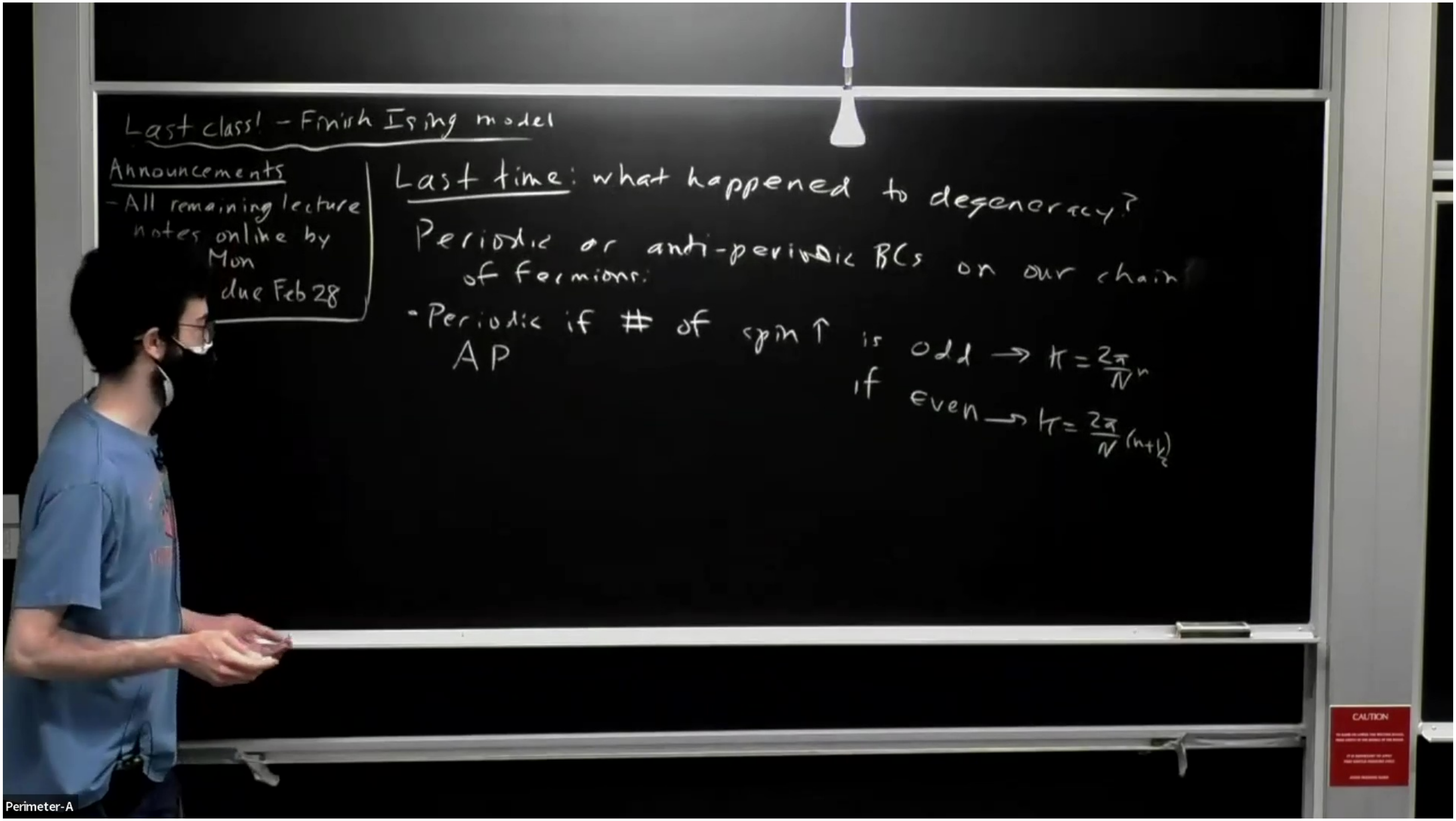
Title: PSI Lecture - Condensed Matter - Lecture 15

