Title: PSI Lecture - Condensed Matter - Lecture 13

Speakers: Aaron Szasz

Collection: PSI Lecture - Condensed Matter

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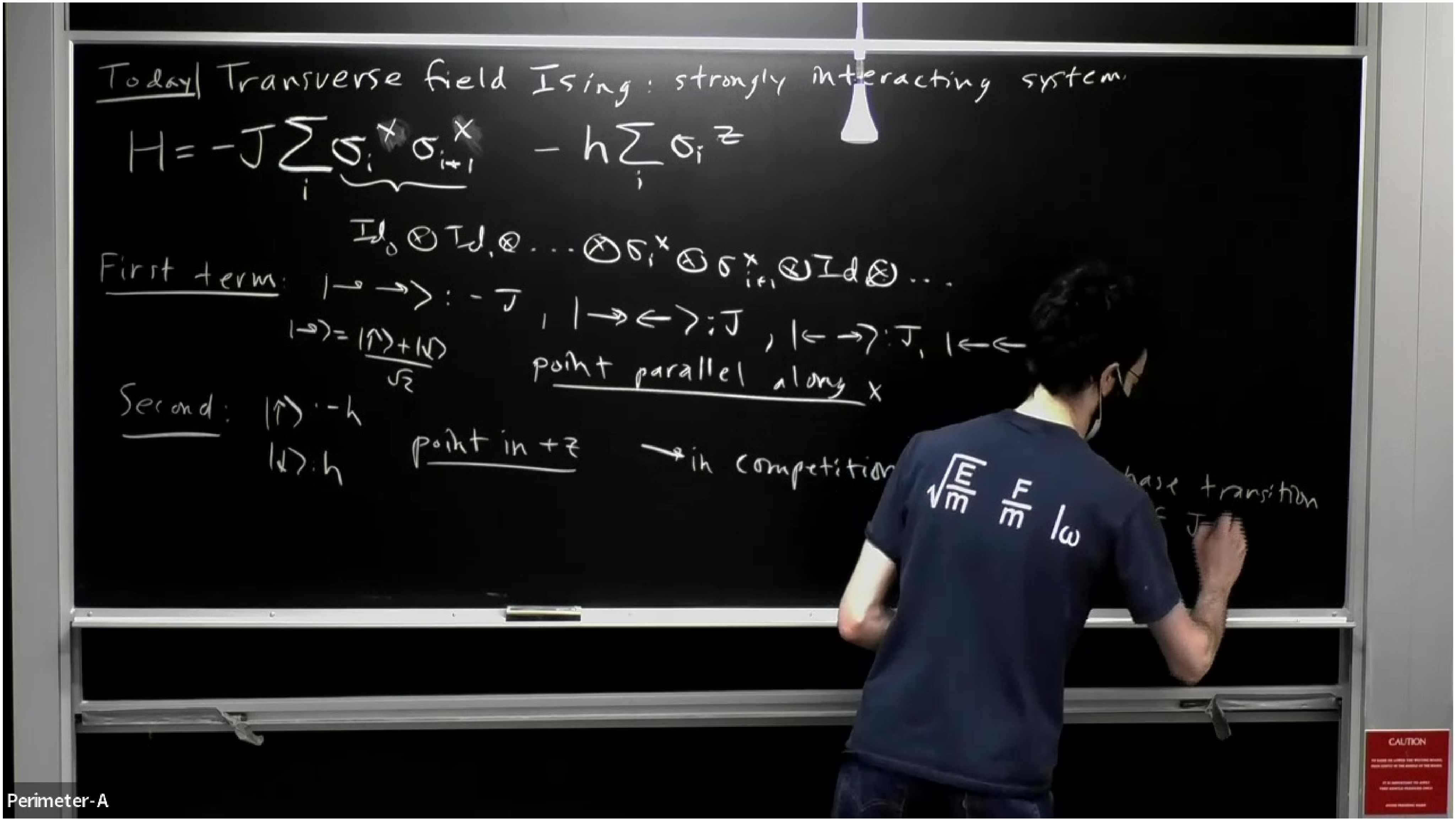
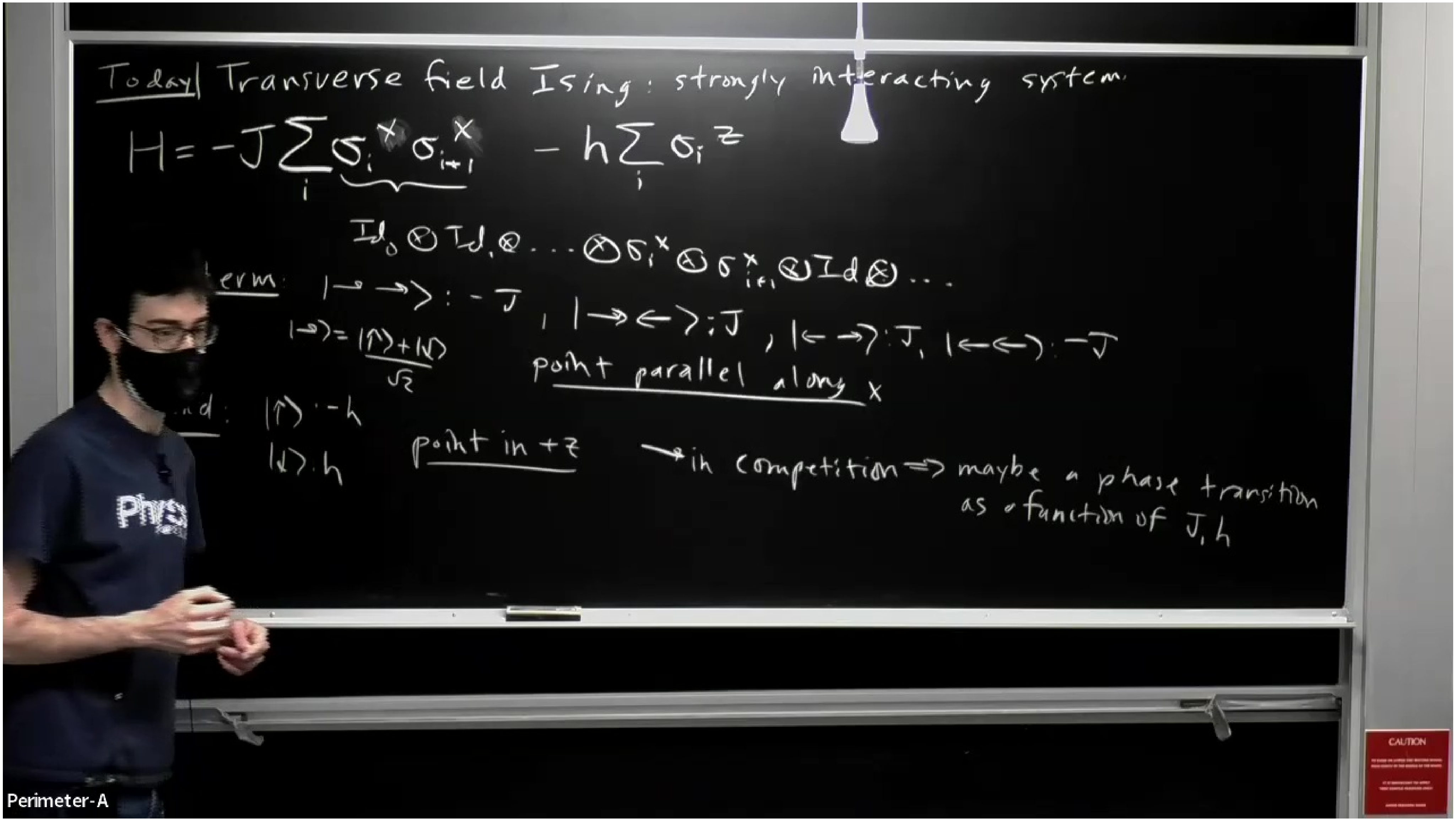
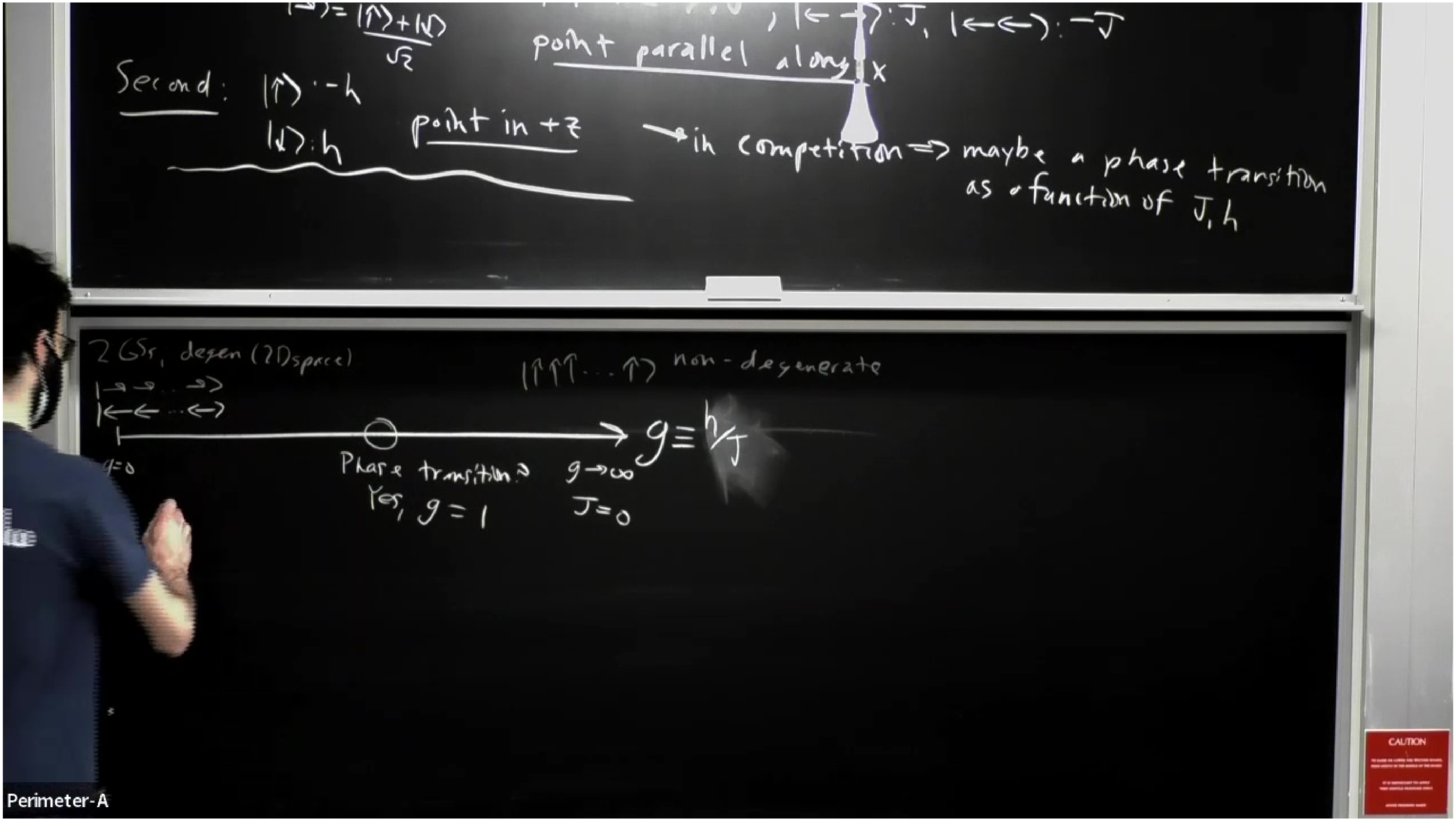
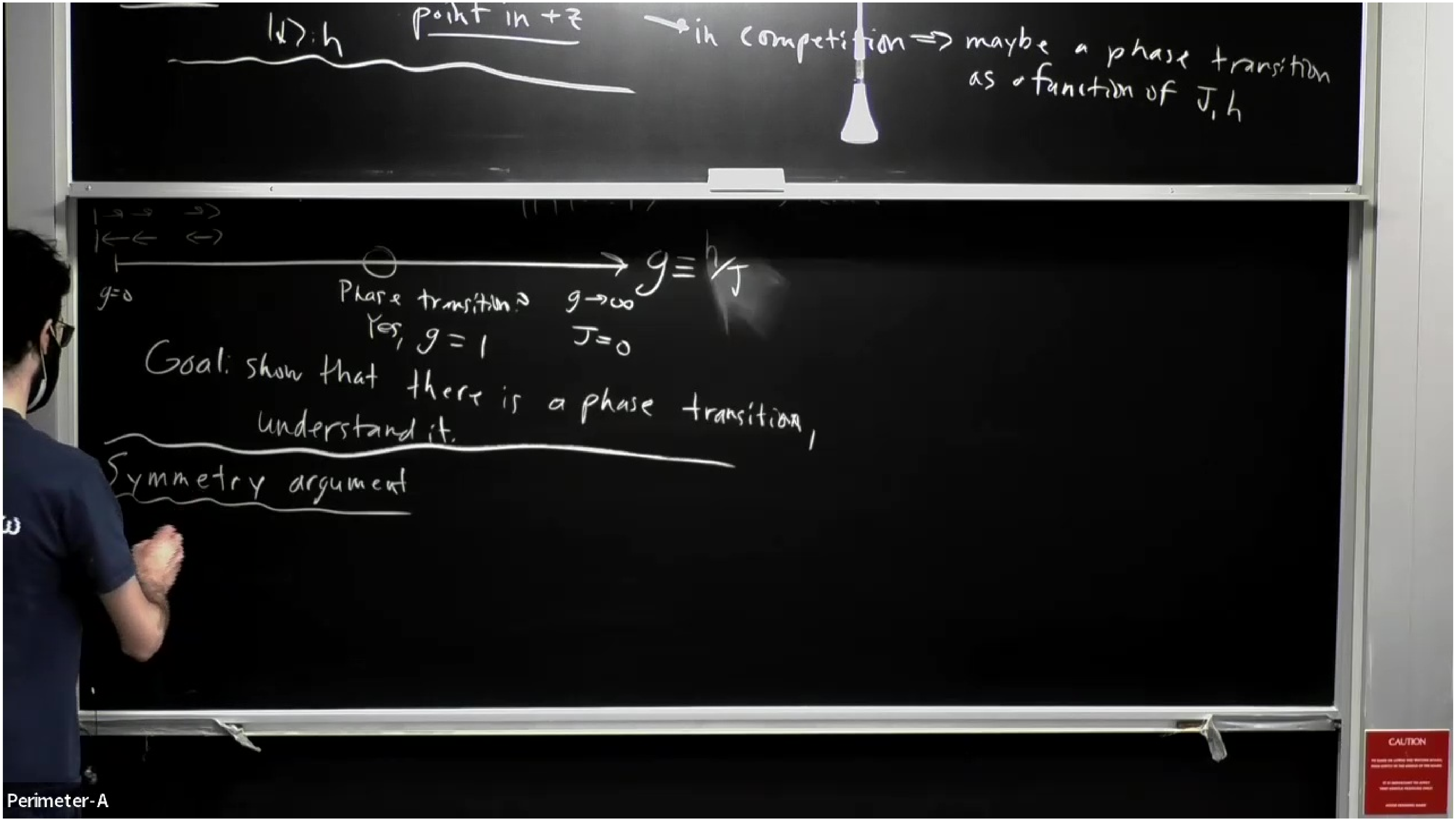

