Title: Detecting nonclassicality in restricted general probabilistic theories

Speakers: Leevi Leppajarvi

Series: Quantum Foundations

Date: January 27, 2022 - 10:00 AM

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Abstract: The formalism of general probabilistic theories provides a universal paradigm that is suitable for describing various physical systems including classical and quantum ones as particular cases. Contrary to the often assumed no-restriction hypothesis, the set of accessible measurements within a given theory can be limited for different reasons, and this raises a question of what restrictions on measurements are operationally relevant. We argue that all operational restrictions must be closed under simulation, where the simulation scheme involves mixing and classical post-processing of measurements. We distinguish three classes of such operational restrictions: restrictions on measurements originating from restrictions on effects; restrictions on measurements that do not restrict the set of effects in any way; and all other restrictions. As a setting to detect nonclassicality in restricted theories we consider generalizations of random access codes, an intriguing class of communication tasks that reveal an operational and quantitative difference between classical and quantum information processing. We formulate a natural generalization of them, called random access tests, which can be used to examine collective properties of collections of measurements. We show that the violation of a classical bound in a random access test is a signature of either measurement incompatibility or super information storability, and that we can use them to detect differences in different restrictions.

DETECTING NONCLASSICALITY IN RESTRICTED GENERAL PROBABILISTIC THEORIES

PI QF seminar 27.1.2022

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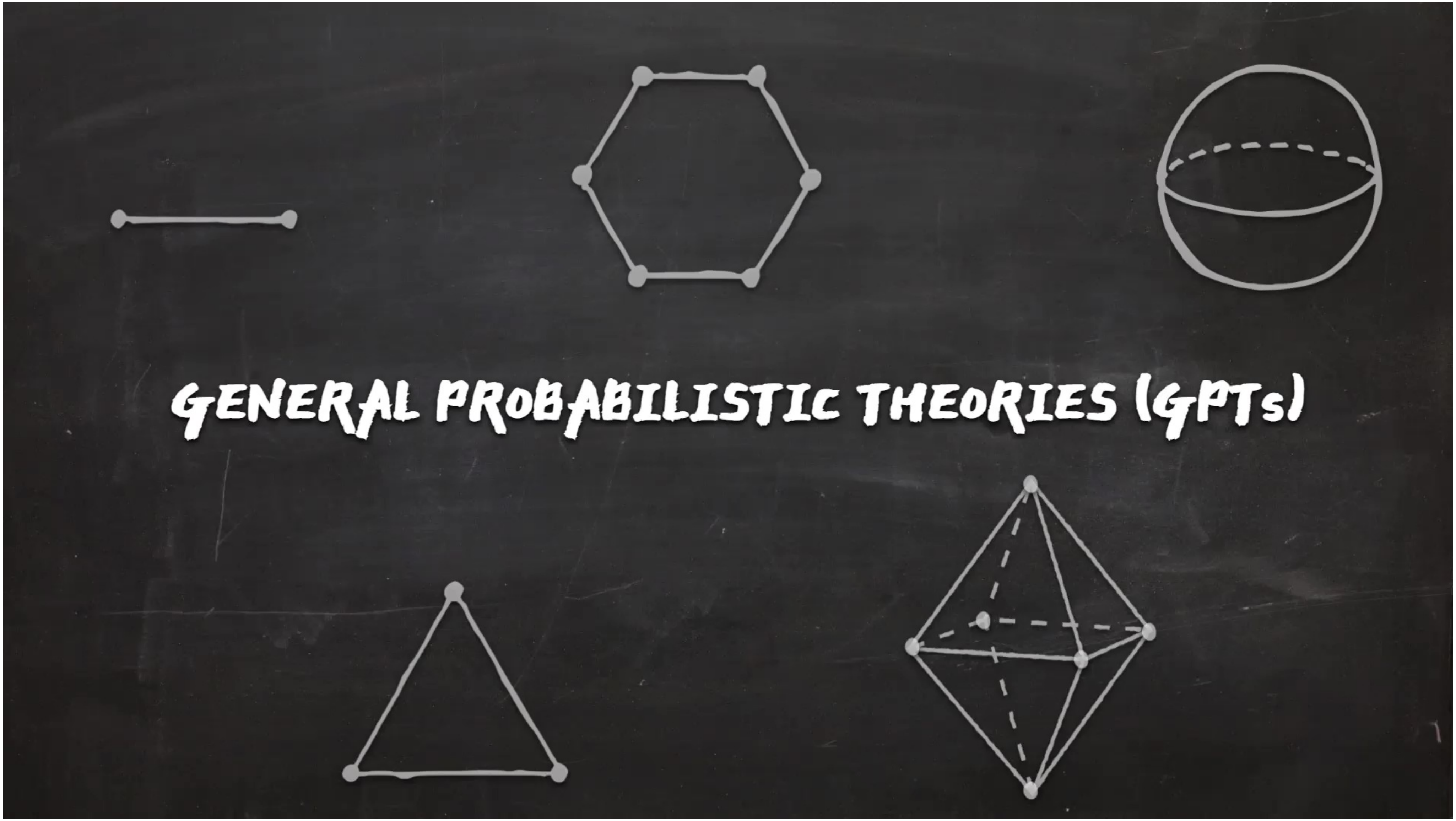
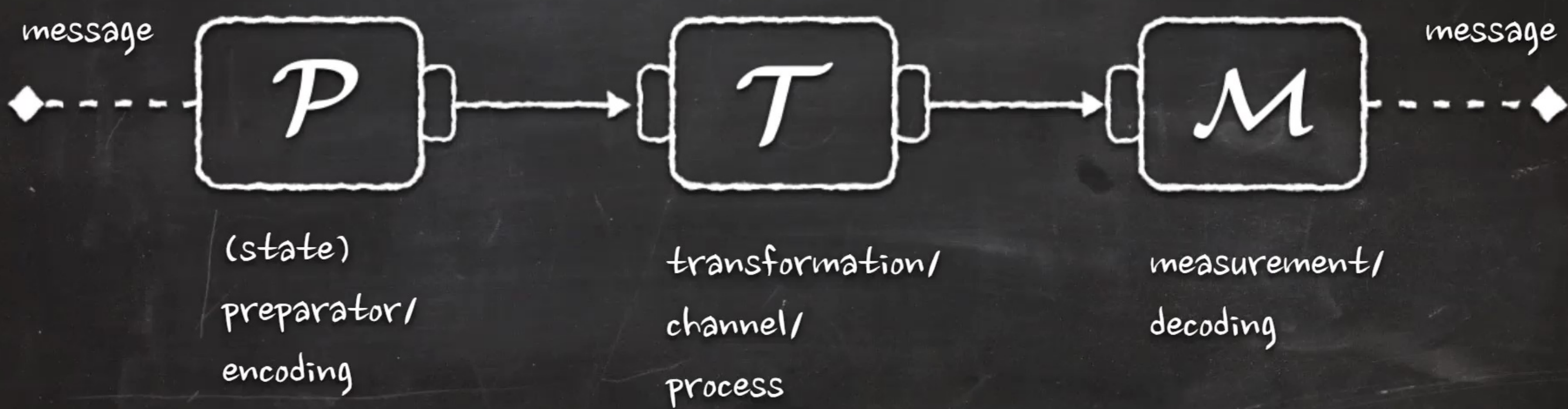
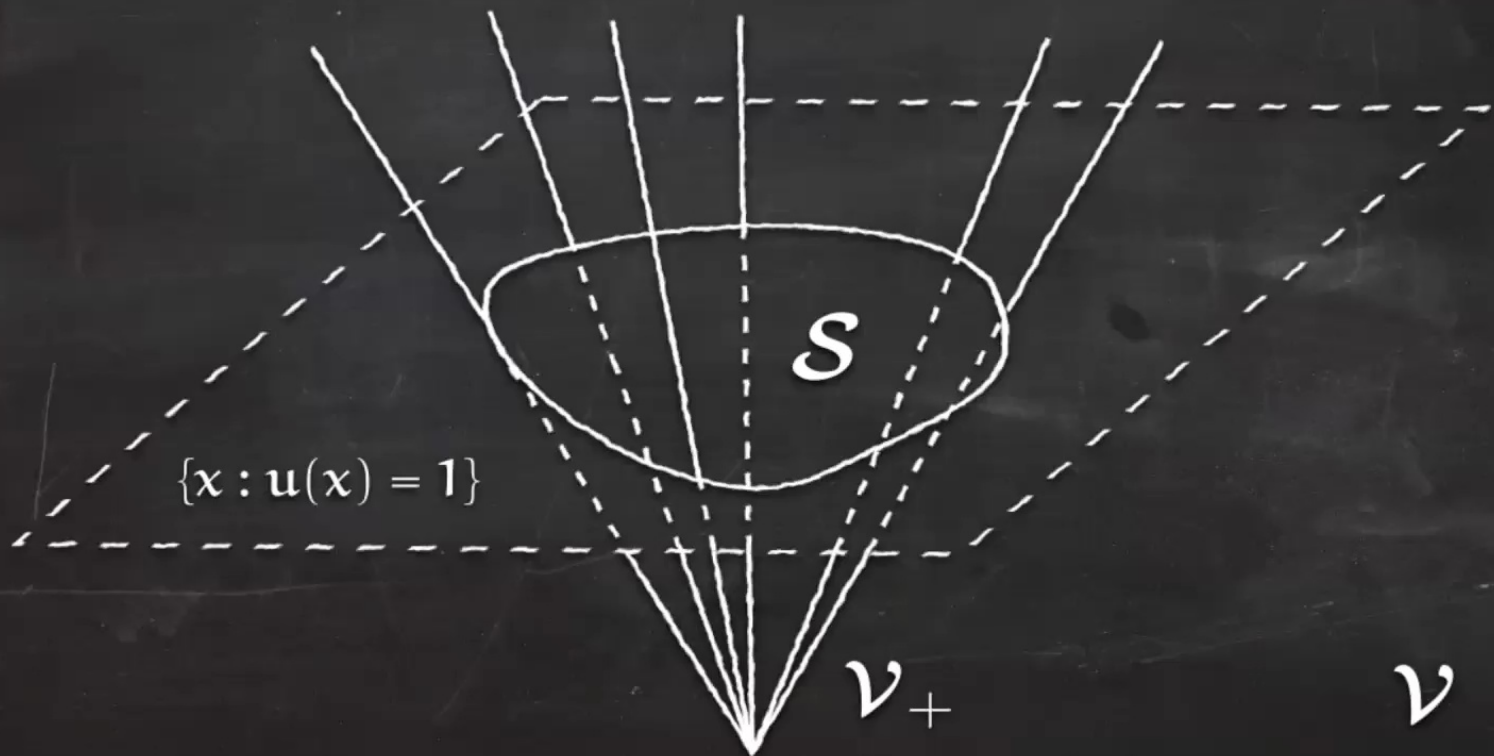