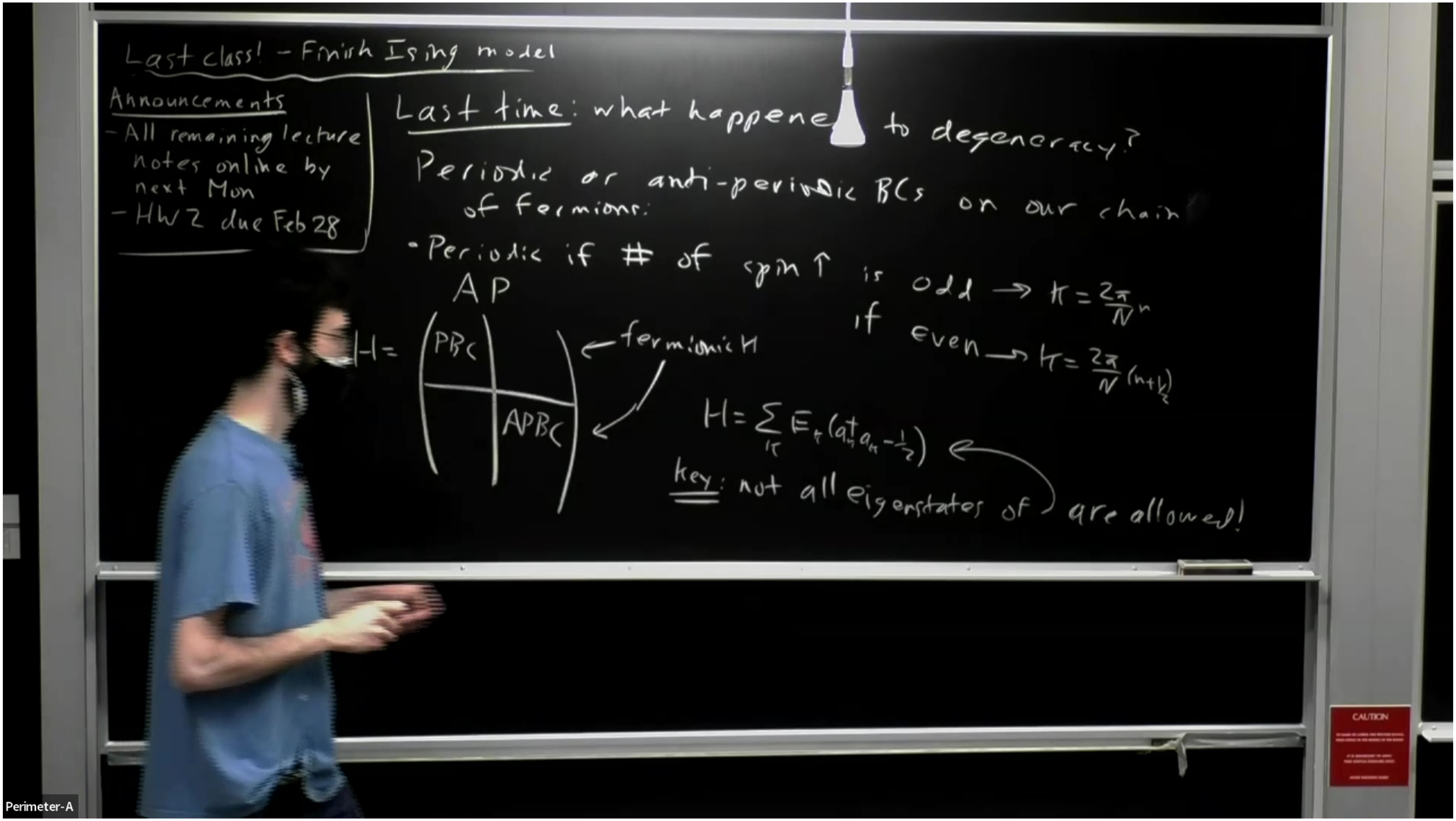
Speakers: Aaron Szasz

Collection: PSI Lecture - Condensed Matter

Date: February 10, 2022 - 9:00 AM

URL: https://pirsa.org/22020012


Last class! - Finish Ising model
Announcements
- All remaining lecture notes online by Mon
due Feb 28
Last time: what happened to degeneracy?
