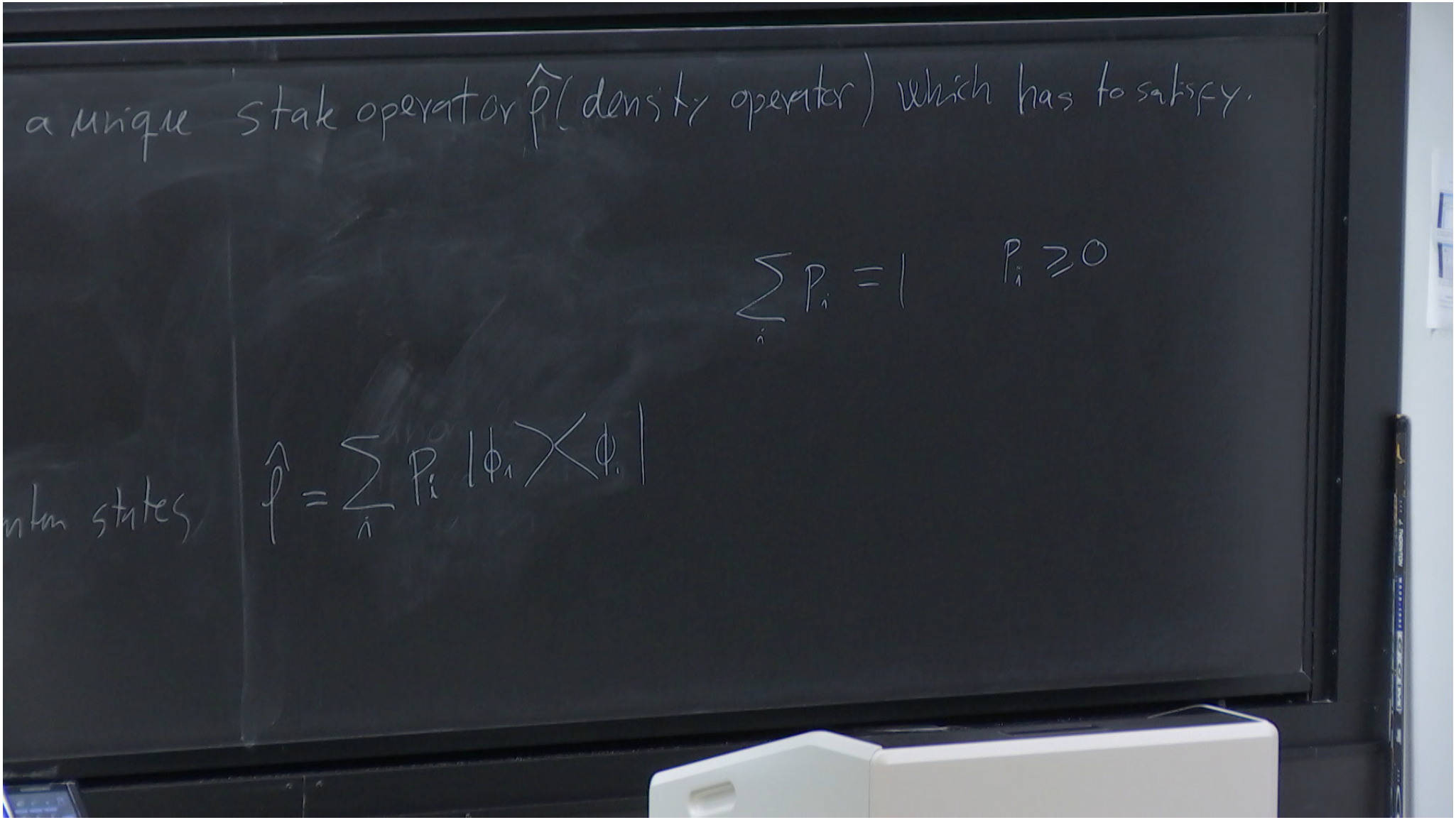
by) $\hat{O}$ will always be an eigenvalue of $\hat{O}$

of classically measure* the outcome a_i when measuring the observable (represented

$\hat{A}|a_i\rangle = a_i|a_i\rangle$

$\Rightarrow \langle\hat{A}\rangle_{|v\rangle} = \langle v|\hat{A}|v\rangle$


$|\psi\rangle = \frac{1}{\sqrt{2}}(|\text{Black}\rangle + |\text{White}\rangle)$


a unique stat operator $\hat{\rho}$ (density operator) which has to satisfy.
$\sum_i p_i = 1 \quad p_i \geq 0$
ntum states $\hat{\rho} = \sum_i p_i |\phi_i\rangle\langle\phi_i|$

$|e_j\rangle = \sum_i \sum_j p_i \langle e_j|\phi_i\rangle\langle\phi_i|\hat{\rho}|e_j\rangle = \sum_{i,j} p_i \langle\phi_i|\hat{\rho}|e_j\rangle\langle e_j|\phi_i\rangle$

$p_i = 1,\ p_{i\neq j} = 0 \Rightarrow \hat{\rho} = |\psi\rangle\langle\psi| \Rightarrow \hat{\rho}^2 = |\psi\rangle\langle\psi| \Rightarrow \mathrm{tr}\,\hat{\rho}^2 =$

Purity: $P(\hat{\rho}) := \mathrm{tr}\,\hat{\rho}^2$ $\quad \frac{1}{d} \leq P(\rho) \leq 1$

$$\langle \hat{O} \rangle_{\hat{\rho}} = tr(\hat{\rho}\hat{O}) = \sum_j \langle e_j | \hat{\rho}\hat{O} | e_j \rangle = \sum_i \sum_j P_i \langle$$

if the state is known with no ignorance: $P_i = 1$, $P_{i \neq j} = 0 \Rightarrow$

We call a state pure IFF $tr\hat{\rho}^2 = 1$. Purity: $P(\hat{\rho}$

$\hat{\rho} = \frac{1}{d}\mathbb{1}$ Maximally mixed state (state of zero knowledge)