GENERAL PROBABILISTIC THEORIES (GPTs)

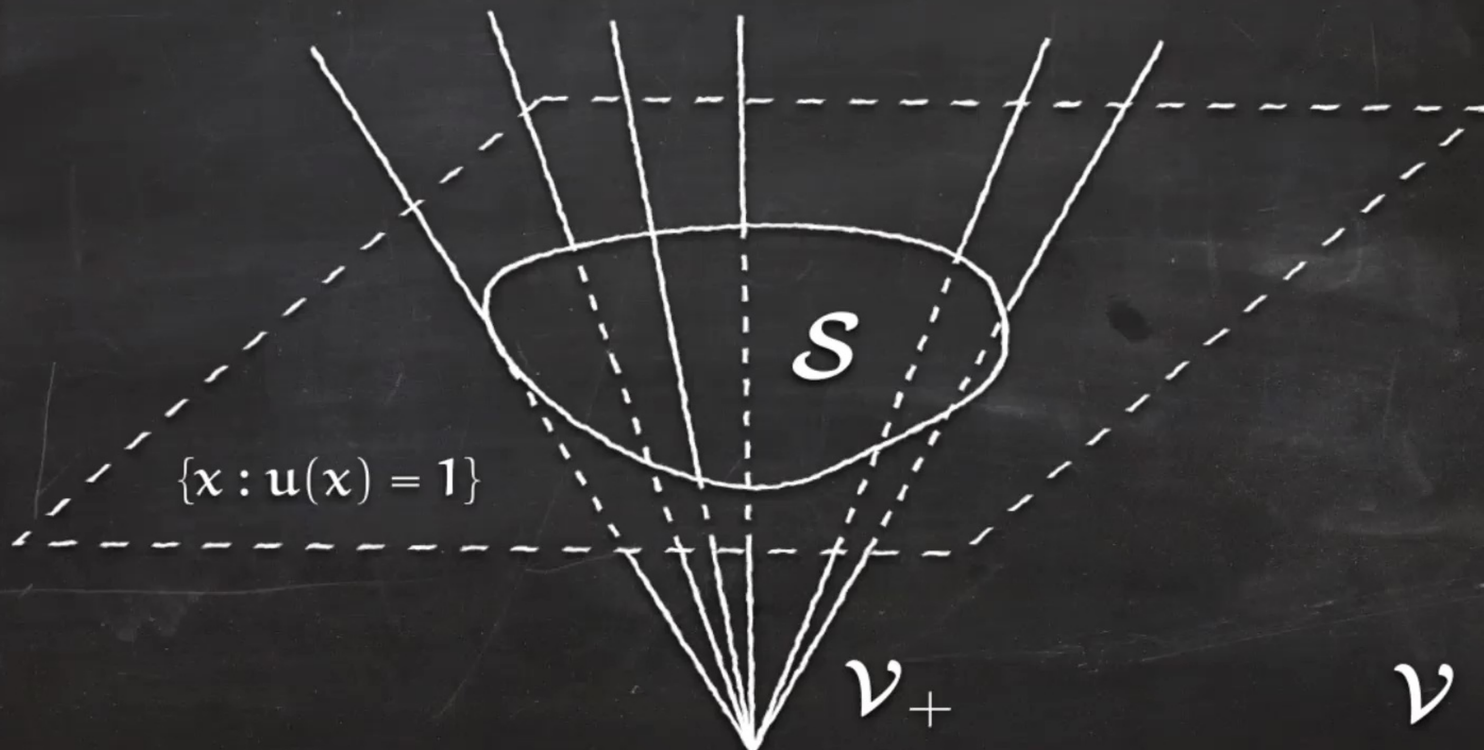
INFORMATION PROCESSING IN OPERATIONAL THEORIES



State space: compact <u>convex</u> subset of a finite-dimensional real vector space



State space: compact convex subset of a finite-dimensional real vector space



$$\mathcal{S} = \{x \in \mathcal{V} \,|\, x \geq 0,\ u(x) = 1\}$$

Effect space:

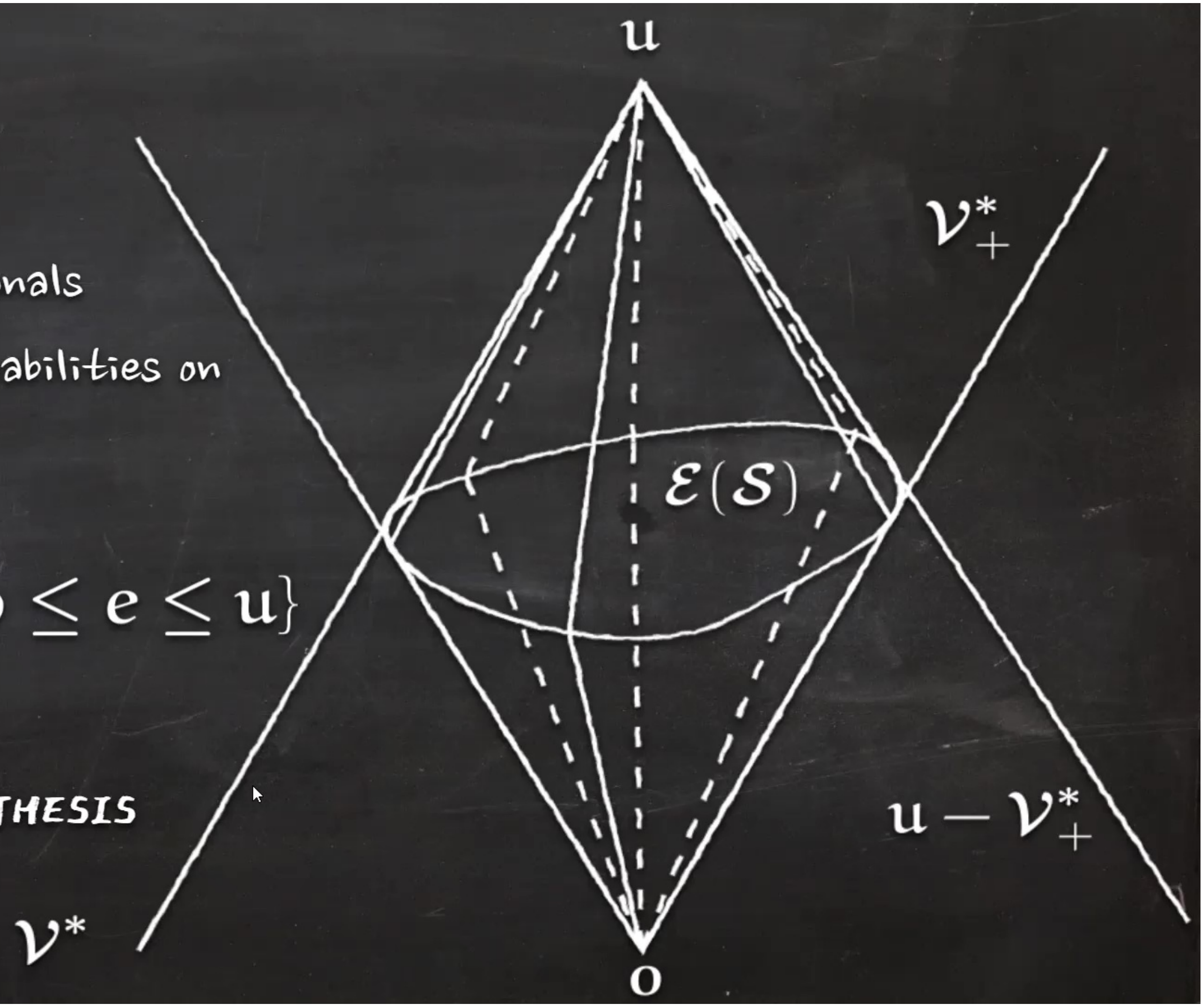
consists of affine functionals

$e : \mathcal{S} \rightarrow [0, 1]$ giving probabilities on

states

$$\mathcal{E}(\mathcal{S}) = \{e \in \mathcal{V}^* \mid o \leq e \leq u\}$$

NO-RESTRICTION HYPOTHESIS



Observables

- An observable A with a finite number of outcomes is a mapping $A : x \mapsto A_x$ from a finite outcome set Ω_A to the set of effects $\mathcal{E}(\mathcal{S})$ such that

$$\sum_{x \in \Omega_A} A_x(s) = 1$$

for all $s \in \mathcal{S}$.

- The set of observables on $\mathcal{S}$ is denoted by $\mathcal{O}(\mathcal{S})$.

Other elements of an operational theory

Channels

Instruments

Composite systems

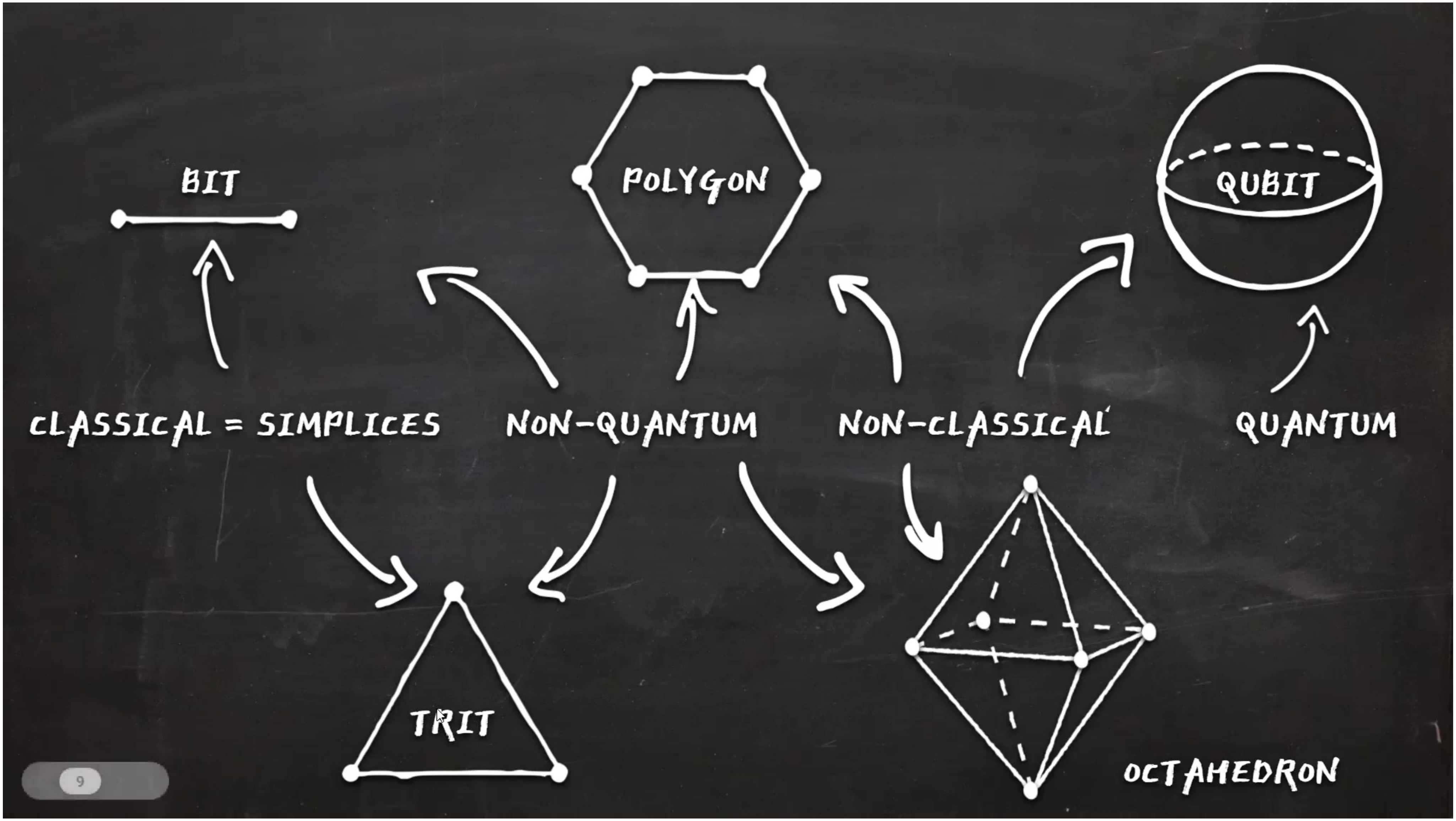
Quantum theory

Let $\mathcal{H}$ be a d-dimensional Hilbert space

States: $\mathcal{S}(\mathcal{H}) = \{\rho \in \mathcal{L}_s(\mathcal{H}) \mid \rho \geq 0,\ \mathrm{tr}[\rho] = 1\}$