Today | Transverse field Ising: strongly interacting system
$H = -J \sum_i \underbrace{\sigma_i^x \sigma_{i+1}^x} - h \sum_i \sigma_i^z$
$Id_0 \otimes Id_1 \otimes \ldots \otimes \sigma_i^x \otimes \sigma_{i+1}^x \otimes Id \otimes \ldots$
First term: $|\rightarrow\rightarrow\rangle : -J$, $|\rightarrow\leftarrow\rangle : J$, $|\leftarrow\rightarrow\rangle : J$, $|\leftarrow\leftarrow$
$|\rightarrow\rangle = \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}$
point parallel along x
Second: $|\uparrow\rangle : -h$
$|\downarrow\rangle : h$
point in +z
→ in competition
hase transition
Perimeter-A
CAUTION


Today | Transverse field Ising: strongly interacting system.
$H = -J \sum_i \sigma_i^x \sigma_{i+1}^x - h \sum_i \sigma_i^z$
$Id_0 \otimes Id_1 \otimes \dots \otimes \sigma_i^x \otimes \sigma_{i+1}^x \otimes Id \otimes \dots$
term: $|\rightarrow\rightarrow\rangle : -J$, $|\rightarrow\leftarrow\rangle : J$, $|\leftarrow\rightarrow\rangle : J$, $|\leftarrow\leftarrow\rangle : -J$
$|\rightarrow\rangle = \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}$
point parallel along x
nd: $|\uparrow\rangle : -h$
$|\downarrow\rangle : h$
point in +z
in competition ⇒ maybe a phase transition as a function of J, h
Perimeter-A


|→⟩ = (|↑⟩ + |↓⟩)/√2
|←→⟩: J, |←←⟩: −J
point parallel along x
Second: |↑⟩: −h
|↓⟩: h
point in +z
in competition ⇒ maybe a phase transition as a function of J, h
2 GSs, degen (2D space)
|→→…→⟩
|←←…←⟩
|↑↑↑…↑⟩ non-degenerate
g ≡ h/J
g=0
Phase transition?
Yes, g = 1
g→∞
J=0
Perimeter-A
CAUTION


point in +z
in competition ⇒ maybe a phase transition
as a function of J, h
g = h/J
g=0
Phase transition?
Yes, g = 1
g→∞
J=0
Goal: show that there is a phase transition,
understand it.
Symmetry argument
Perimeter-A
CAUTION

$|\downarrow\rangle : h$ point in $+z$ → in competition ⇒ maybe a phase transition as a function of J, h

$g=0$ ⟶ $g \equiv h/J$

Phase transition? $g \to \infty$

Yes, $g = 1$ $J = 0$

Goal: show that there is a phase transition, understand it.