Gibbs thermality since in spontaneous processes $\Delta S \geq 0$

Now, what is the state that maximizes Von Neumann entropy?

$$\frac{\partial}{\partial \hat{\rho}}\left[S - \beta\left(\mathrm{Tr}(\hat{\rho}\hat{H}) - E\right) - \mu\left[\mathrm{Tr}(\hat{\rho}) - 1\right]\right] = 0$$

$$\hat{\rho} = \sum_i p_i \hat{P}_i \Rightarrow S = -\sum_i n_i p_i \mathrm{Log}\, p_i$$

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$$\hat{\rho} = \sum_i p_i \hat{P}_i \Rightarrow S = -\sum_i n_i p_i \mathrm{Log}\, p_i, \quad \sum_i n_i E_i p_i = E$$

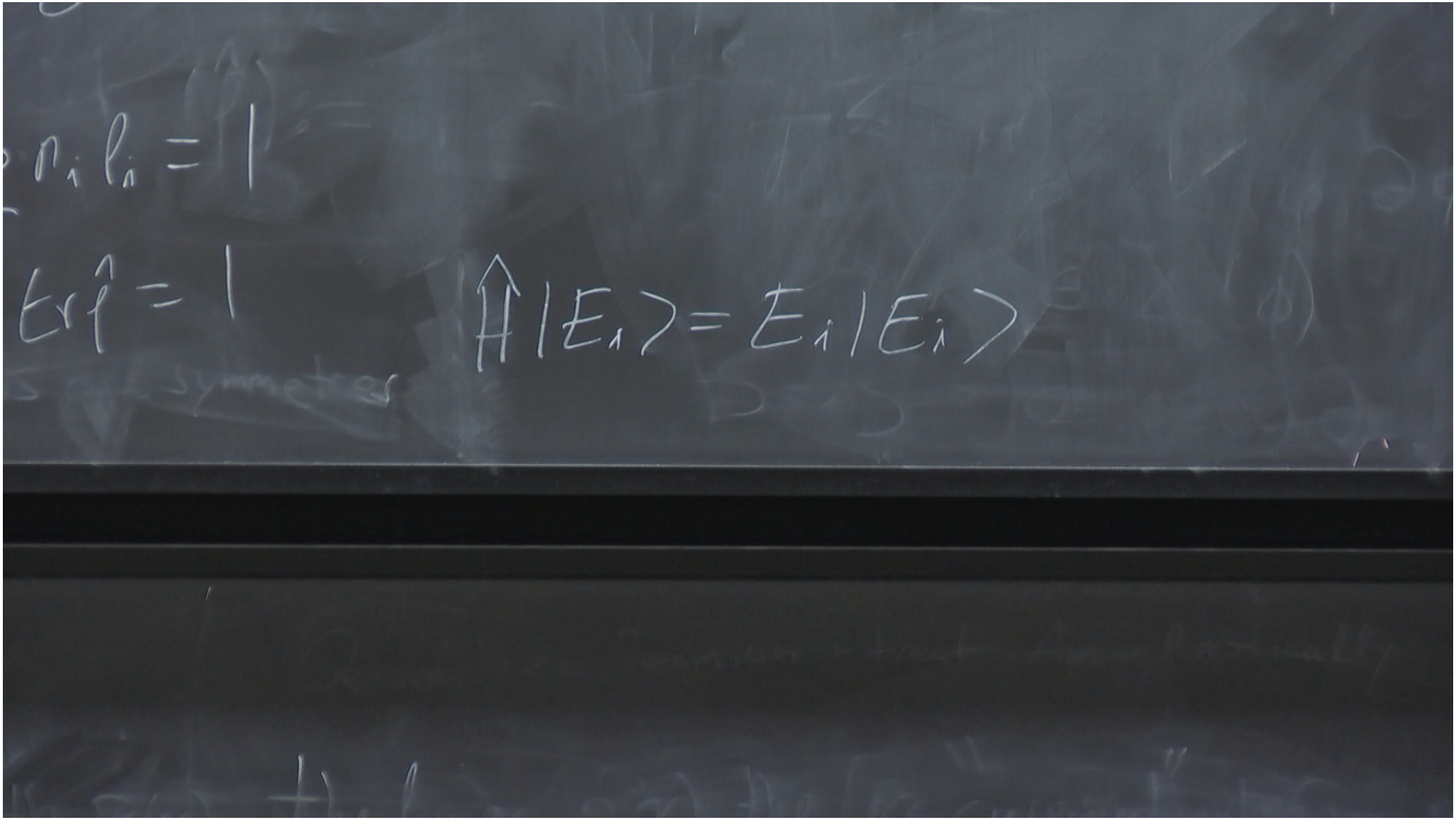
$\langle \hat{A} \rangle$?