Effects: $\mathcal{E}(\mathcal{S}(\mathcal{H})) \simeq \mathcal{E}(\mathcal{H}) = \{E \in \mathcal{L}_s(\mathcal{H}) \mid 0 \leq E \leq I\}$

Observables = POVMs: $A : x \mapsto A(x)$ such that $\sum_{x \in \Omega_A} A(x) = I$

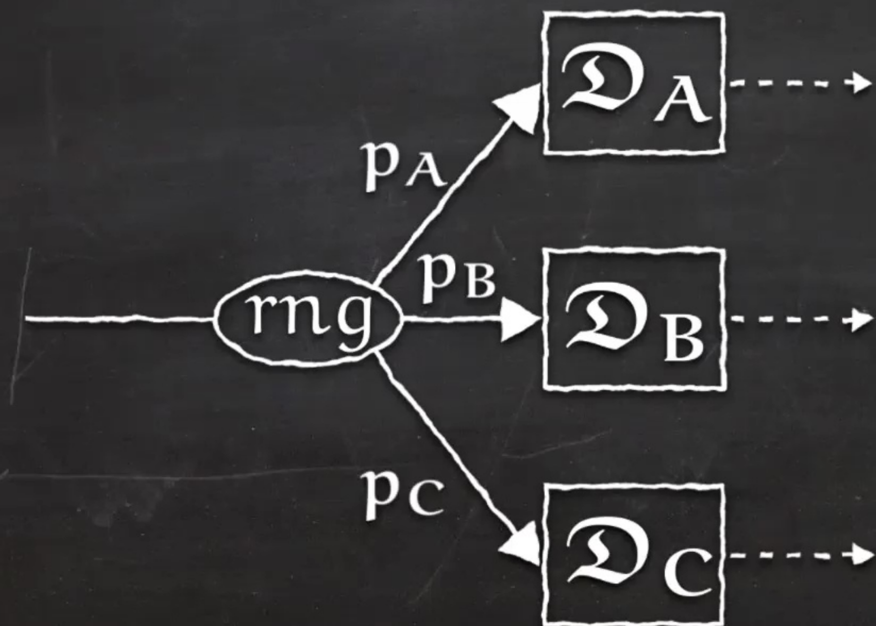

BIT
POLYGON
QUBIT
CLASSICAL = SIMPLICES
NON-QUANTUM
NON-CLASSICAL
QUANTUM
TRIT
OCTAHEDRON
9

SIMULATION OF MEASUREMENTS

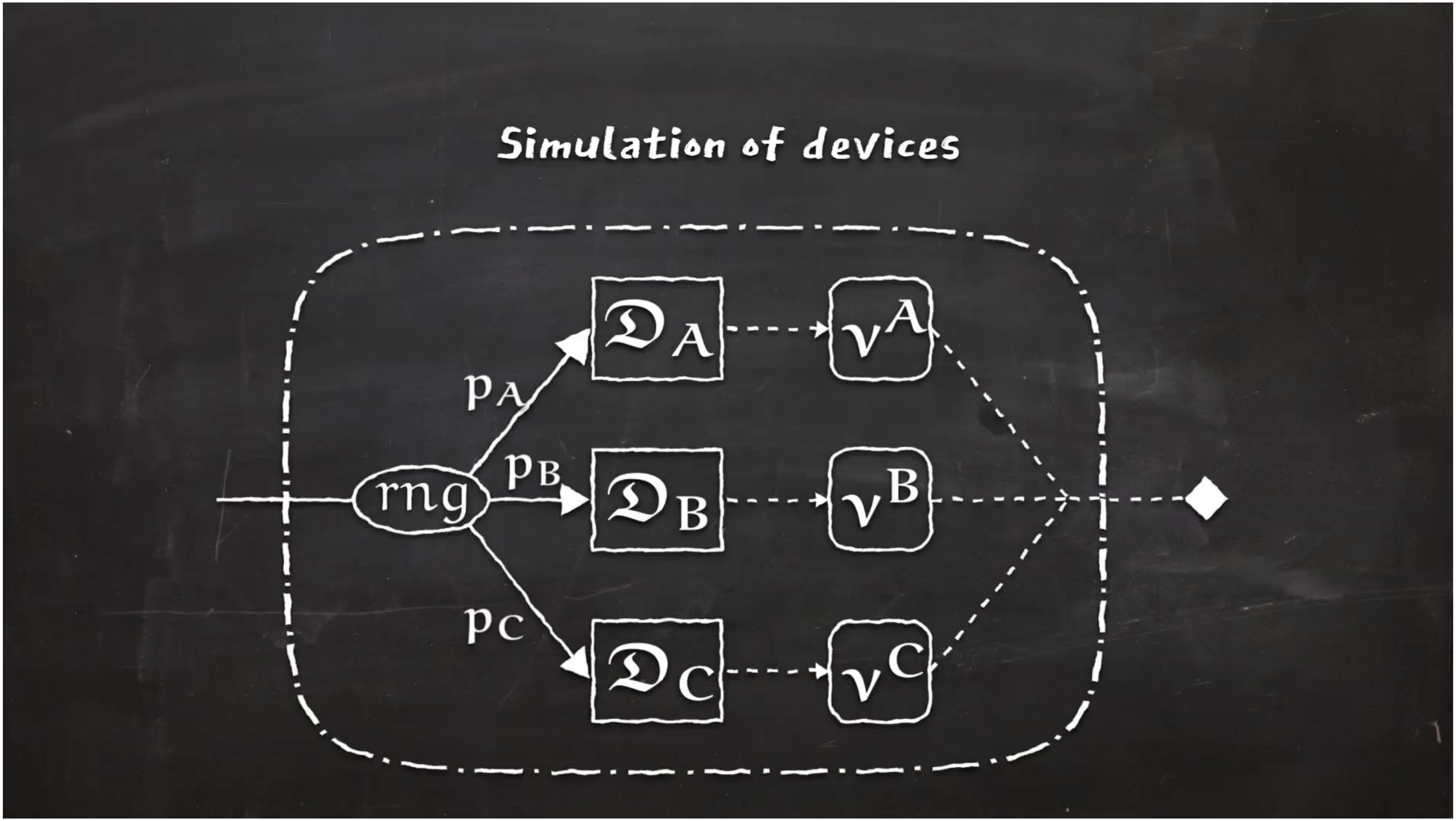
Filippov, Heinosaari and Leppäjärvi, Simulability of observables in general probabilistic theories, Phys. Rev. A 97, 062102 (2018).