Periodic or anti-periodic BCs on our chain of fermions:
- Periodic if # of spin ↑ is odd → $k = \frac{2\pi}{N} n$
AP if even → $k = \frac{2\pi}{N}(n+\frac{1}{2})$
Perimeter-A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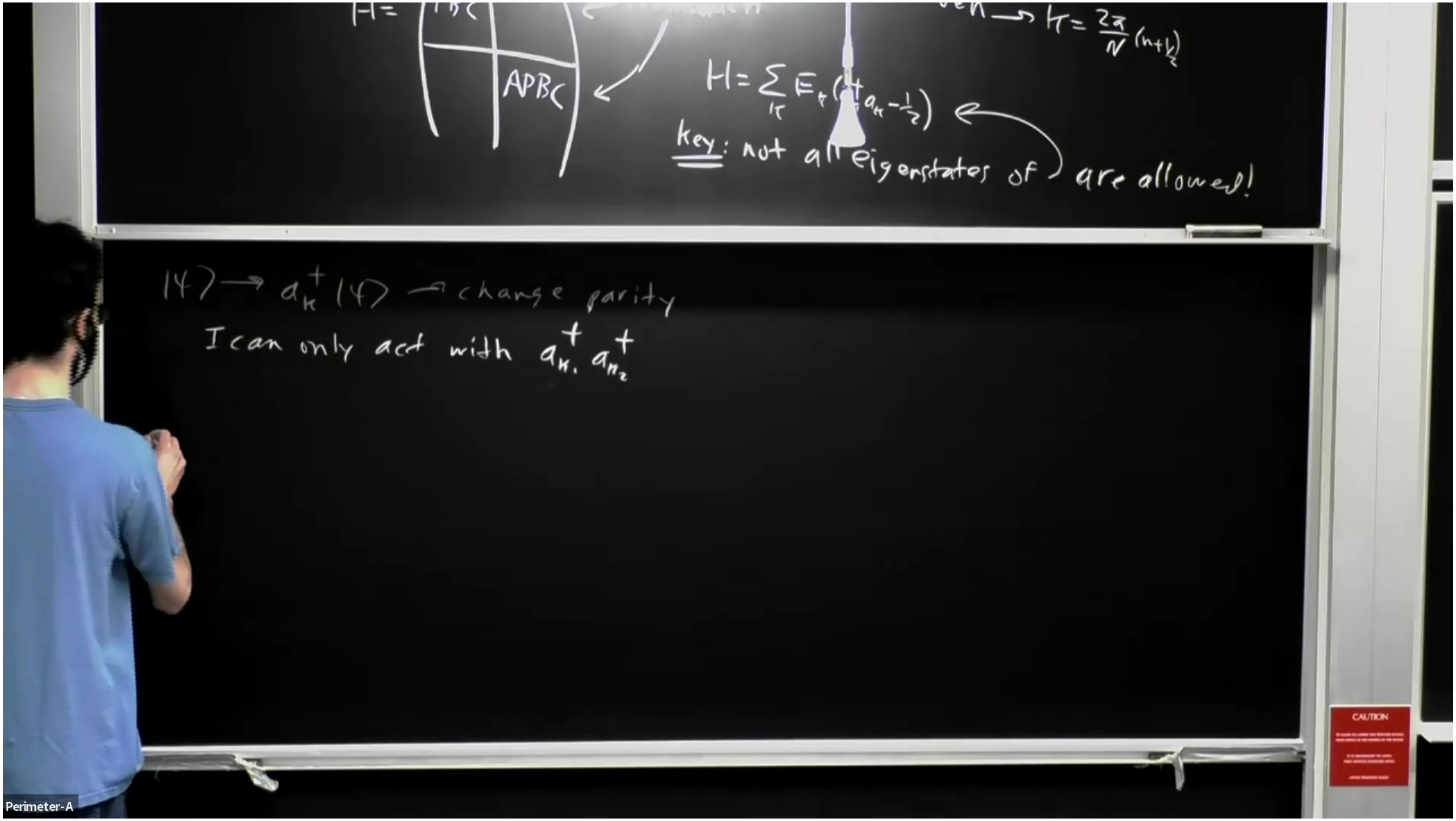
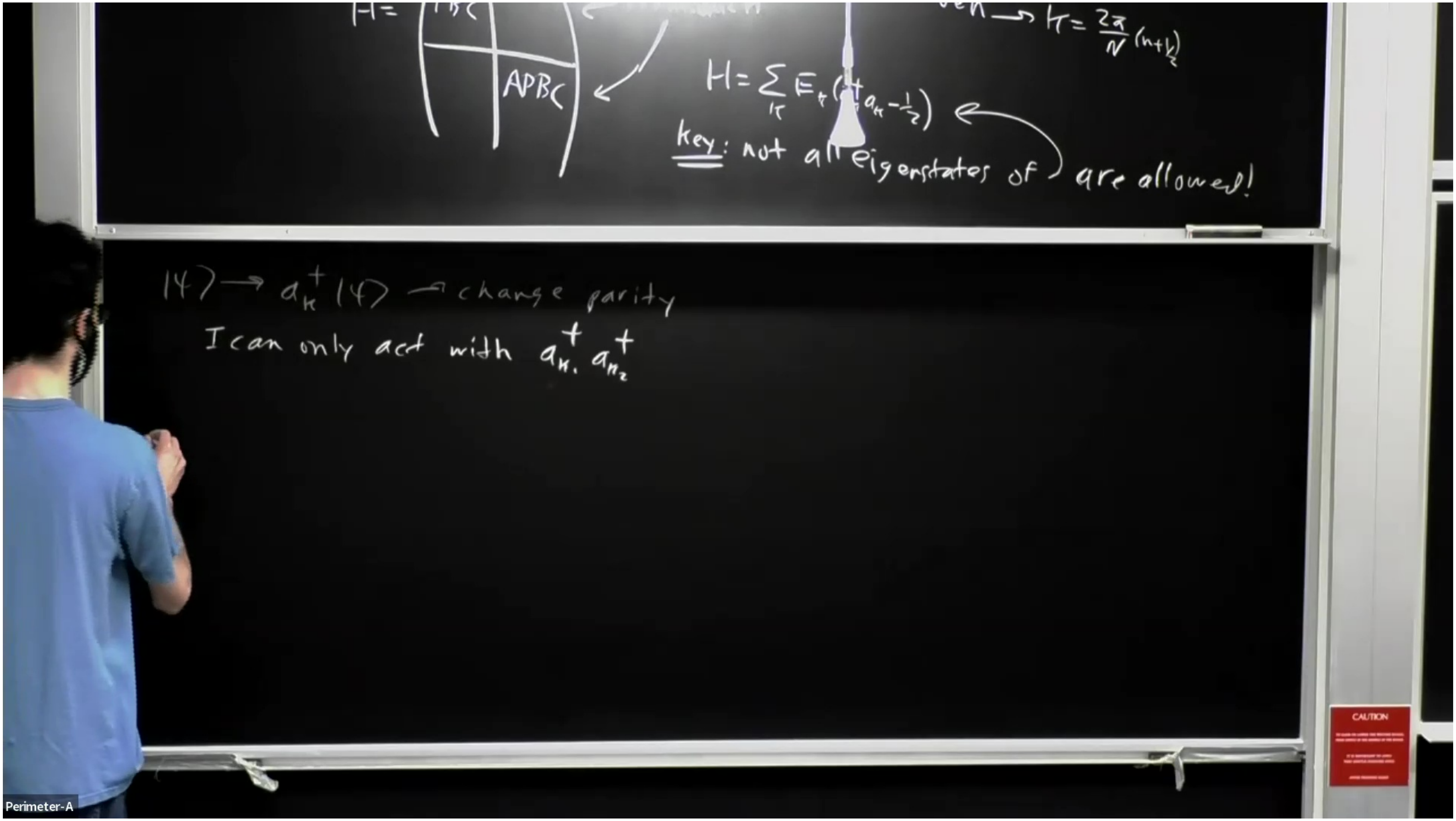