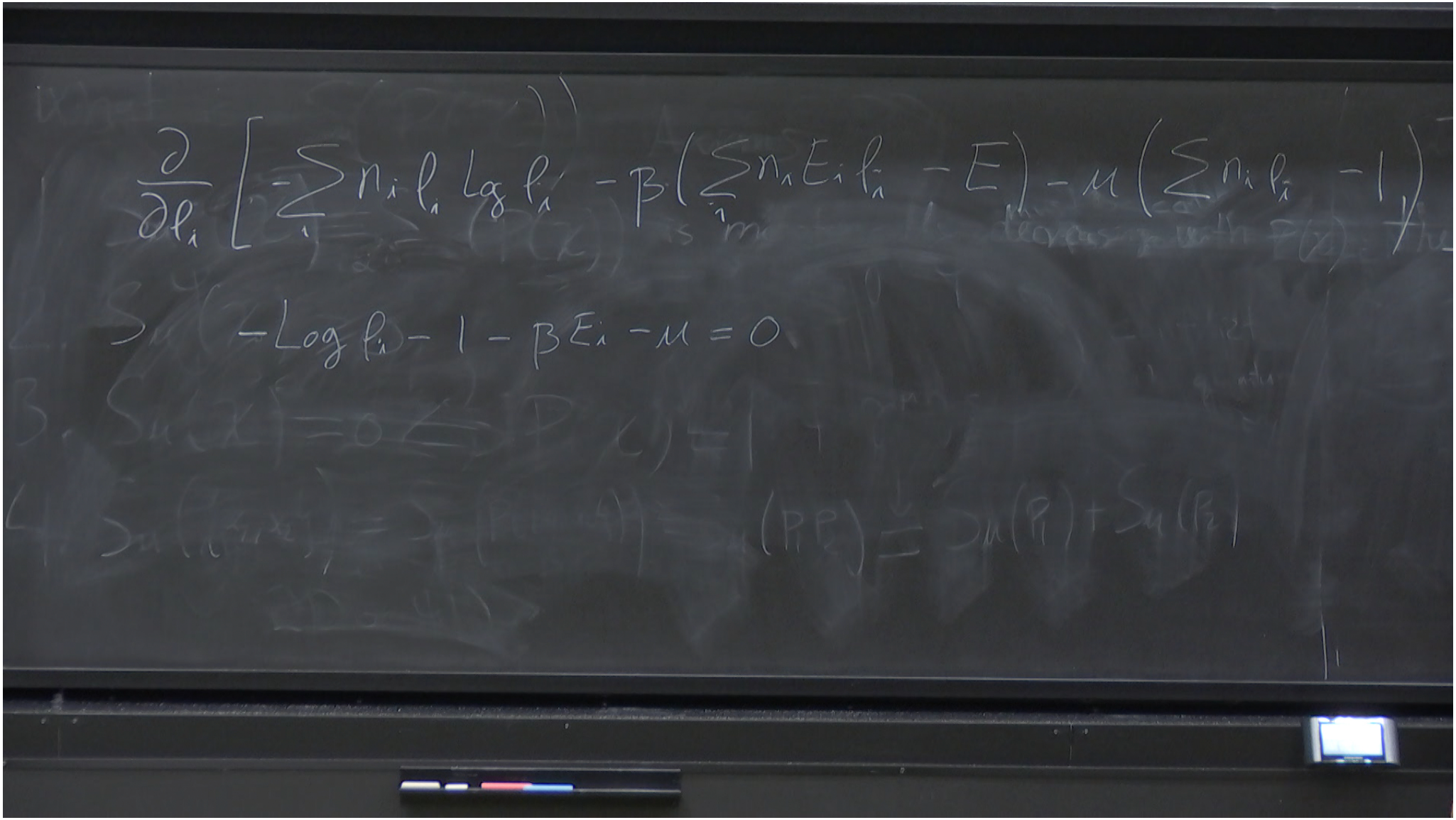
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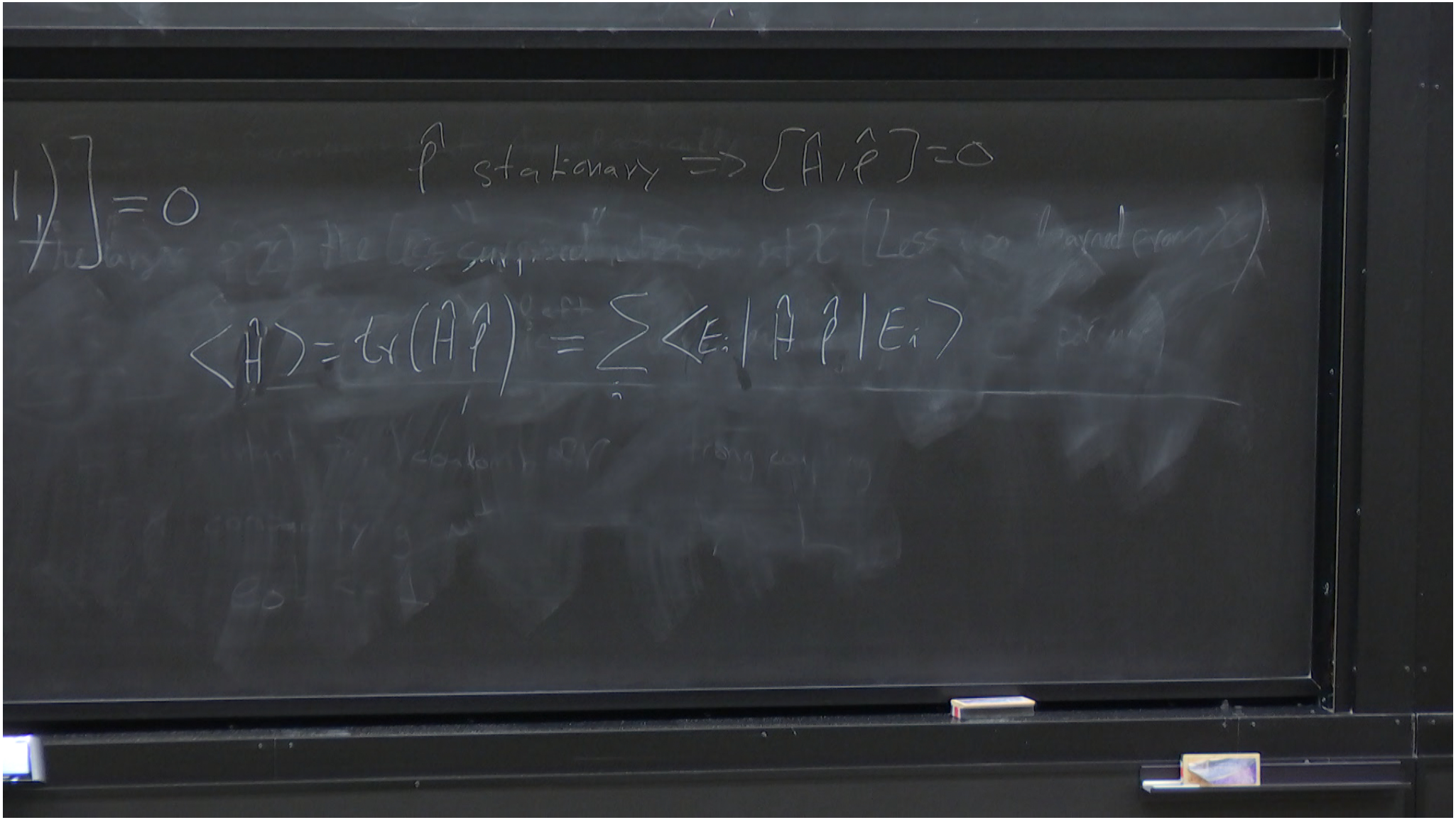
and thermalization is spontaneous $\Rightarrow$ Thermal states sit at max(S)