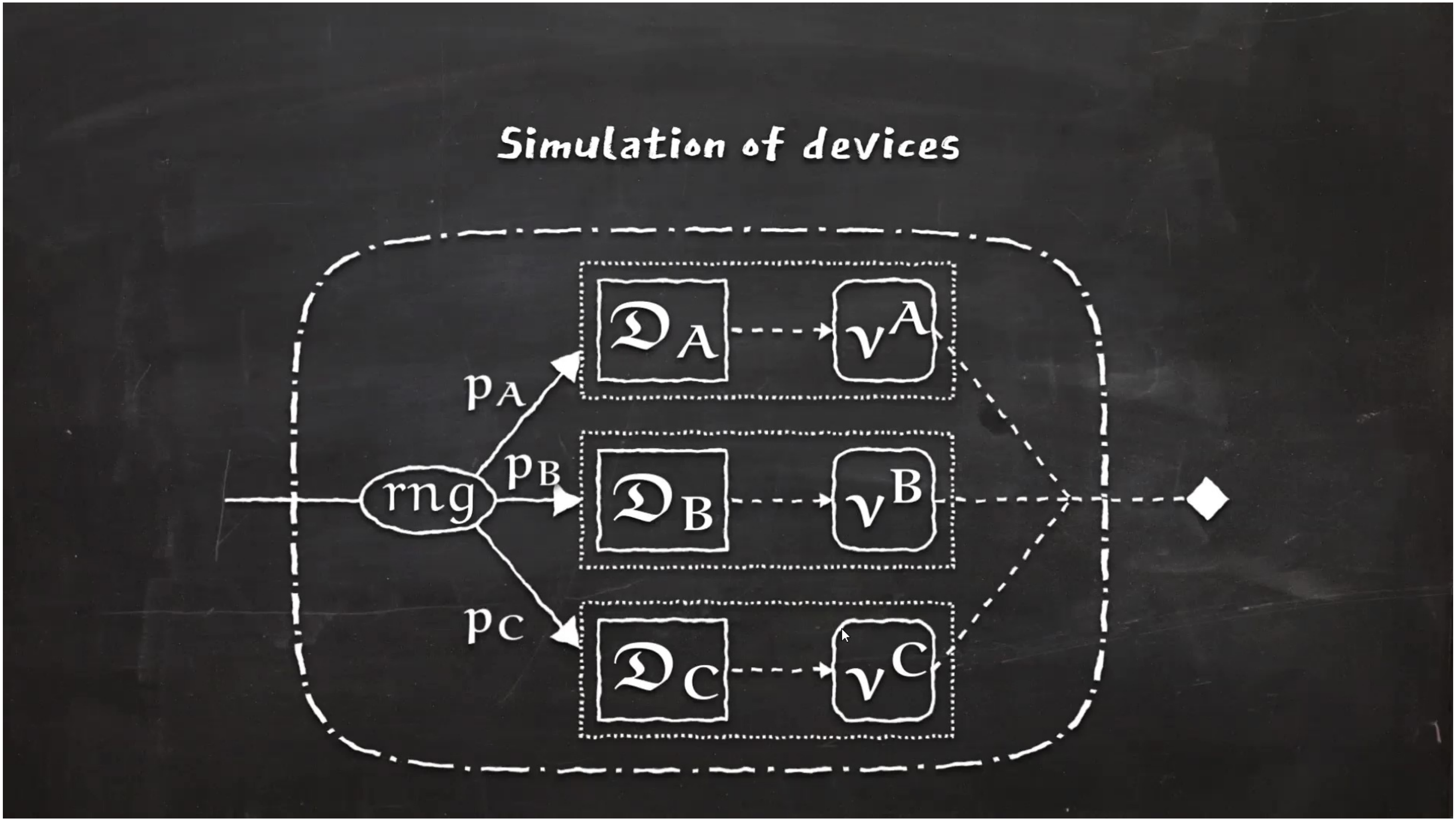
Oszmaniec, Guerini, Wittek and Acín, Phys. Rev. Lett. 119, 190501 (2017)
Guerini, Bavaresco, Terra Cunha and Acín, J. Math. Phys. 58, 092102 (2017)

Mixing of devices



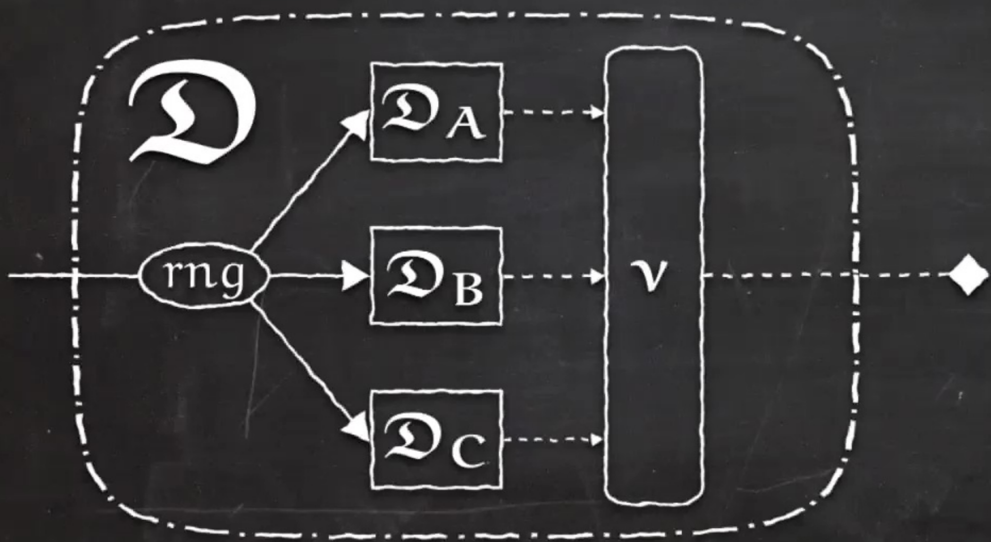
$$p_A + p_B + p_C = 1$$


Simulation of devices
rng
p_A
p_B
p_C
$\mathfrak{D}_A$
$\mathfrak{D}_B$
$\mathfrak{D}_C$
ν^A
ν^B
ν^C



Simulation of devices
rng
pA
pB
pC
DA
DB
DC
νA
νB
νC

Simulation of devices

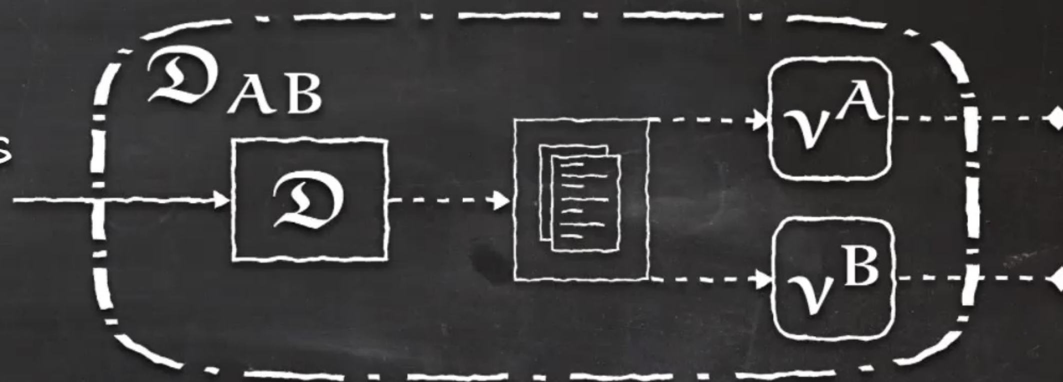


$$\subset \mathrm{sim}(\{\mathfrak{D}_A, \mathfrak{D}_B, \mathfrak{D}_C\})$$

Directions of applications

Limit the number of simulators

-> Compatibility of observables



Limit the number of outcomes

-> Effectively dichotomic measurements

$$\mathcal{O}_{2-\mathrm{eff}} = \mathrm{sim}(\{A \in \mathcal{O}(\mathcal{S}) \mid \#\Omega_A = 2\})$$

OPERATIONAL RESTRICTIONS

JOINT WORK WITH SERGEY FILIPPOV, STAN GUDDER AND TEIKO HEINOSAARI

FILIPPOV, GUDDER, HEINOSAARI AND LEPPÄJÄRVI, OPERATIONAL RESTRICTIONS IN GENERAL PROBABILISTIC THEORIES, FOUND. PHYS. 50, 850-876 (2020).

No-restriction hypothesis

$$\mathcal{E}(\mathcal{S}) = \{e : \mathcal{S} \to [0,1] \mid e \text{ affine}\}$$

all mathematically valid functions are taken to be physical effects in the theory

NO OPERATIONAL JUSTIFICATION

Janotta and Lal, Generalized probabilistic theories without the no-restriction hypothesis, Phys. Rev. A, 87:052131 (2013).

No-restriction hypothesis

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NO OPERATIONAL JUSTIFICATION

JANOTTA AND LAL, GENERALIZED PROBABILISTIC THEORIES WITHOUT THE NO-RESTRICTION HYPOTHESIS, PHYS. REV. A, 87:052131 (2013).

Also, how about observables?