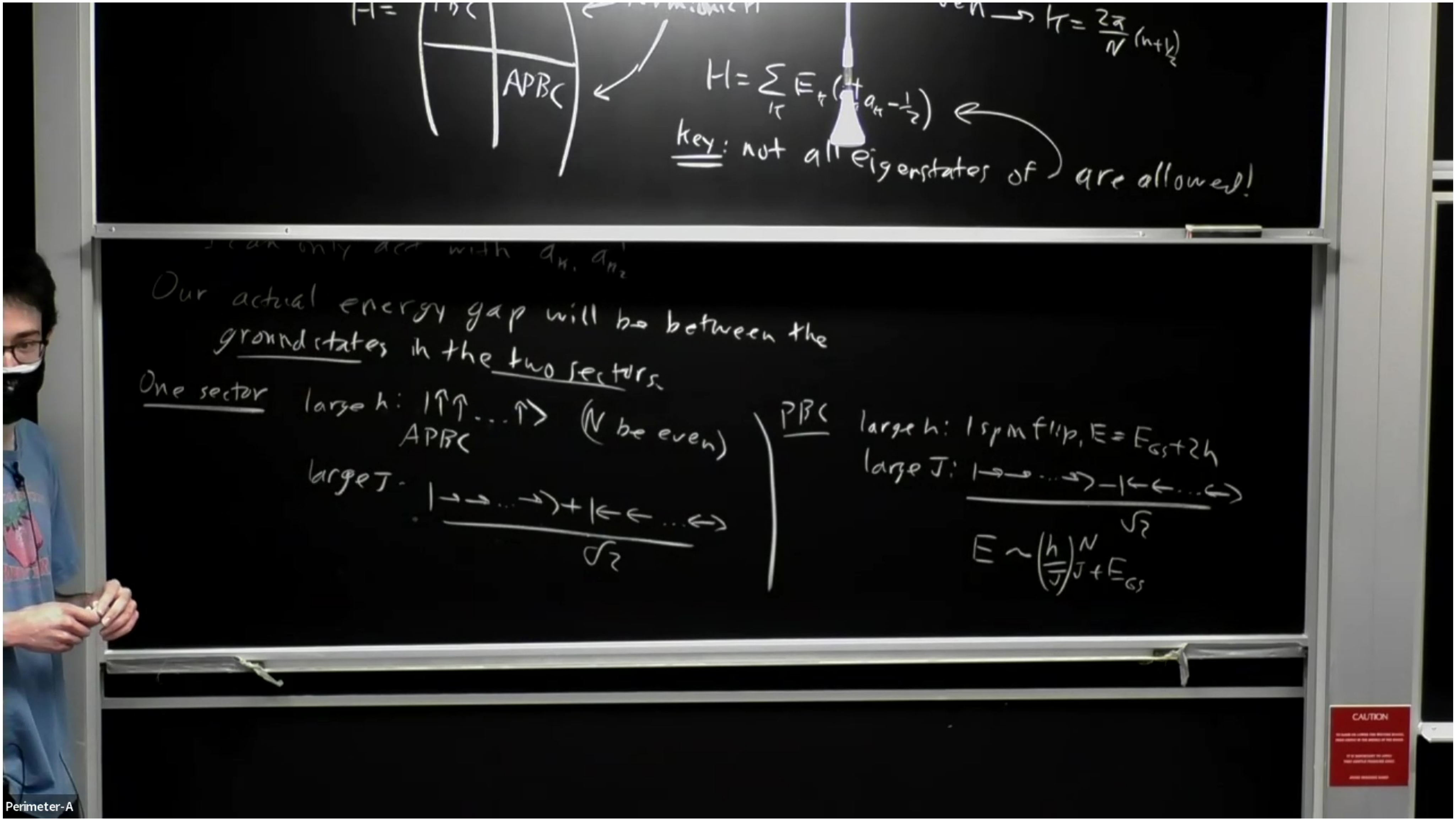
• Periodic if # of spin ↑ is odd → $k = \frac{2\pi}{N} n$
if even → $k = \frac{2\pi}{N}(n+\frac{1}{2})$
AP
$H = \begin{pmatrix} PBC & \\ & APBC \end{pmatrix}$
← fermionic H
$H = \sum_k E_k (a_k^\dagger a_k - \frac{1}{2})$
key: not all eigenstates of are allowed!
Perimeter-A
CAUTION