(under $\langle \hat{H} \rangle_{\hat{\rho}} = E$, $\mathrm{tr}\,\hat{\rho} = 1$)

p_i are the eigenvalues of $\hat{\rho}$ with multiplicity n_i


$\sum_i p_i l_i = 1$
$\mathrm{Tr}\,\hat{\rho} = 1$
$\hat{A}|E_i\rangle = E_i|E_i\rangle$


$\frac{\partial}{\partial p_i}\left[-\sum_i n_i p_i \log p_i - \beta\left(\sum_i n_i E_i p_i - E\right) - \mu\left(\sum n_i p_i - 1\right)\right]$
$-\log p_i - 1 - \beta E_i - \mu = 0$

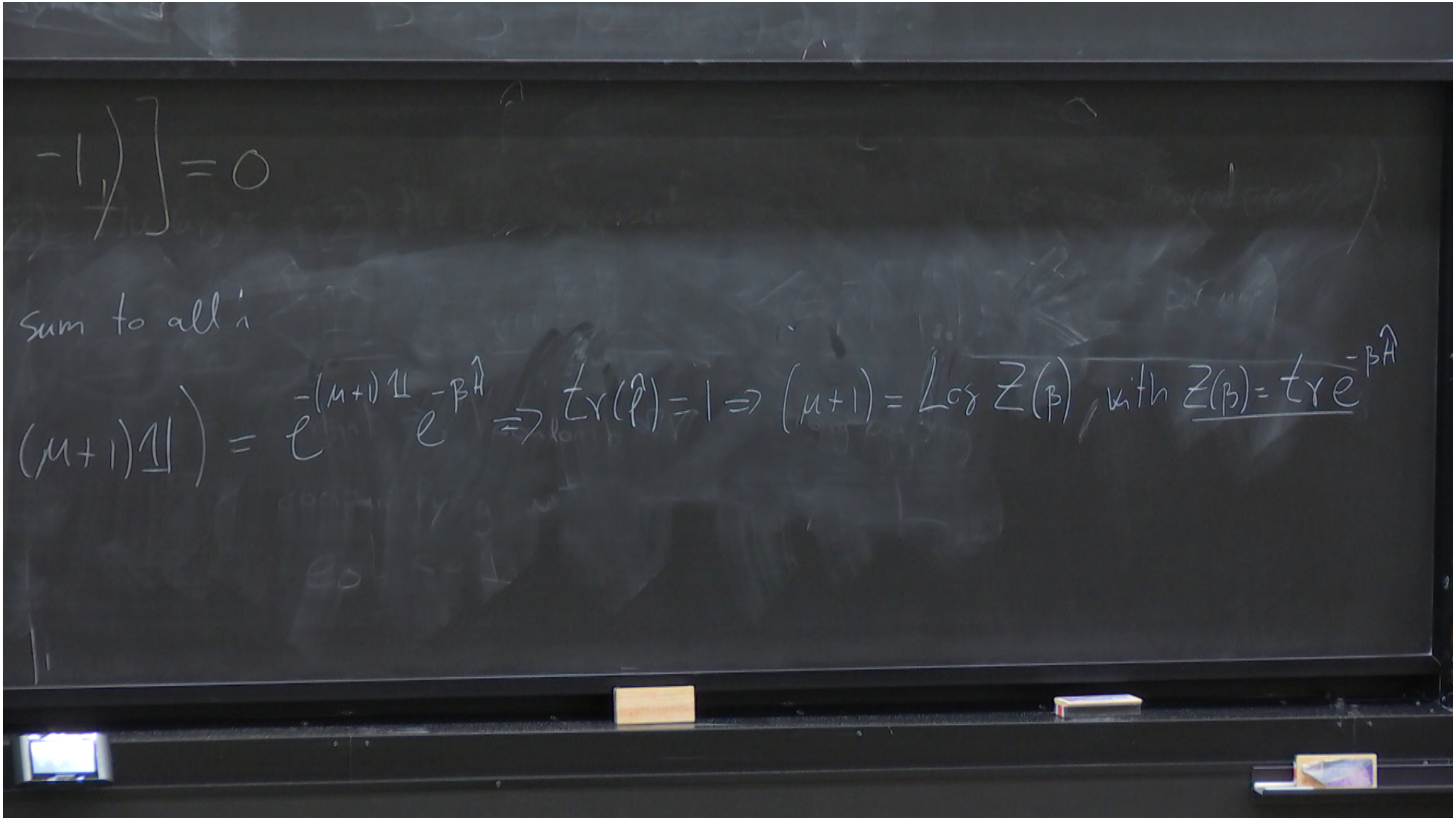

$\hat{\rho}$ stationary $\Rightarrow [\hat{A}, \hat{\rho}] = 0$
$\langle \hat{H} \rangle = \mathrm{tr}(\hat{H}\hat{\rho}) = \sum \langle E_i | \hat{H}\hat{\rho} | E_i \rangle$