Simulation closed restrictions

$^{*}\,\mathrm{ran}(\mathbf{A}) = \left\{ \sum_{x\in\tilde{\Omega}} \mathbf{A}_x \;\middle|\; \tilde{\Omega} \subseteq \Omega_{\mathbf{A}} \right\}$

Simulation closed restrictions

Def: An operational restriction of measurements is a subset $\tilde{\mathcal{O}} \subset \mathcal{O}(\mathcal{S})$ of observables that is simulation closed, i.e.,

$$\mathrm{sim}(\tilde{\mathcal{O}}) = \tilde{\mathcal{O}}$$

Simulation closed restrictions

For a subset of observables $\tilde{\mathcal{O}} \subseteq \mathcal{O}(\mathcal{S})$ we denote

$$\mathcal{E}_{\tilde{\mathcal{O}}} = \{ \mathrm{e} \in \mathcal{E}(\mathcal{S}) \mid \exists \mathrm{A} \in \tilde{\mathcal{O}} : \ \mathrm{e} \in \mathrm{ran}(\mathrm{A}) \}$$

- the set of available effects

For a subset of effects $\tilde{\mathcal{E}} \subseteq \mathcal{E}(\mathcal{S})$ we denote

$$\mathcal{O}_{\tilde{\mathcal{E}}} = \{ \mathrm{A} \in \mathcal{O}(\mathcal{S}) \mid \mathrm{ran}(\mathrm{A}) \subset \tilde{\mathcal{E}} \}$$

- the induced set of feasible observables

$$^{*}\,\mathrm{ran}(\mathrm{A}) = \left\{ \sum_{x \in \tilde{\Omega}} \mathrm{A}_x \mid \tilde{\Omega} \subseteq \Omega_{\mathrm{A}} \right\}$$

Types of restrictions

$(R1)\quad \tilde{\mathcal{O}} = \mathcal{O}_{\tilde{\mathcal{E}}}$ for some $\tilde{\mathcal{E}} \subset \mathcal{E}(\mathcal{S})$

restriction purely by effect restrictions

$(R2)\quad \mathcal{E}_{\tilde{\mathcal{O}}} = \mathcal{E}(\mathcal{S})$ but $\tilde{\mathcal{O}} \neq \mathcal{O}(\mathcal{S})$

restriction purely on observables

$(R3)\quad \mathcal{E}_{\tilde{\mathcal{O}}} \subset \mathcal{E}(\mathcal{S})$ and $\tilde{\mathcal{O}} \neq \mathcal{O}_{\tilde{\mathcal{E}}}$ for any $\tilde{\mathcal{E}} \subset \mathcal{E}(\mathcal{S})$

restriction on both effects and observables

(R1): restrictions induced by effect restrictions

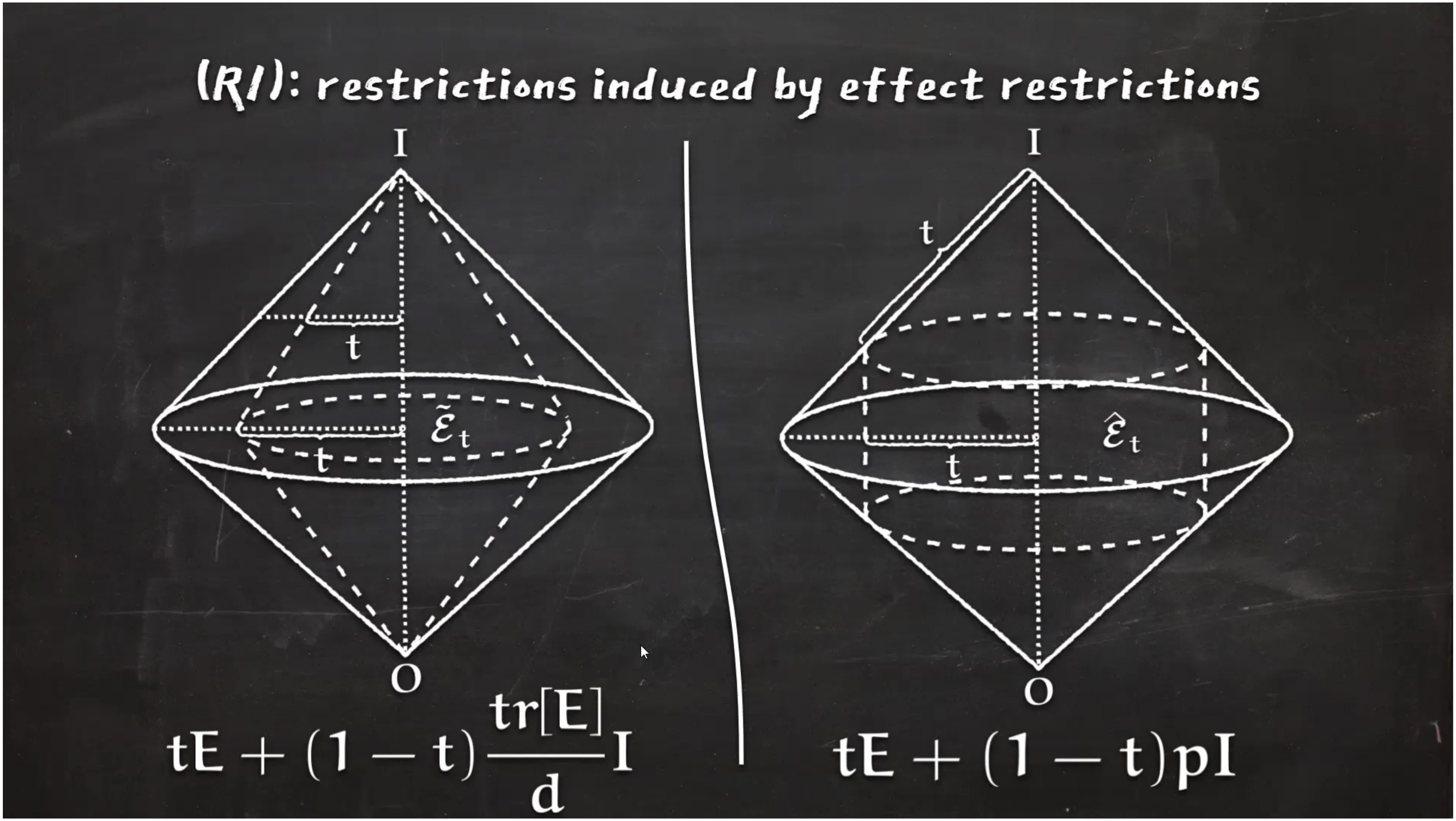
For a valid effect restriction $\tilde{\mathcal{E}} \subseteq \mathcal{E}(\mathcal{S})$ we require that

E1) $\mathbf{u} \in \tilde{\mathcal{E}}$

E2) $\forall \mathrm{e} \in \tilde{\mathcal{E}},\ \exists \mathrm{A} \in \mathcal{O}_{\tilde{\mathcal{E}}} :\ \mathrm{e} \in \mathrm{ran}(\mathrm{A})$

Prop: An effect restriction $\tilde{\mathcal{E}} \subseteq \mathcal{E}(\mathcal{S})$ satisfying E1) and E2) induces a simulation closed operational restriction $\mathcal{O}_{\tilde{\mathcal{E}}}$ if and only if $\tilde{\mathcal{E}}$ is convex.

This includes but is not limited to convex effect subalgebras of $\mathcal{E}(\mathcal{S})$.