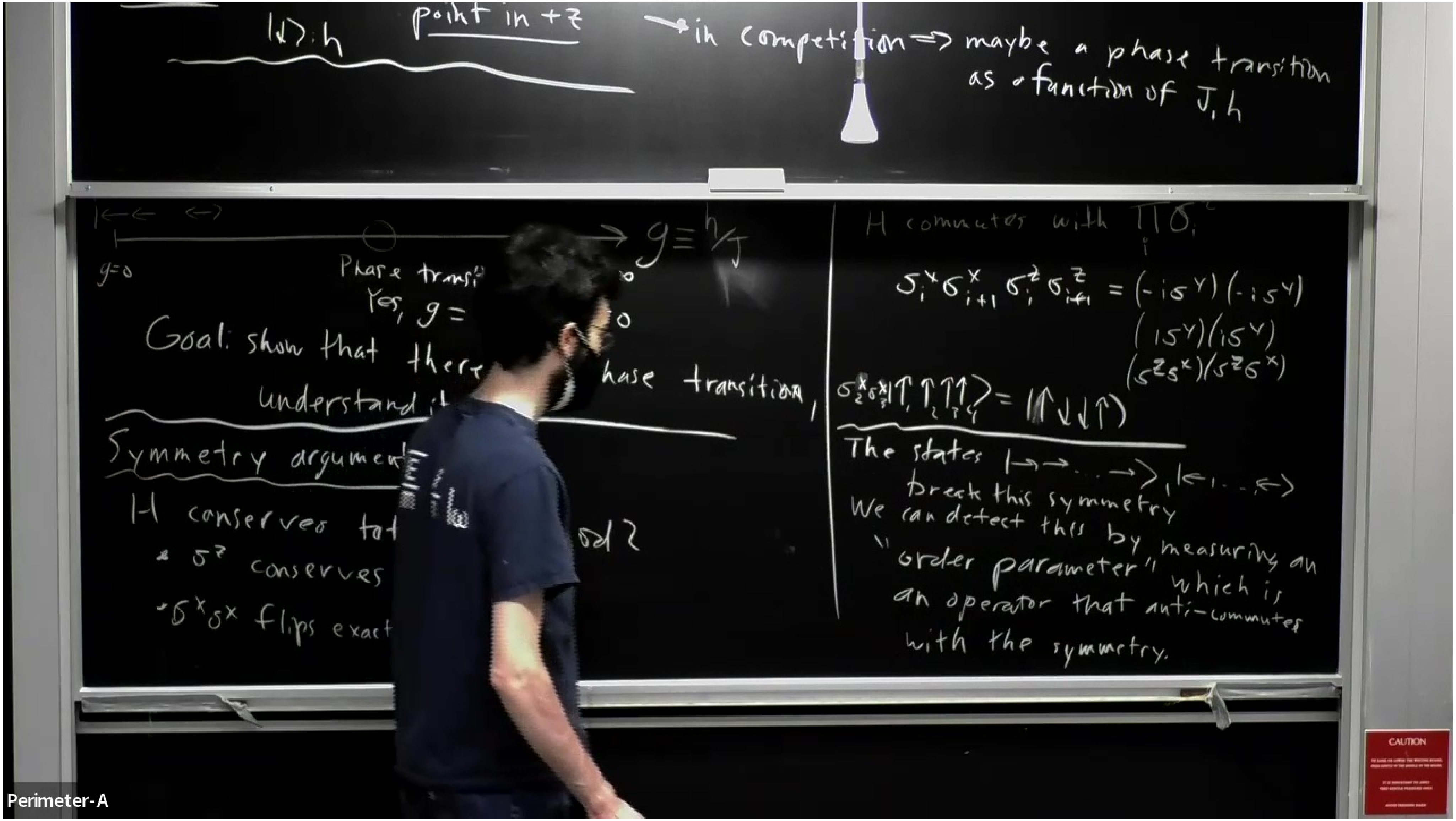
Symmetry argument

H conserves total S_z, mod 2

- σ^z conserves S_z
- $\sigma^x \sigma^x$ flips exactly 2 spins

H commutes with $\prod_i \sigma_i^z$

$\sigma_i^x \sigma_{i+1}^x \sigma_i^z \sigma_{i+1}^z = (-i\sigma^y)$



point in +z
in competition ⇒ maybe a phase transition as a function of J, h
g=0
Phase transi
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phase transition,
understand i
Symmetry argument
H conserves tot
od 2
σ^z conserves
σ^x σ^x flips exact
H commutes with ∏ σ_i^z
σ_i^x σ_{i+1}^x σ_i^z σ_{i+1}^z = (-iσ^y)(-iσ^y)
(iσ^y)(iσ^y)
(σ^z σ^x)(σ^z σ^x)
σ_2^x σ_3^x |↑_1 ↑_2 ↑_3 ↑_4⟩ = |↑↓↓↑⟩
The states |→→...→⟩, |←...←⟩
break this symmetry
We can detect this by measuring an
"order parameter" which is
an operator that anti-commutes
with the symmetry.
Perimeter-A
CAUTION

understand it.

Symmetry argument

H conserves total ...

... $= |\uparrow\downarrow\downarrow\uparrow\rangle$

The states $|\rightarrow\rightarrow\ldots\rightarrow\rangle$, $|\leftarrow\ldots\leftarrow\rangle$ break this symmetry

We can detect this ...

Example. $\langle\sigma_i^x\rangle$ because $\{\sigma_i^x, \prod_j \sigma_j^z\} = 0$

$\frac{|\rightarrow\rangle^{\otimes N} + |\leftarrow\rangle^{\otimes N}}{\sqrt{2}}$

$|\rightarrow\rightarrow\ldots\rightarrow\rangle$, $\langle\sigma_i^x\rangle = 1$

$|\leftarrow\leftarrow\ldots\leftarrow\rangle$, $\langle\sigma_i^x\rangle = -1$ $\longleftarrow$ $h=0 \Rightarrow$ spontaneous symmetry breaking (SSB)

On the other hand, $|\uparrow\uparrow\ldots\uparrow\rangle$, $\rightarrow \langle\sigma_i^x\rangle = 0 \longleftarrow J=0$, no SSB

More generally, for order parameter O, compute the correlation function $\langle O_i O_j\rangle$ in the limit $|i-j|\to\infty$, if the state is ordered, this will be $\neq 0$.

Intuition: if there was no correlation, $\langle O_i O_j\rangle \sim \langle O_i\rangle\langle O_j\rangle$

Perimeter-A

$\neq 0.$, If the state is ordered, this will be

Intuition: if there was no correlation, $\langle O_i O_j \rangle \sim \langle O_i \rangle \langle O_j \rangle$

$\langle \sigma_0^x \sigma_\infty^x \rangle \neq 0$ $\qquad\qquad$ $\langle \sigma_0^x \sigma_\infty^x \rangle = 0$

$g=0$ $\longrightarrow$ $g \equiv h/J$

2 possibilities:

no PT (1) $\langle \sigma^x \sigma^x \rangle_\infty \neq 0$ for all finite g

PT (2) $\exists\ g_c$ where $\langle \sigma_0^x \sigma_\infty^x \rangle = 0$ for $g > g_c$

H commutes with $\prod_i \sigma_i^z$

$$\sigma_i^x \sigma_{i+1}^x \sigma_i^z \sigma_{i+1}^z = (-i\sigma^y)(-i\sigma^y)$$
$$(i\sigma^y)(i\sigma^y)$$
$$(\sigma^z\sigma^x)(\sigma^z\sigma^x)$$

$\sigma_2^x \sigma_3^x |\uparrow_1 \uparrow_2 \uparrow_3 \uparrow_4\rangle = |\uparrow\downarrow\downarrow\uparrow\rangle$

The states $|\rightarrow\rightarrow\ldots\rightarrow\rangle$, $|\leftarrow\ldots\leftarrow\rangle$ break this symmetry

We can detect this by measuring an "order parameter" which is an operator that anti-commutes with the symmetry.

Perimeter-A

$\neq 0.$

Intuition: if there was no correlation, $\langle O_i O_j \rangle \sim \langle O_i \rangle \langle O_j \rangle$, if the state is ordered, this will be

$$g = 0 \longrightarrow g \equiv h/J$$

2 possibilities:

no PT (1) $\langle \sigma^x \sigma^x \rangle_\infty \neq 0$ for all finite g

PT (2) $\exists\ g_c$ where $\langle \sigma_0^x \sigma_\infty^x \rangle = 0$ for $g > g_c$

(If eig fcts are cts)

(3) 1st order PT, where eig fcts not cts.