APBC
$k = \frac{2\pi}{N}(n+\frac{1}{2})$
key: not all eigenstates of are allowed!
$|\psi\rangle \to a_k^\dagger |\psi\rangle$ change parity
I can only act with $a_{k_1}^\dagger a_{k_2}^\dagger$
Perimeter-A


APBC
k = 2π/N (n+1/2)
H = Σ_k E_k (a_k† a_k − 1/2)
key: not all eigenstates of are allowed!
|ψ⟩ → a_k† |ψ⟩ → change parity
I can only act with a_k1† a_k2†
Perimeter-A
CAUTION


APBC
$\to k = \frac{2\pi}{N}(n+\frac{1}{2})$
$H = \sum_k E_k (a_k^\dagger a_k - \frac{1}{2})$
key: not all eigenstates of are allowed!
can only act with a_{k_1}, a_{k_2}
Our actual energy gap will be between the groundstates in the two sectors.
One sector
large h: $|\uparrow\uparrow\ldots\uparrow\rangle$ (N be even)
APBC
large J: $\frac{|\rightarrow\rightarrow\ldots\rightarrow\rangle + |\leftarrow\leftarrow\ldots\leftarrow\rangle}{\sqrt{2}}$
PBC
large h: 1 spin flip, $E = E_{GS} + 2h$
large J: $\frac{|\rightarrow\rightarrow\ldots\rightarrow\rangle - |\leftarrow\leftarrow\ldots\leftarrow\rangle}{\sqrt{2}}$
$E \sim \left(\frac{h}{J}\right)^N J + E_{GS}$
Perimeter-A
CAUTION

$H = $ PBC / APBC

$\to k = \frac{2\pi}{N}(n+\frac{1}{2})$

$H = \sum_k E_k (a_k^\dagger a_k - \frac{1}{2})$

key: not all eigenstates of are allowed!

can only act with $a_{k_1} a_{k_2}$

Our actual energy gap will be between the groundstates in the two sectors.