(RI): restrictions induced by effect restrictions
I
t
t
$\tilde{\mathcal{E}}_t$
O
$tE + (1-t)\frac{tr[E]}{d}I$
I
t
t
$\hat{\mathcal{E}}_t$
O
$tE + (1-t)pI$

(R2): restrictions only on observables

$$\boxed{\text{(R2)} \quad \mathcal{E}_{\tilde{\mathcal{O}}} = \mathcal{E}(\mathcal{S}) \text{ but } \tilde{\mathcal{O}} \neq \mathcal{O}(\mathcal{S})}$$

Effectively dichotomic observables

$$\mathcal{O}_{2-\mathrm{eff}} = \mathrm{sim}(\{A \in \mathcal{O}(\mathcal{S}) \mid \#\Omega_A = 2\})$$

Prop: If $\tilde{\mathcal{O}} \subset \mathcal{O}(\mathcal{S})$ is an operational restriction of type (R2), then

$$\mathcal{O}_{2-\mathrm{eff}} \subset \tilde{\mathcal{O}}$$

(R3): restrictions on both effects and observables

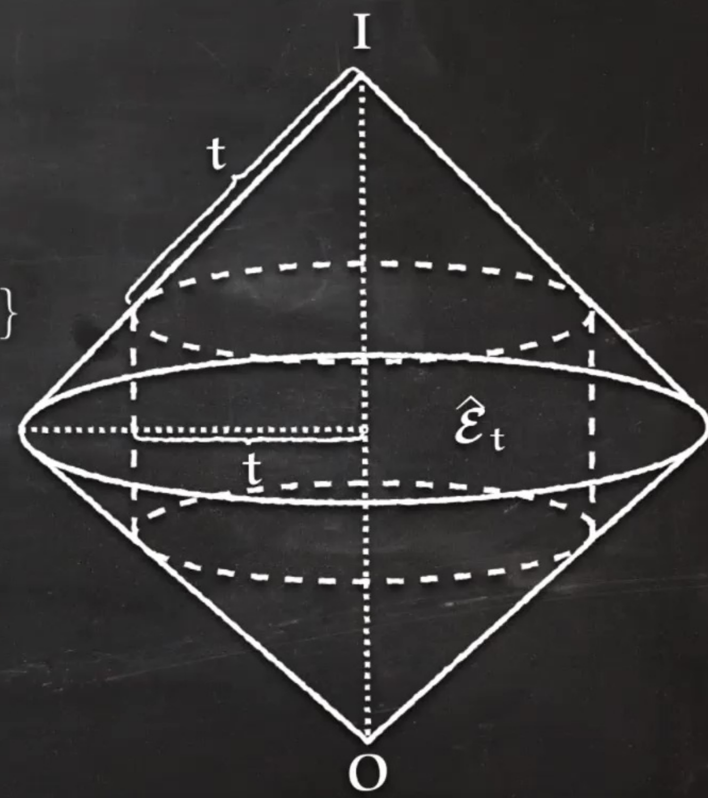
(not R1 nor R2)

Noisy observables (random noise)

$$\tilde{\mathcal{O}}_t = \{tA + (1-t)pu \mid A \in \mathcal{O}(\mathcal{S}), p \in \mathcal{P}(\Omega_A)\}$$

Available effects

$$\mathcal{E}_{\tilde{\mathcal{O}}_t} = \{te + (1-t)pe \mid e \in \mathcal{E}(\mathcal{S}), p \in [0,1]\}$$



(R3): restrictions on both effects and observables

(not R1 nor R2)

Noisy observables (random noise)

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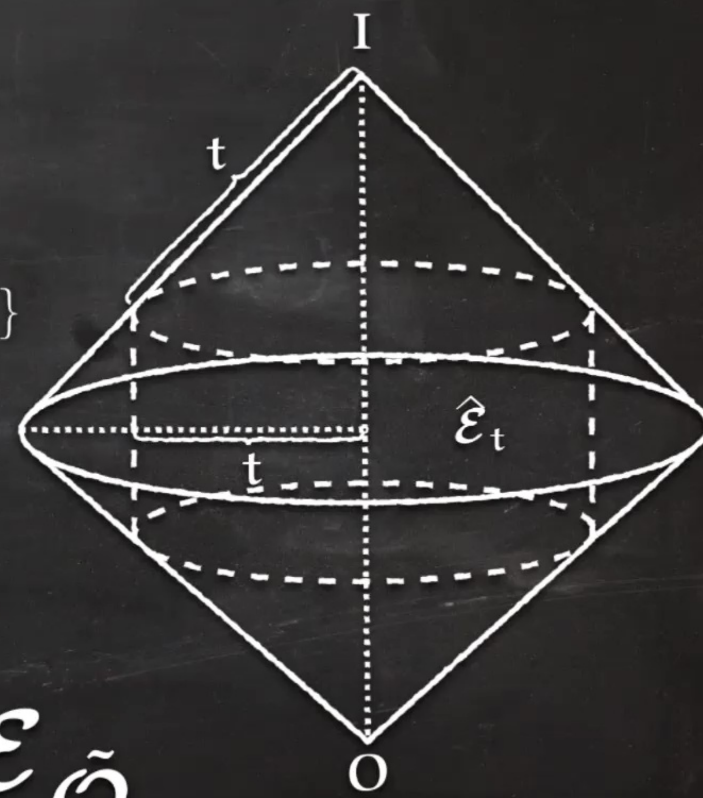
Available effects

$$\mathcal{E}_{\tilde{\mathcal{O}}_t} = \{te + (1-t)pe \mid e \in \mathcal{E}(\mathcal{S}), p \in [0,1]\}$$



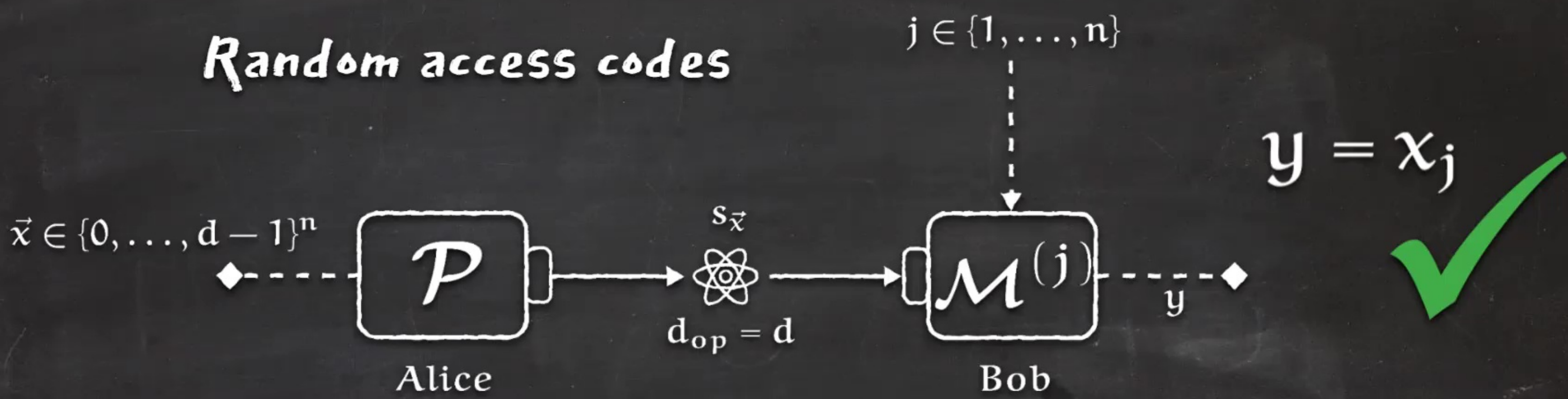
$\tilde{\mathcal{O}}_t$ IS NOT INDUCED BY $\mathcal{E}_{\tilde{\mathcal{O}}_t}$

DETECTING NONCLASSICALITY WITH RANDOM ACCESS TESTS

JOINT WORK WITH TEIKO HEINOSAARI

HEINOSAARI AND LEPPÄJÄRVI, RANDOM ACCESS TEST AS AN IDENTIFIER OF NONCLASSICALITY, ARXIV:2112.03781 [QUANT-PH] (2021).