H commutes with $\prod_i \sigma_i^z$

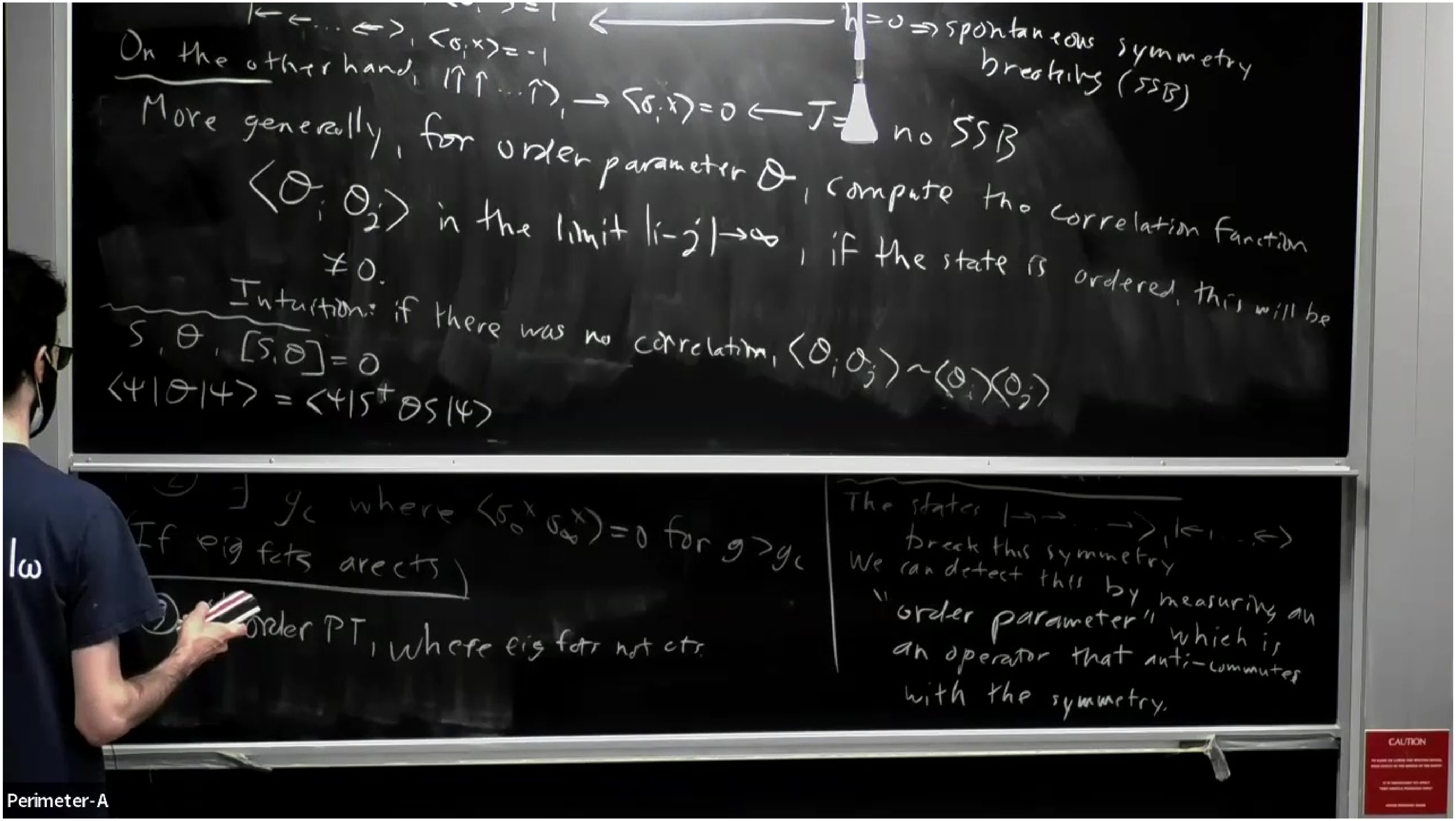
$$\sigma_i^x \sigma_{i+1}^x \sigma_i^z \sigma_{i+1}^z = (-i\sigma^y)(-i\sigma^y)$$
$$(i\sigma^y)(i\sigma^y)$$
$$(\sigma^z\sigma^x)(\sigma^z\sigma^x)$$

$$\sigma_2^x \sigma_3^x |\uparrow_1 \uparrow_2 \uparrow_3 \uparrow_4\rangle = |\uparrow\downarrow\downarrow\uparrow\rangle$$

The states $|\rightarrow\rightarrow\ldots\rightarrow\rangle, |\leftarrow\ldots\leftarrow\rangle$ break this symmetry

We can detect this by measuring an "order parameter" which is an operator that anti-commutes with the symmetry.

Perimeter-A



|←←...←>, ⟨σ_i^x⟩ = -1
h = 0 ⇒ spontaneous symmetry breaking (SSB)
On the other hand, |↑↑...↑>, → ⟨σ_i^x⟩ = 0 ← J = no SSB
More generally, for order parameter O, compute the correlation function
⟨O_i O_j⟩ in the limit |i-j| → ∞, if the state is ordered, this will be ≠ 0.
Intuition: if there was no correlation, ⟨O_i O_j⟩ ~ ⟨O_i⟩⟨O_j⟩
S, O, [S, O] = 0
⟨ψ|O|ψ⟩ = ⟨ψ|S† O S|ψ⟩
∃ g_c where ⟨σ_0^x σ_∞^x⟩ = 0 for g > g_c
(If eig fcts are cts)
order PT, where eig fcts not cts
The states |→→...→>, |←...←> break this symmetry
We can detect this by measuring an "order parameter" which is an operator that anti-commutes with the symmetry.
Perimeter-A

$|\leftarrow\leftarrow\ldots\leftarrow\rangle$, $\langle\sigma_i^x\rangle = -1$ $\longleftarrow$ $h = 0 \Rightarrow$ spontaneous symmetry breaking (SSB)

On the other hand, $|\uparrow\uparrow\ldots\uparrow\rangle$, $\rightarrow \langle\sigma_i^x\rangle = 0 \longleftarrow J = 0$ no SSB

More generally, for order parameter $\mathcal{O}$, compute the correlation function $\langle \mathcal{O}_i \mathcal{O}_j \rangle$ in the limit $|i-j| \to \infty$, if the state is ordered, this will be $\neq 0$.

Intuition: if there was no correlation, $\langle \mathcal{O}_i \mathcal{O}_j \rangle \sim \langle \mathcal{O}_i \rangle \langle \mathcal{O}_j \rangle$

A, $\{A, S\} = 0$

$|\psi\rangle$, respects symmetry

$$\langle\psi|A|\psi\rangle = \langle\psi|S^\dagger A S|\psi\rangle = -\langle\psi|A|\psi\rangle = 0$$

$\exists\ g_c$ where $\langle \sigma_0^x \sigma_\infty^x \rangle = 0$ for $g > g_c$

(If eig fcts are cts)

③ 1^{st} order PT, where eig fcts not cts.