One sector large h: $|\uparrow\uparrow\ldots\uparrow\rangle$ (N be even) APBC

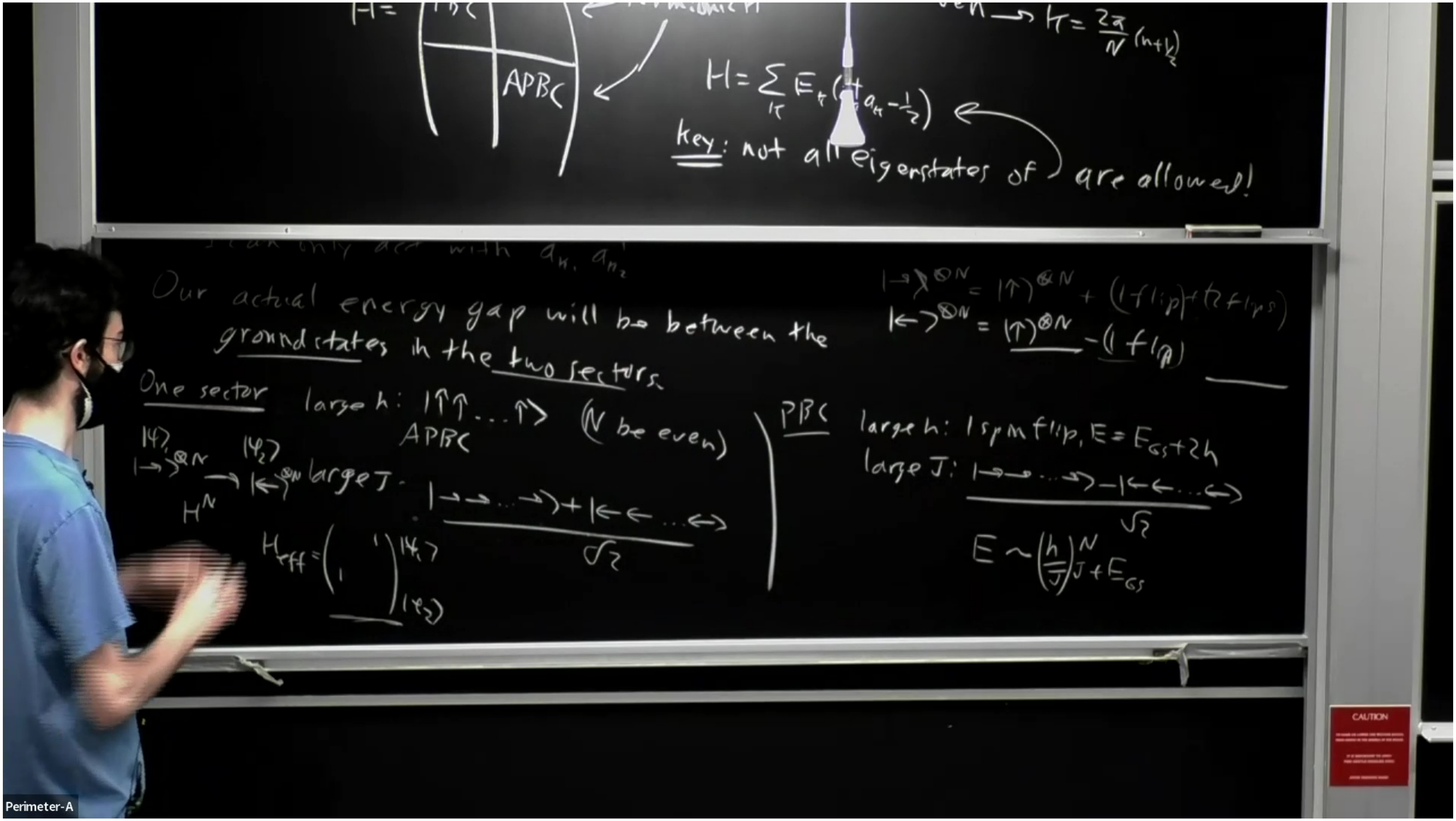
$|\rightarrow\rangle^{\otimes N} \to |\leftarrow\rangle^{\otimes N}$ large J. $\frac{|\rightarrow\rightarrow\ldots\rightarrow\rangle + |\leftarrow\leftarrow\ldots\leftarrow\rangle}{\sqrt{2}}$

H^N

PBC large h: 1 spin flip, $E = E_{GS} + 2h$

large J: $\frac{|\rightarrow\rightarrow\ldots\rightarrow\rangle - |\leftarrow\leftarrow\ldots\leftarrow\rangle}{\sqrt{2}}$

$E \sim \left(\frac{h}{J}\right)^N J + E_{GS}$


APBC
H = ∑ E(a a - 1/2)
key: not all eigenstates of are allowed!
Our actual energy gap will be between the
groundstates in the two sectors.
One sector
large h: |↑↑...↑⟩ (N be even)
APBC
large J
H_eff
PBC
large h: 1 spin flip, E = E_GS + 2h
large J
E ~ (h/J)^N J + E_GS
Perimeter-A
CAUTION

Last class! - Finish Ising model

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Translate to fermions:

$|\uparrow\rangle^{\otimes N} \to |0\rangle_c$ for $c^\dagger$

GS is $\left(\prod_k a_k^\dagger\right)|0\rangle_a$, $a_k|0\rangle_a = 0 \ \forall k$

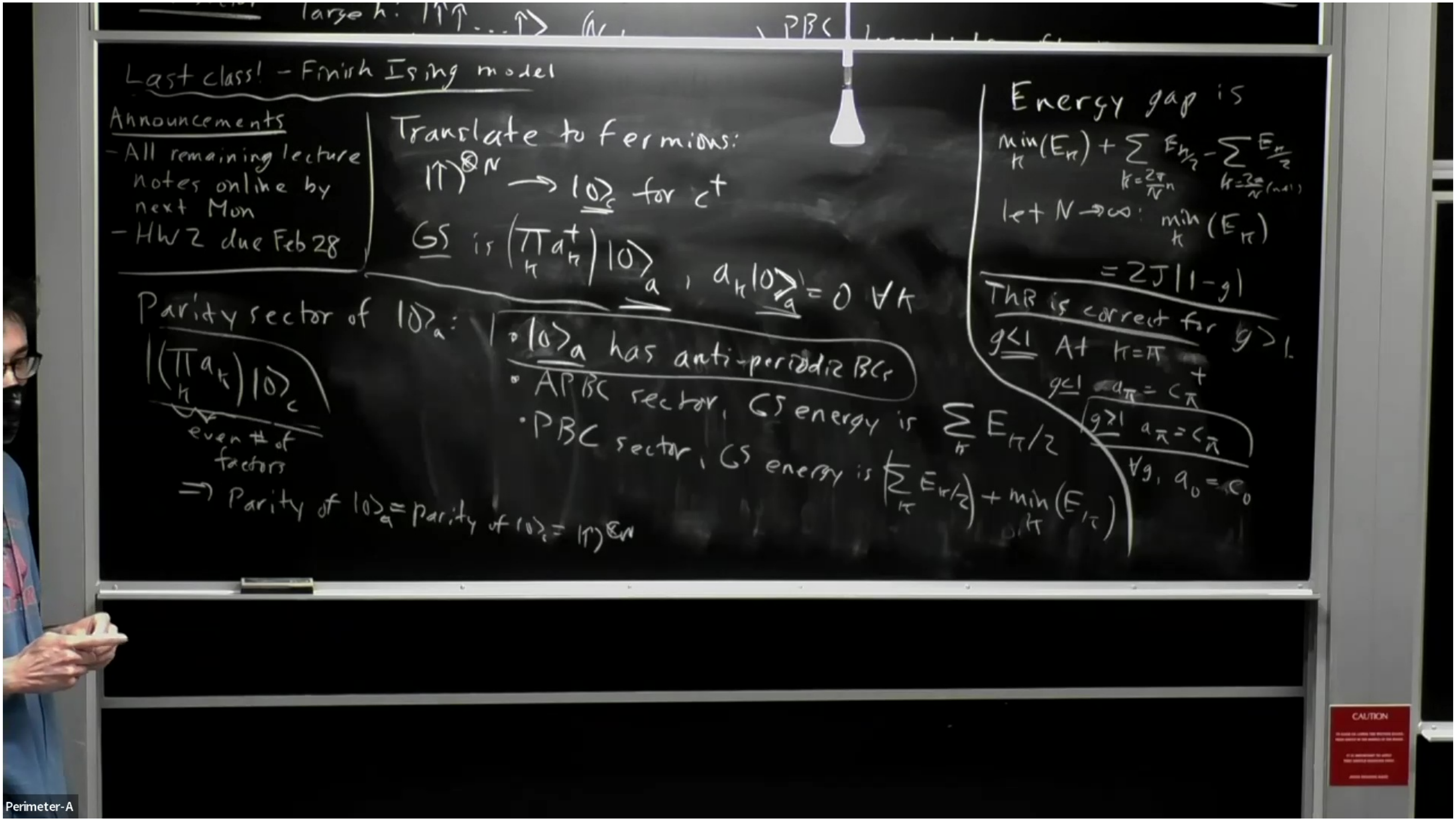
Parity sector of $|0\rangle_a$:

$\underbrace{\left(\prod_k a_k\right)}_{\text{even \# of factors}}|0\rangle_c$

$\Rightarrow$ Parity of $|0\rangle_a$ = parity of $|0\rangle_c = |\uparrow\rangle^{\otimes N}$

- $|0\rangle_a$ has anti-periodic BCs
- APBC sector, GS energy is $\sum_k E_k/2$
- PBC sector, GS energy is $\left(\sum_k E_k/2\right) + \min_k (E_k)$

Perimeter-A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Translate to fermions:
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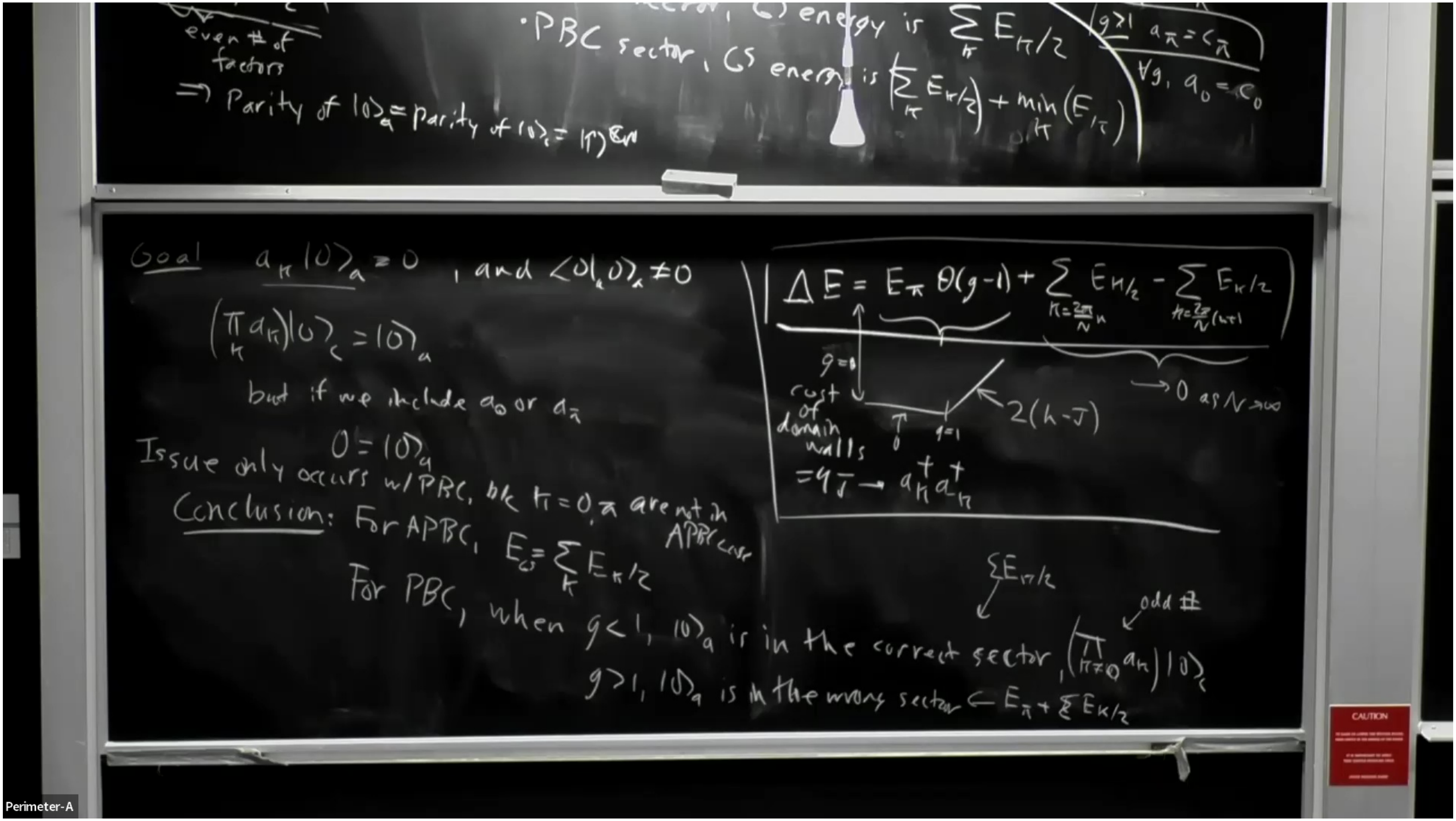
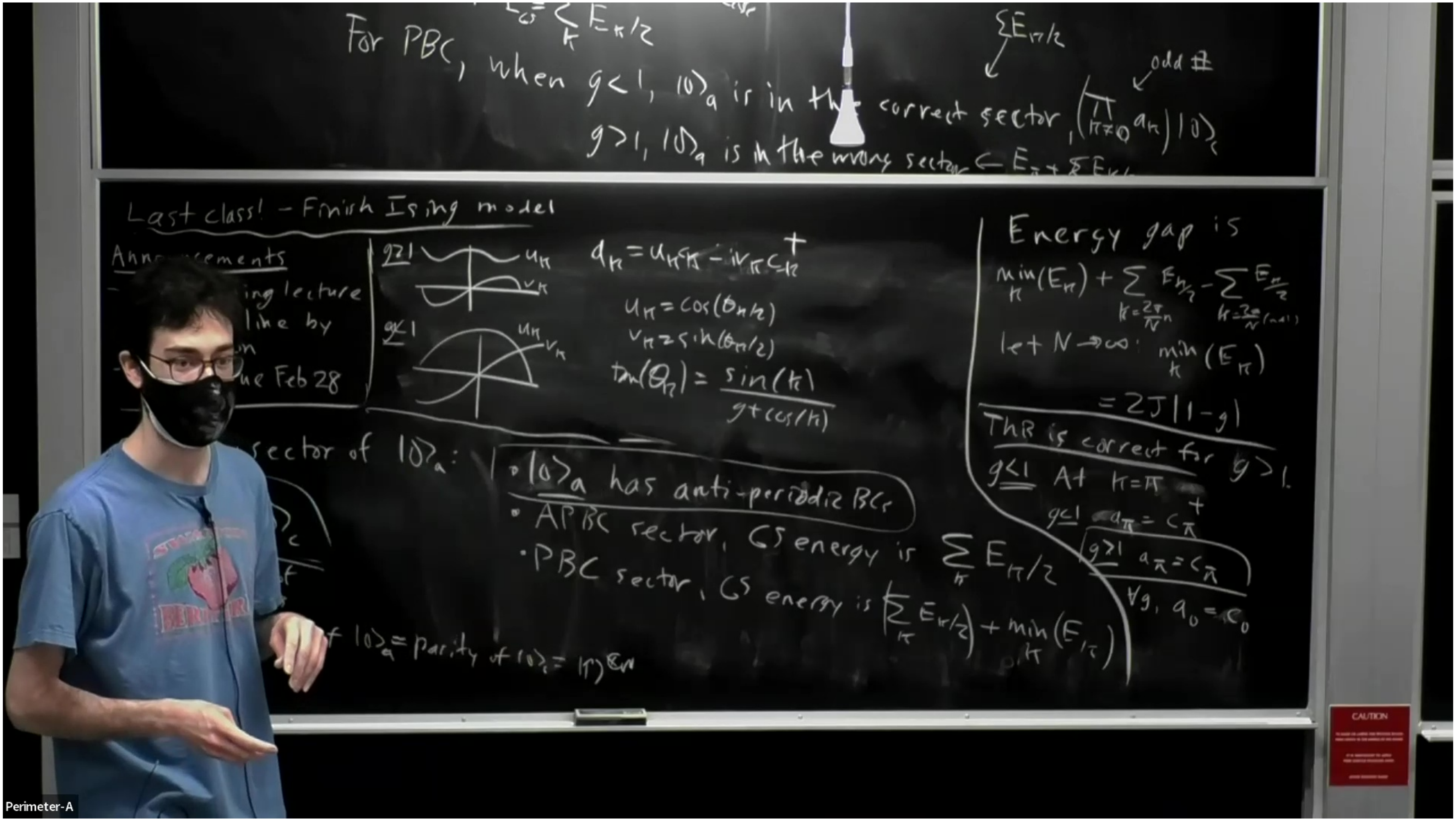
Parity sector of $|0\rangle_a$:
$\left(\prod_k a_k\right)|0\rangle_c$
even # of factors
$\Rightarrow$ Parity of $|0\rangle_a$ = Parity of $|0\rangle_c = |\uparrow\rangle^{\otimes N}$
• $|0\rangle_a$ has anti-periodic BCs
• APBC sector, GS energy is $\sum_k E_k/2$
• PBC sector, GS energy is $\left(\sum_k E_k/2\right) + \min_k(E_k)$
Energy gap is
$\min_k(E_k) + \sum_{k=\frac{2\pi}{N}n} E_k/2 - \sum_{k=\frac{2\pi}{N}(n+\frac{1}{2})} E_k/2$
let $N\to\infty$: $\min_k(E_k)$
$= 2J|1-g|$
This is correct for $g>1$.
$g<1$ At $k=\pi$
$g<1$ $a_\pi = c_\pi^\dagger$
$g>1$ $a_\pi = c_\pi$
$\forall g$, $a_0 = c_0$
Perimeter-A
CAUTION


even # of factors
⇒ Parity of |0⟩_a = parity of |0⟩_c = |π⟩ ...
• PBC sector, GS energy is Σ_k E_k/2
• PBC sector, GS energy is (Σ_k E_k/2) + min_k (E_k/2)
g>1 a_π = c_π
∀g, a_0 = c_0
Goal a_k|0⟩_a = 0 , and ⟨0|_c 0⟩_a ≠ 0
(Π_k a_k)|0⟩_c = |0⟩_a
but if we include a_0 or a_π
0 = |0⟩_a
Issue only occurs w/ PBC, b/c k = 0, π are not in APBC case
Conclusion: For APBC, E_GS = Σ_k E_k/2
For PBC, when g<1, |0⟩_a is in the correct sector, (Π_{k≠0} a_k)|0⟩_c
g>1, |0⟩_a is in the wrong sector ← E_π + Σ E_k/2
ΔE = E_π θ(g-1) + Σ_{k=2π/N n} E_k/2 - Σ_{k=2π/N (n+1)} E_k/2
g=0 cost of domain walls = 4J → a_k† a_-k†
2(h-J)
0 g=1
→ 0 as N→∞
ΣE_k/2
odd #
Perimeter-A
CAUTION