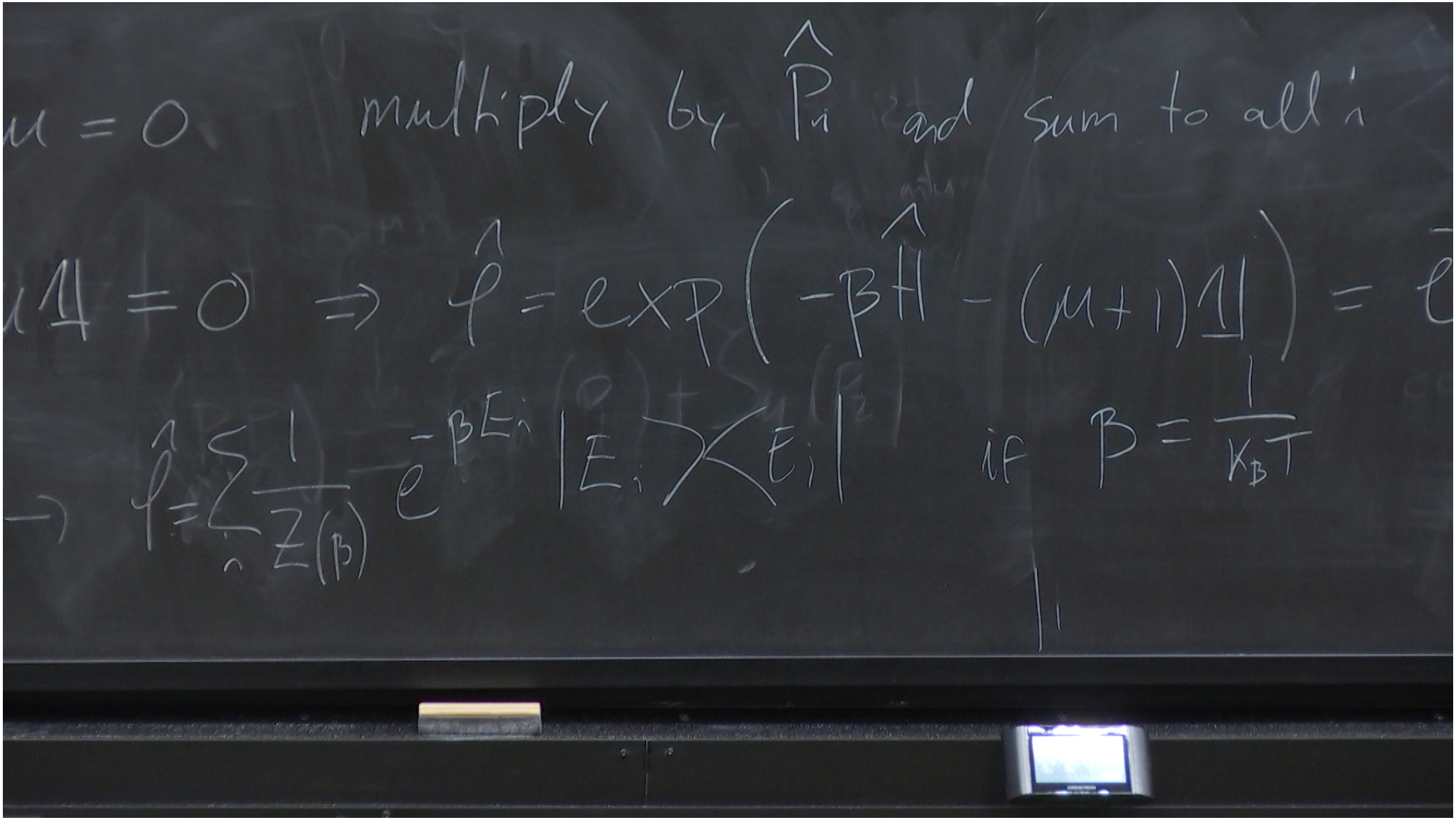
$$\frac{\partial}{\partial p_i}\left[-\sum_i n_i p_i \log p_i - \beta\left(\sum_i n_i E_i p_i - E\right) - \mu\left(\sum n_i p_i -\right.\right.$$

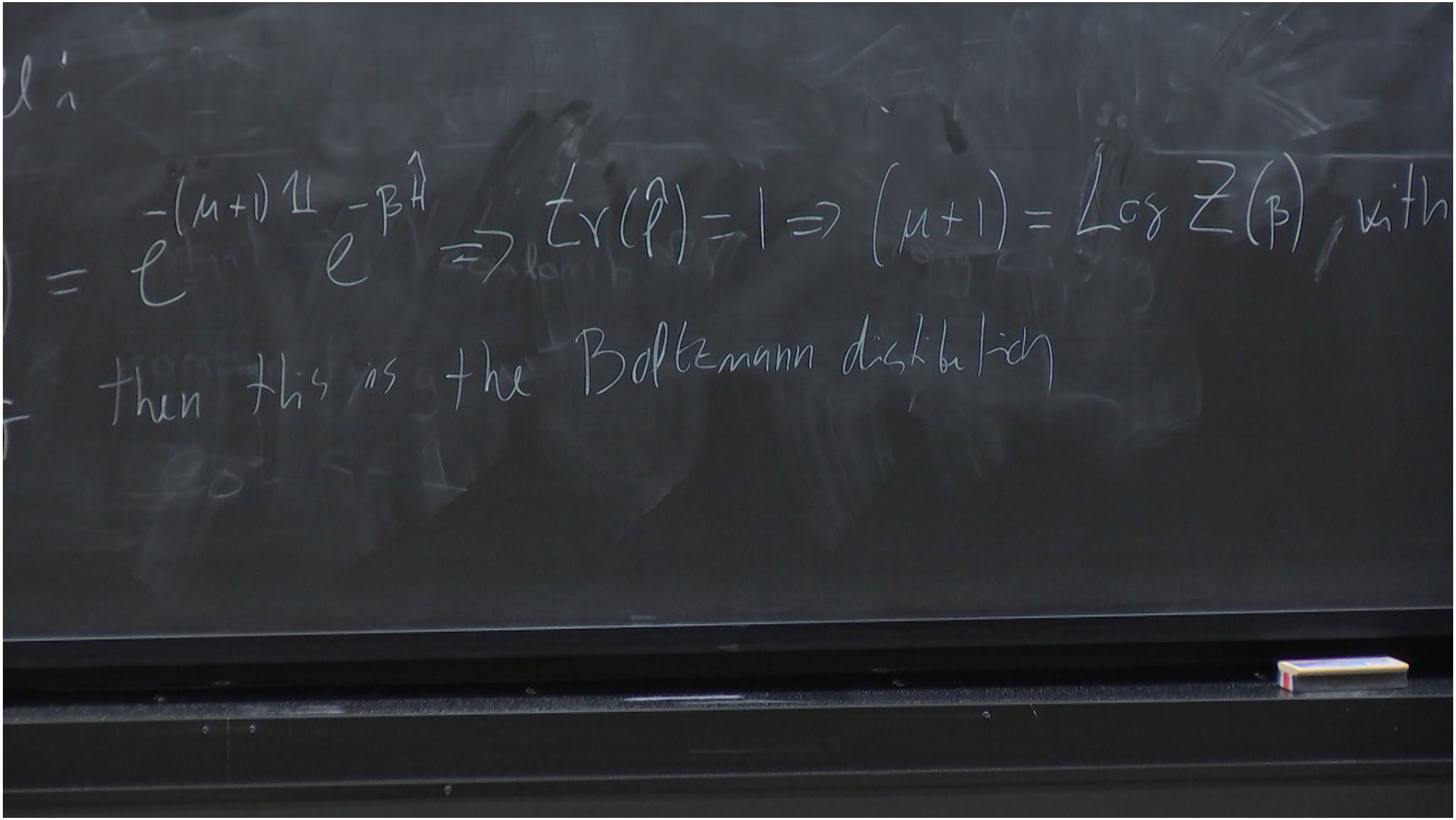
$$-\log p_i - 1 - \beta E_i - \mu = 0 \qquad \text{multiply by } \hat{P}_i \text{ and sum}$$