The states $|\rightarrow\rightarrow\ldots\rightarrow\rangle$, $|\leftarrow\ldots\leftarrow\rangle$ break this symmetry

We can detect this by measuring an "order parameter" which is an operator that anti-commutes with the symmetry.

Perimeter-A

Suppose A anticommutes with symmetry S, $S|\psi\rangle = |\psi\rangle$

then $\langle\psi|A|\psi\rangle = \langle S\psi|A|S\psi\rangle = \langle\psi|S^\dagger A S|\psi\rangle = -\langle\psi|A|\psi\rangle$

$\therefore \langle\psi|A|\psi\rangle = 0$

If $\langle\psi|A|\psi\rangle \neq 0$, then one of the premises is false

$\Rightarrow$ if we pick $\{A, S\} = 0$, then any $\langle\psi|A|\psi\rangle \neq 0 \Rightarrow |\psi\rangle$ breaks S

Perimeter-A

Suppose A anticommutes with symmetry S, $S|\psi\rangle = |\psi\rangle$

then $\langle\psi|A|\psi\rangle = \langle S\psi|A|S\psi\rangle = \langle\psi|S^{\dagger}AS|\psi\rangle = -\langle\psi|A|\psi\rangle$

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If $\langle\psi|A|\psi\rangle \neq 0$, then one of the promises is false

$\Rightarrow$ if we pick $\{A, S\} = 0$, then any $\langle\psi|A|\psi\rangle \neq 0 \Rightarrow |\psi\rangle$ breaks S

To find whether there is a phase transition, exactly solve the model.

Procedure: ① Jordan-Wigner transformation

② Fourier transform

③ Bogoliubov transformation.

3) 1st order PT, where eig fcts not cts

order parameter" by measuring an an operator that anti-commutes which is with the symmetry.

Jordan-Wigner transformation

② Fourier transform

③ Bogoliubov transformation.

① JW transformation: from spins to fermions.