For PBC, when g<1, |0⟩a is in the correct sector, (∏ k≠0 ak)|0⟩c
g>1, |0⟩a is in the wrong sector
ΣEk/2
odd #
Last class! - Finish Ising model
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ing lecture
line by
Feb 28
g≥1 uk vk
g<1 uk vk
ak = uk ck - i vk c†-k
uk = cos(θk/2)
vk = sin(θk/2)
tan(θk) = sin(k) / (g+cos(k))
sector of |0⟩a:
• |0⟩a has anti-periodic BCs
• APBC sector, GS energy is Σk Ek/2
• PBC sector, GS energy is (Σk Ek/2) + min k (Ek)
parity of |0⟩a = parity of |0⟩c = |↑⟩⊗N
Energy gap is
min k (Ek) + Σ Ek/2 - Σ Ek/2
let N→∞: min k (Ek)
= 2J|1-g|
This is correct for g>1.
g<1 At k=π
g<1 aπ = c†π
g>1 aπ = cπ
∀g, a0 = c0
Perimeter-A
CAUTION

$E_0 = \sum_k E_k/2$ $\quad$ $\sum E_k/2$

odd #

For PBC, when $g<1$, $|0\rangle_a$ is in the correct sector $\left(\prod_{k\geq 0} a_k\right)|0\rangle$

Claim $g=1$ is a quantum phase transition.

Last thing to do: compute order parameter / correlation function

$$M=\sqrt{\langle \sigma_0^x \sigma_n^x\rangle}\,,\quad M^2 = \langle \underbrace{(c_0^+ - c_0)}_{B_0}\underbrace{(c_1^+ + c_1)}_{A_1}\underbrace{(c_1^+ - c_1)}_{B_1}\cdots\underbrace{(c_{n-1}^+ - c_{n-1})}_{B_{n-1}}\underbrace{(c_n^+ + c_n)}_{A_n}\rangle$$

and H is quadratic $\rightarrow$ Wick's theorem

$$\Rightarrow M^2 = \det\begin{pmatrix} G_{-1} & G_{-2} & \cdots & G_{-n} \\ G_0 & & & \\ & & & \\ G_{n-2} & & & G_{-1}\end{pmatrix},\quad G_n = \frac{1}{\pi}\int_0^\pi \frac{g\cos(kn) + \cos(k(n+1))}{\sqrt{1+g^2+2g\cos(k)}}\,dk$$

Toeplitz determinant

$\rightarrow$ some hypergeometric fct.

Perimeter-A

Ising_correlations_v2.nb * - Wolfram Mathematica 12.1

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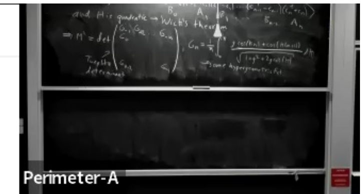

In[ ]:= $MinPrecision = 64

Out[ ]= 64

In[ ]:= g =.;

$$Gn = \text{Integrate}\left[\frac{g\,\text{Cos}[k\,n] + \text{Cos}[k\,(n+1)]}{\sqrt{1+g^2+2\,g\,\text{Cos}[k]}}, \{k, 0, \pi\}, \text{Assumptions} \to \{n > 0, n \in \text{Integers}, g \neq 1, g > 0\}\right] \Big/ \pi$$

Out[ ]=
$$\frac{g\,\text{HypergeometricPFQRegularized}\left[\left\{\frac{1}{2}, \frac{1}{2}, 1\right\}, \{1-n, 1+n\}, -\frac{4\,g}{(-1+g)^2}\right] + \text{HypergeometricPFQRegularized}\left[\left\{\frac{1}{2}, \frac{1}{2}, 1\right\}, \{-n, 2+n\}, -\frac{4\,g}{(-1+g)^2}\right]}{\text{Abs}[-1+g]}$$

In[ ]:=
$$G[n\_, g\_] := \text{If}\left[g \neq 1, \frac{g\,\text{HypergeometricPFQRegularized}\left[\left\{\frac{1}{2}, \frac{1}{2}, 1\right\}, \{1-n, 1+n\}, -\frac{4\,g}{(-1+g)^2}\right] + \text{HypergeometricPFQRegularized}\left[\left\{\frac{1}{2}, \frac{1}{2}, 1\right\}, \{-n, 2+n\}, -\frac{4\,g}{(-1+g)^2}\right]}{\text{Abs}[-1+g]}, \frac{2}{\pi}\,\frac{(-1)^n}{2\,n+1}\right]$$

```
In[ ]:= gvals = Table[g, {g, 0.001, 2.001, 0.01}];
        numgvals = Dimensions[gvals][[1]];

In[ ]:= GVals = Table[N[G[n, gvals[[gIndex]]]], {n, -100, 100}, {gIndex, 1, numgvals}]
```

··· General: $-472966.\ 1.151008212224907671643504890291774935438022002413651236850296756 \times 10^{-314}$ is too small to represent as a normalized machine number; precision may be lost.

··· General: $-55243.7\ 1.151008212224907671643504890291774935438022002413651236850296756 \times 10^{-314}$ is too small to represent as a normalized machine number; precision may be lost.

··· General: $-11389.4\ 1.151008212224907671643504890291774935438022002413651236850296756 \times 10^{-314}$ is too small to represent as a normalized machine number; precision may be lost.

··· General: Further output of General::munfl will be suppressed during this calculation.

```
Out[ ]= {{0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0.,
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```

large output | show less | show more | show all | set size limit...

```
In[ ]:= dims = Dimensions[GVals];

In[ ]:= Correlations = Table[Det[
```

150%

9:58 AM 2/10/2022