$$-\log \hat{p}_i - \mathbb{1} - \beta \hat{H} - \mu \mathbb{1} = 0 \Rightarrow \hat{\rho} = \exp\left(-\beta \hat{H} - (\mu\right.$$

$$Z(\beta) = \mathrm{tr}\, e^{-\beta \hat{H}} \qquad \hat{\rho} = \frac{1}{Z(\beta)} e^{-\beta \hat{H}}$$

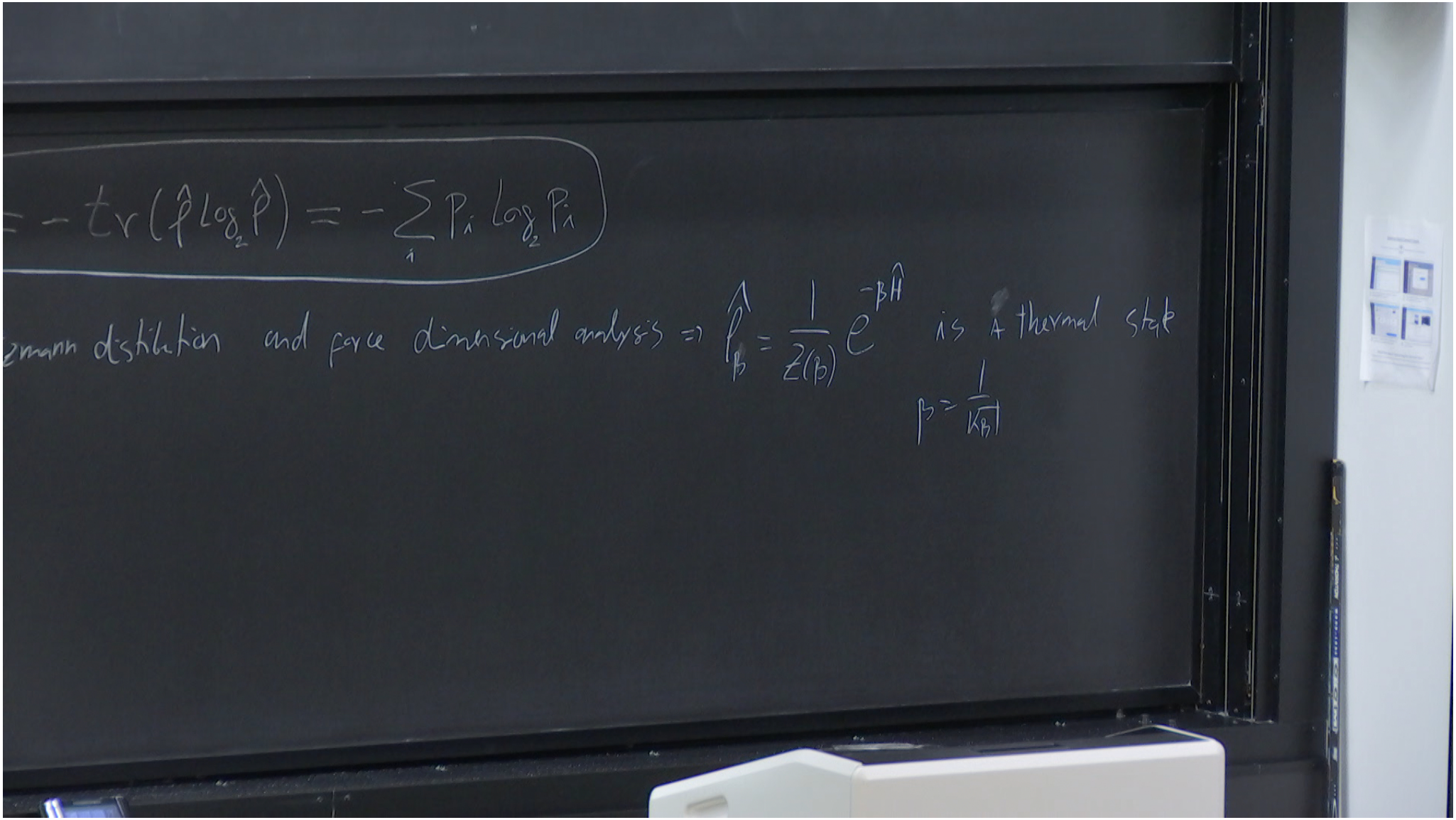

$-1_i)\Big] = 0$
sum to all i
$(\mu+1)\mathbb{1}) = e^{-(\mu+1)\mathbb{1}}\, e^{-\beta\hat{H}} \Rightarrow Tr(\rho) = 1 \Rightarrow (\mu+1) = Log\, Z(\beta)$, with $Z(\beta) = Tr\, e^{-\beta\hat{H}}$