Notice: $|\uparrow\rangle, |\downarrow\rangle$ is a 2D Hilbert space

$|0\rangle, c^\dagger|0\rangle$ is too

1st attempt: $|\uparrow\rangle \to |0\rangle$

$|\downarrow\rangle \to c^\dagger|0\rangle \Longrightarrow$

$\sigma^+ \longrightarrow c$

$\sigma^- \longrightarrow c^\dagger$

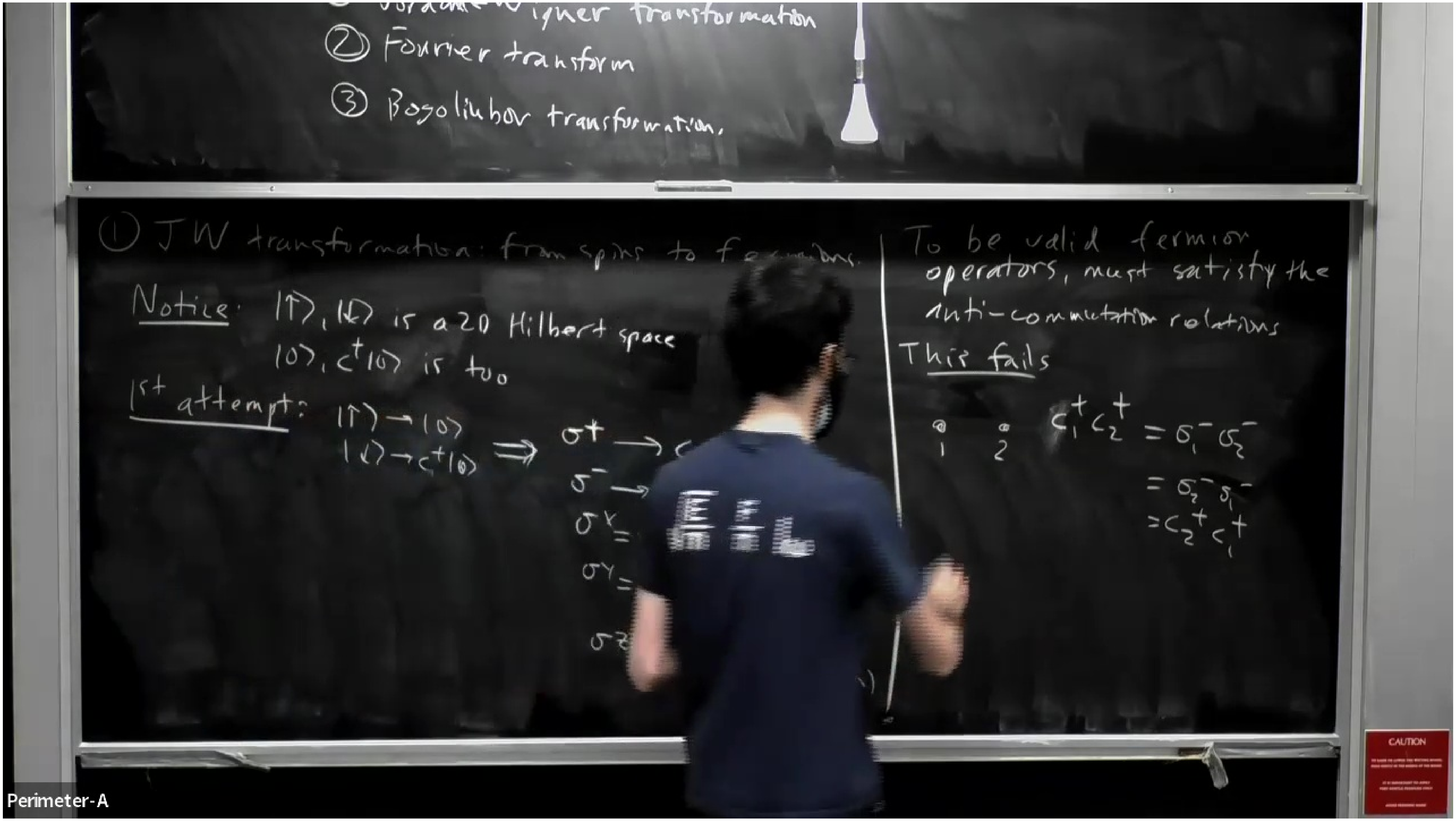
$\sigma^x = \sigma^+ + \sigma^-$

$\sigma^y = \frac{\sigma^+ - \sigma^-}{i}$

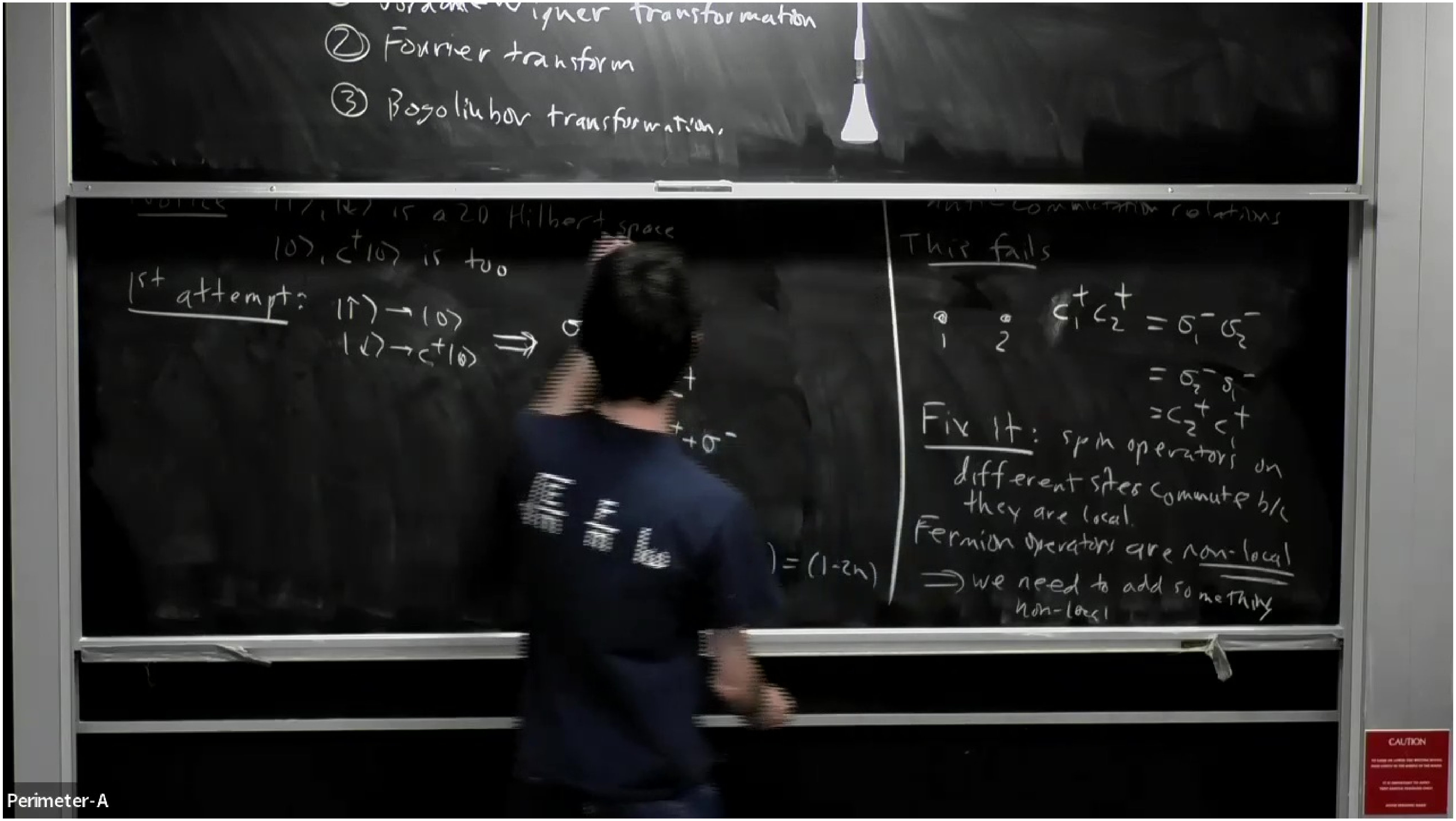
$\sigma^z = (1 - 2c^\dagger c) = (1-2n)$

To be valid fermion operators, must sat

Perimeter-A



② Fourier transform
③ Bogoliubov transformation,
① JW transformation: from spins to fermions.
Notice: |↑⟩, |↓⟩ is a 2D Hilbert space
|0⟩, c†|0⟩ is too
1st attempt: |↑⟩ → |0⟩
|↓⟩ → c†|0⟩ ⟹ σ+ →
σ− →
σx =
σy =
σz
To be valid fermion operators, must satisfy the anti-commutation relations
This fails
1 2
c1† c2† = σ1− σ2−
= σ2− σ1−
= c2† c1†
Perimeter-A
CAUTION


② Fourier transform
③ Bogoliubov transformation.
$|0\rangle, c^\dagger|0\rangle$ is too
1st attempt: $|\uparrow\rangle \to |0\rangle$
$|\downarrow\rangle \to c^\dagger|0\rangle \Rightarrow$
$) = (1-2n)$
This fails
$c_1^\dagger c_2^\dagger = \sigma_1^- \sigma_2^-$
$= \sigma_2^- \sigma_1^-$
$= c_2^\dagger c_1^\dagger$
Fix it: spin operators on different sites commute b/c they are local.
Fermion operators are non-local
$\Rightarrow$ we need to add something non-local
Perimeter-A

Solution $c_j^\dagger = \left(\prod_{l<j} \sigma_l^z\right)\sigma_j^-$

$$c_j = \left(\prod_{l<j} \sigma_l^z\right)\sigma_j^+$$

Could likewise check $\{c_i, c_j^\dagger\} = \delta_{ij}$

Invert to write σ operators with $c, c^\dagger$, substitute to H.

$$\sigma_j^+ = \left(\prod_{l<j} \sigma_l^z\right) c_j = \left(\prod_{l<j}(1-2n_l)\right) c_j$$

Let's check:

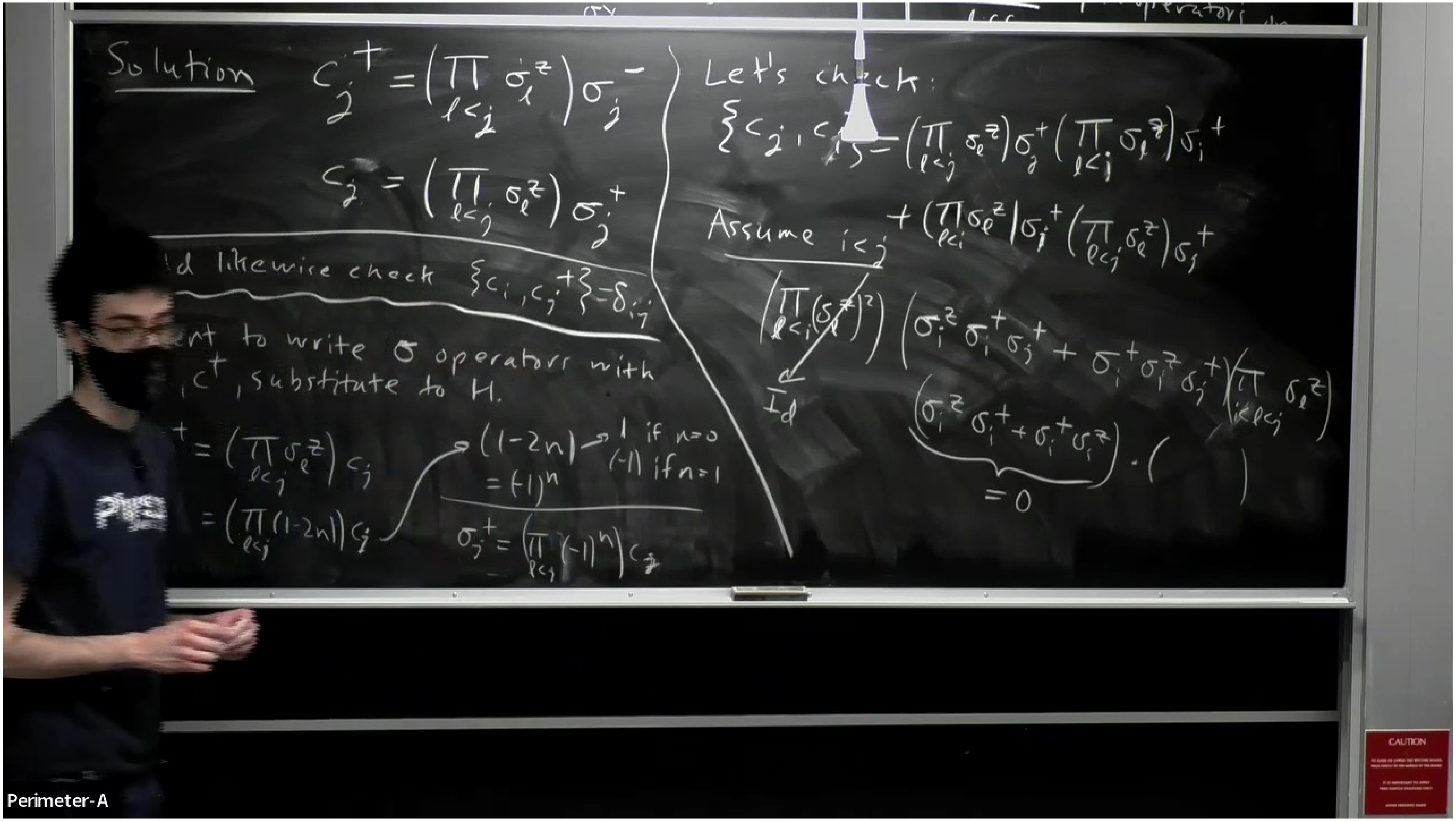
$$\{c_j, c_i\} = \left(\prod_{l<j} \sigma_l^z\right)\sigma_j^+\left(\prod_{l<i} \sigma_l^z\right)\sigma_i^+ + \left(\prod_{l<i} \sigma_l^z\right)\sigma_i^+\left(\prod_{l<j} \sigma_l^z\right)\sigma_j^+$$

Assume $i<j$

$$\underbrace{\left(\prod_{l<i} (\sigma_l^z)^2\right)}_{I_d}\left(\sigma_i^z\sigma_i^+\sigma_j^+ + \sigma_i^+\sigma_i^z\sigma_j^+\right)\left(\prod_{i<l<j} \sigma_l^z\right)$$

$$\underbrace{\left(\sigma_i^z\sigma_i^+ + \sigma_i^+\sigma_i^z\right)}_{=0}\cdot\left(\quad\right)$$

Perimeter-A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d likewise check $\{c_i, c_j^\dagger\} = \delta_{ij}$
nt to write σ operators with
, $c^\dagger$, substitute to H.
$^+ = \left(\prod_{l<j} \sigma_l^z\right) c_j$
$= \left(\prod_{l<j} (1-2n)\right) c_j$
$(1-2n) \to 1$ if $n=0$
(-1) if $n=1$
$= (-1)^n$
$\sigma_j^+ = \left(\prod_{l<j} (-1)^n\right) c_j$
Let's check:
$\{c_j, c_i\} = \left(\prod_{l<j} \sigma_l^z\right)\sigma_j^+\left(\prod_{l<i} \sigma_l^z\right)\sigma_i^+ + \left(\prod_{l<i} \sigma_l^z\right)\sigma_i^+\left(\prod_{l<j} \sigma_l^z\right)\sigma_j^+$
Assume $i<j$
$\left(\prod_{l<i} (\sigma_l^z)^2\right)\left(\sigma_i^z\sigma_i^+\sigma_j^+ + \sigma_i^+\sigma_i^z\sigma_j^+\right)\left(\prod_{i<l<j} \sigma_l^z\right)$
I_d
$\underbrace{\left(\sigma_i^z\sigma_i^+ + \sigma_i^+\sigma_i^z\right)}_{=0} \cdot (\quad)$
Perimeter-A
CAUTION

Perimeter-A

Invert to write σ operators with $c, c^\dagger$, substitute to H.

$$\sigma_j^+ = \Big(\prod_{l<j}\sigma_l^z\Big)c_j$$
$$= \Big(\prod_{l<j}(1-2n_l)\Big)c_j$$

$(1-2n) \to 1$ if $n=0$, (-1) if $n=1$

$= (-1)^n$

$$\sigma_j^+ = \Big(\prod_{l<j}(-1)^{n_l}\Big)c_j$$

$$\Big(\sigma_i^z\sigma_i^+\sigma_j^+ + \sigma_i^+\sigma_i^z\sigma_j^+\Big)\Big(\prod_{i\le l<j}\sigma_l^z\Big)$$

I_d

$$\underbrace{\Big(\sigma_i^z\sigma_i^+ + \sigma_i^+\sigma_i^z\Big)}_{=0}\cdot\Big(\quad\Big)$$

Substitute: $\sigma_j^x \to \Big(\prod_{l<j}(-1)^{n_l}\Big)(c_j + c_j^\dagger)$

$\sigma_j^y \to \Big(\qquad\Big)(c_j - c_j^\dagger)\cdot(-i)$

$\sigma_j^z \to (1-2n_j)$

$$\Rightarrow H = -J\sum_i\Big((c_j + c_j^\dagger)(1-2n_j)(c_{j+1} + c_{j+1}^\dagger) + g(1-2n_j)\Big)$$

$c_j n_j = c_j$

$c_j^\dagger n_j = 0$

$$\boxed{-J\sum_i\Big((c_j^\dagger - c_j)(c_{j+1}^\dagger + c_{j+1}) + g - 2g\,c_j^\dagger c_j\Big)}$$

$c_j^\dagger c_{j+1}^\dagger$

To be valid fermion operators, must satisfy the anti-commutation relations

This fails

1 2

$$c_1^\dagger c_2^\dagger = \sigma_1^-\sigma_2^- = \sigma_2^-\sigma_1^- = c_2^\dagger c_1^\dagger$$

Fix it: spin operators on different sites commute b/c they are local.

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