multiply by $\hat{P}_i$ and sum to all i
$\mu = 0$
$\mu \mathbb{1} = 0 \Rightarrow \hat{\rho} = \exp\left(-\beta \hat{H} - (\mu+1)\mathbb{1}\right) = e$
$\rightarrow \hat{\rho} = \sum_i \frac{1}{Z(\beta)} e^{-\beta E_i} |E_i\rangle\langle E_i|$ if $\beta = \frac{1}{k_B T}$


$= e^{-(\mu+1)\mathbb{1}} e^{-\beta\hat{H}} \Rightarrow Tr(\hat{\rho}) = 1 \Rightarrow (\mu+1) = \log Z(\beta)$, with
then this is the Boltzmann distribution

Von Neumann Entropy: the Shannon entropy of $\hat{\rho}$: $S := -$

If you compare $\hat{\rho}$ that maximizes S with the $\hat{\rho}$ given by the Boltzman


$= -\mathrm{tr}(\hat{\rho}\log_2\hat{\rho}) = -\sum_i p_i \log_2 p_i$
zmann distribution and force dimensional analysis $\Rightarrow \hat{\rho}_\beta = \frac{1}{Z(\beta)} e^{-\beta\hat{H}}$ is a thermal state
$\beta = \frac{1}{k_B T}$

$$\sum_i p_i \log_2 p_i$$

dimensional analysis $\Rightarrow$ $\hat{\rho}_\beta = \frac{1}{Z(\beta)} e^{-\beta \hat{H}}$ is a thermal state

$$\beta = \frac{1}{k_B T}$$

Von Neumann Entropy: the Shannon entropy of $\hat{\rho}$

If you compare $\hat{\rho}$ that maximizes S with the $\hat{\rho}$ given by

$$\hat{\rho}_\beta := \frac{1}{Z(\beta)} e^{-\beta \hat{H}} \qquad \lim_{\beta \to 0} \hat{\rho}_\beta =$$

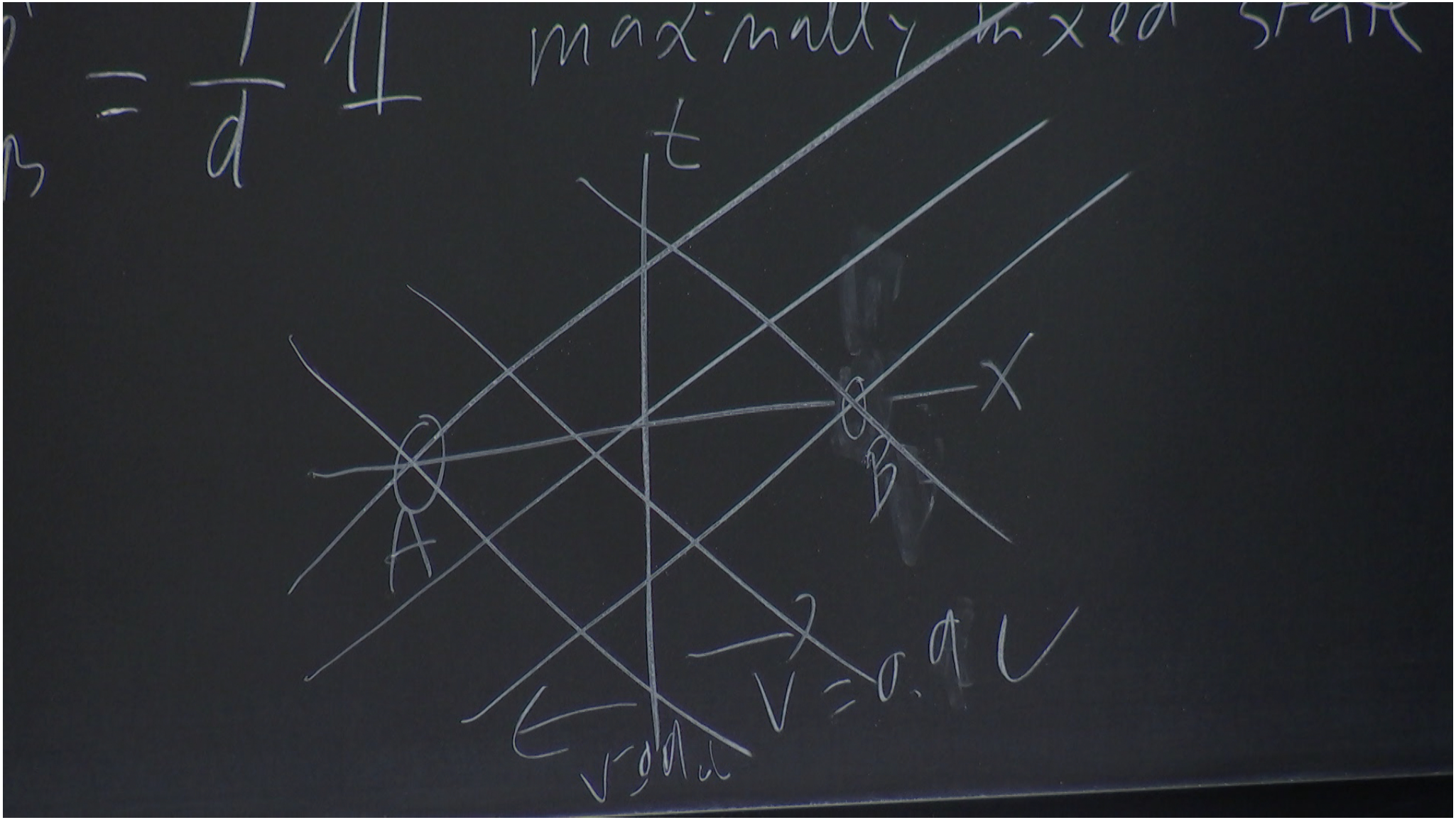
$$\lim_{T \to 0} \hat{\rho}_\beta = \lim_{\beta \to \infty} \hat{\rho}_\beta = |E_0\rangle\langle E_0|$$

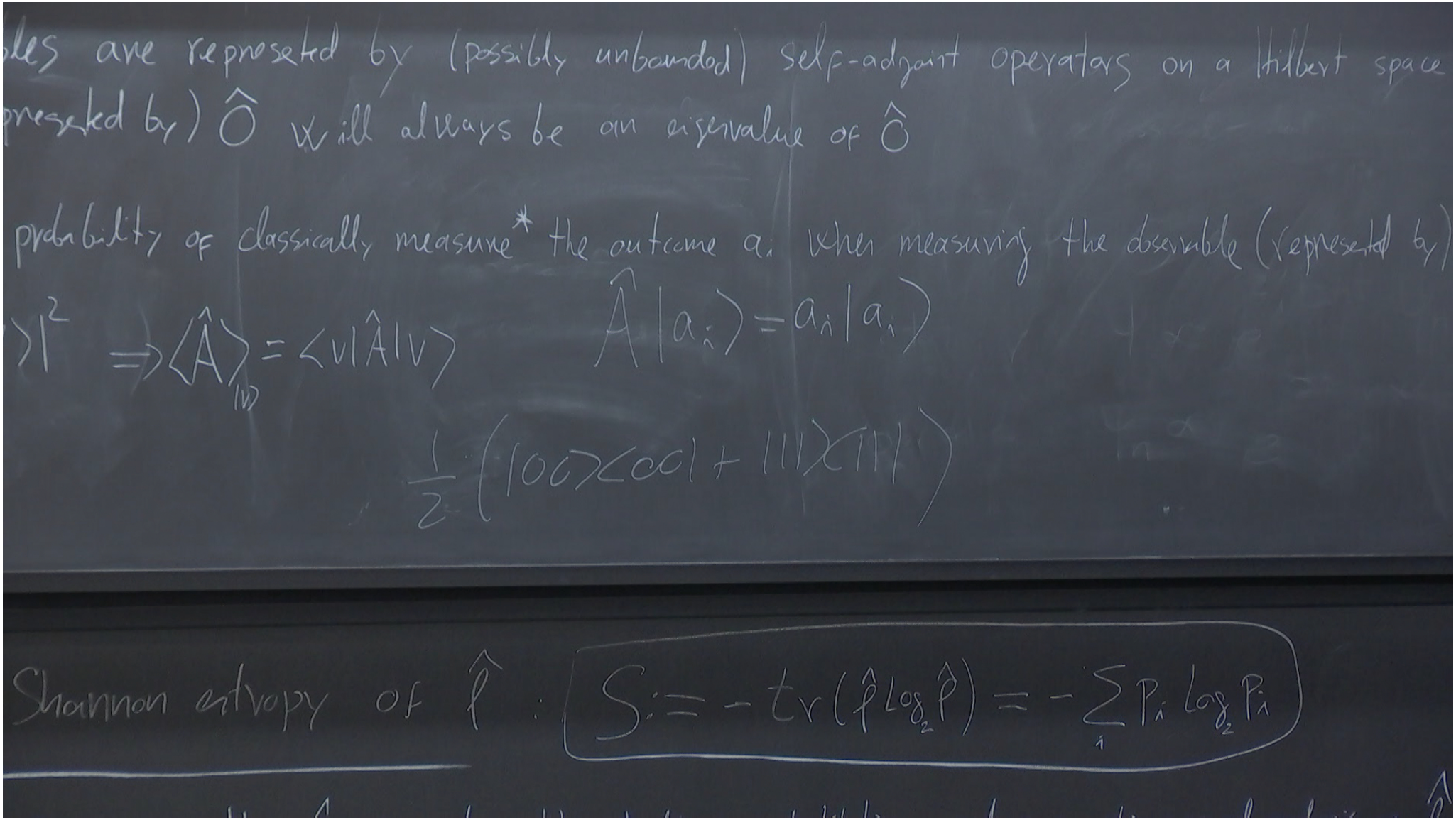
y: the Shannon entropy of $\hat{\rho}$: $S := -\mathrm{tr}(\hat{\rho}\log_2\hat{\rho})$ =

the maximizes S with the $\hat{\rho}$ given by the Boltzmann distibution and

$\hat{\rho}_\beta = \lim_{t\to\infty} \hat{\rho}_\beta = \frac{1}{d}\mathbb{1}$ maximally mixed state full ignorance.

$|E_0\rangle\langle E_0|$


t
x
A
B
V = 0.9 c


bles are represented by (possibly unbounded) self-adjoint operators on a Hilbert space
presented by) $\hat{O}$ will always be an eigenvalue of $\hat{O}$
probability of classically measure* the outcome a_i when measuring the observable (represented by)
$\hat{A}|a_i\rangle = a_i|a_i\rangle$
$|\cdot|^2 \Rightarrow \langle\hat{A}\rangle_{|v\rangle} = \langle v|\hat{A}|v\rangle$
$\frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$
Shannon entropy of $\hat{\rho}$: $S := -\mathrm{tr}(\hat{\rho}\log_2\hat{\rho}) = -\sum_i p_i \log_2 p_i$