Random access codes



Average success probability

$$\bar{P}^{(n,d)}\left(M^{(1)}, \ldots, M^{(n)}; \{s_{\vec{x}}\}_{\vec{x}}\right) = \frac{1}{nd^n} \sum_{\vec{x}} \left[M^{(1)}_{x_1} + \cdots + M^{(n)}_{x_n}\right](s_{\vec{x}})$$

Optimal: $$\bar{P}^{(2,d)}_q = \frac{1}{2}\left(1 + \frac{1}{\sqrt{d}}\right) > \frac{1}{2}\left(1 + \frac{1}{d}\right) = \bar{P}^{(2,d)}_c$$

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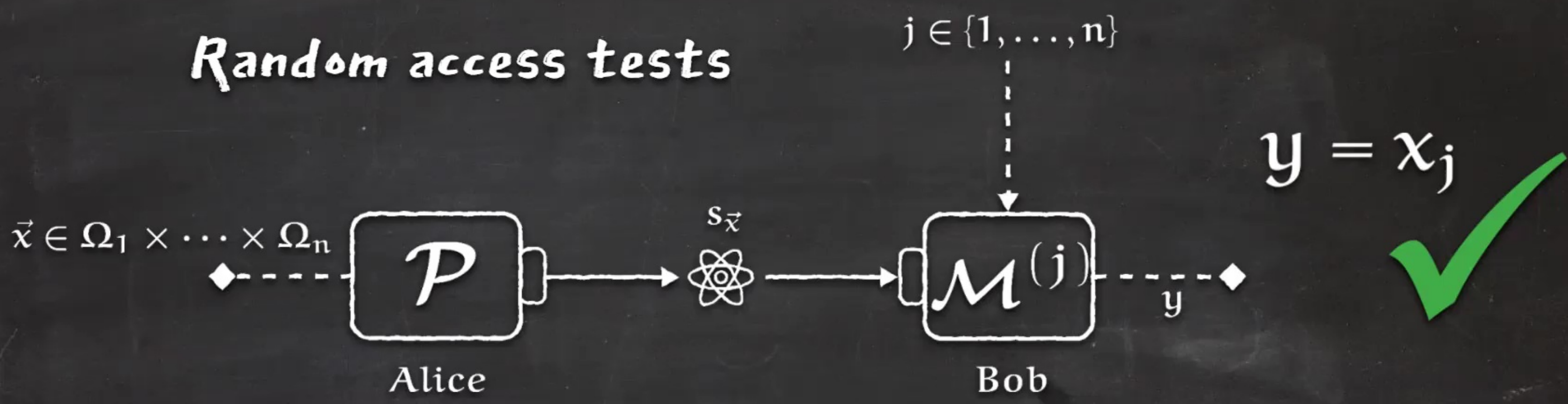
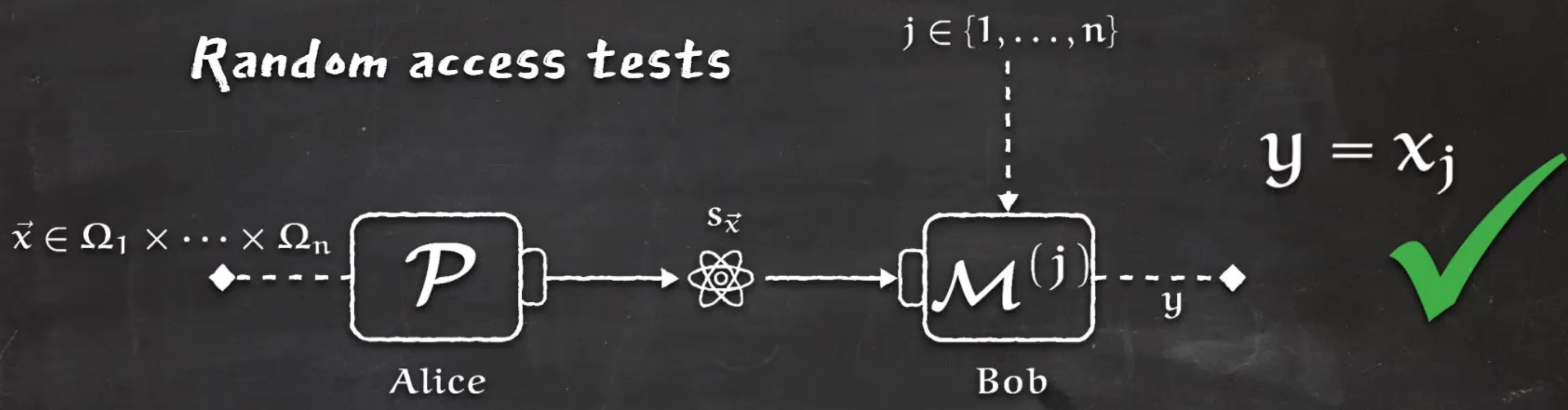
Random access tests



Average success probability ($\#\Omega_j = m_j$)

$$\bar{P}\left(M^{(1)}, \ldots, M^{(n)}; \{s_{\vec{x}}\}_{\vec{x}}\right) = \frac{1}{n m_1 \cdots m_n} \sum_{\vec{x}} \left[M^{(1)}_{x_1} + \cdots + M^{(n)}_{x_n} \right] (s_{\vec{x}})$$

Random access tests



Maximum average success probability for fixed observables

$$\bar{P}\left(M^{(1)}, \ldots, M^{(n)}\right) = \frac{1}{n m_1 \cdots m_n} \sum_{\vec{x}} \left\| M^{(1)}_{x_1} + \cdots + M^{(n)}_{x_n} \right\|$$

Information storability

Decoding power of an observable

$$\boxed{\lambda_{max}(A) = \sum_{x\in\Omega_A} \|A_x\| = \sum_{x\in\Omega_A} \sup_{s\in\mathcal{S}} A_x(s)}$$

$\dfrac{\lambda_{max}(A)}{\#\Omega_A}$ is the maximal probability of decoding $\#\Omega_A$ messages by using A

Information storability* of a subset $\tilde{\mathcal{O}} \subseteq \mathcal{O}(\mathcal{S})$

$$\boxed{\lambda_{max}(\tilde{\mathcal{O}}) = \sup_{A\in\tilde{\mathcal{O}}} \lambda_{max}(A)}$$

*Matsumoto and Kimura, Information storing yields a point-asymmetry of state space in general probabilistic theories, arXiv:1802.01162 [quant-ph], (2018).

Connecting RATs and incompatibility of observables

Def: Observables $M^{(1)}, \ldots, M^{(n)}$ are $\tilde{\mathcal{O}}$-compatible if they are compatible, i.e., jointly measurable, and their joint measurement belongs to $\tilde{\mathcal{O}} \subseteq \mathcal{O}(\mathcal{S})$.

Prop.: If two observables $M^{(1)}$ and $M^{(2)}$ with m_1 and m_2 outcomes respectively are $\tilde{\mathcal{O}}$-compatible, then

$$\bar{P}\left(M^{(1)}, M^{(2)}\right) \leq \frac{1}{2}\left(1 + \frac{\lambda_{max}(\tilde{\mathcal{O}})}{m_1 m_2}\right)$$

Detecting incompatibility

Prop.: If

$$\bar{P}\left(M^{(1)}, M^{(2)}\right) > \frac{1}{2}\left(1 + \frac{\lambda_{max}(\tilde{O})}{m_1 m_2}\right)$$

then $M^{(1)}$ and $M^{(2)}$ are $\tilde{O}$-incompatible.

Note: the observables can still be compatible in the unrestricted theory.

If $\lambda_{max}(\tilde{O}) = m_1 = m_2 = d$, then RHS equals $\bar{P}_c^{(2,d)}$!

Detecting incompatibility

Prop.: If

$$\bar{P}\left(M^{(1)}, M^{(2)}\right) > \frac{1}{2}\left(1 + \frac{\lambda_{max}(\tilde{O})}{m_1 m_2}\right)$$

then $M^{(1)}$ and $M^{(2)}$ are $\tilde{O}$-incompatible.

Note: the observables can still be compatible in the unrestricted theory.

If $\lambda_{max}(\tilde{O}) = m_1 = m_2 = d$, then RHS equals $\bar{P}_c^{(2,d)}$!

$\longrightarrow$ OPERATIONAL CONNECTION TO NONCLASSICALITY

Let $\tilde{\mathcal{O}} \subseteq \mathcal{O}(\mathcal{S})$ be a restriction with an operational dimension d.

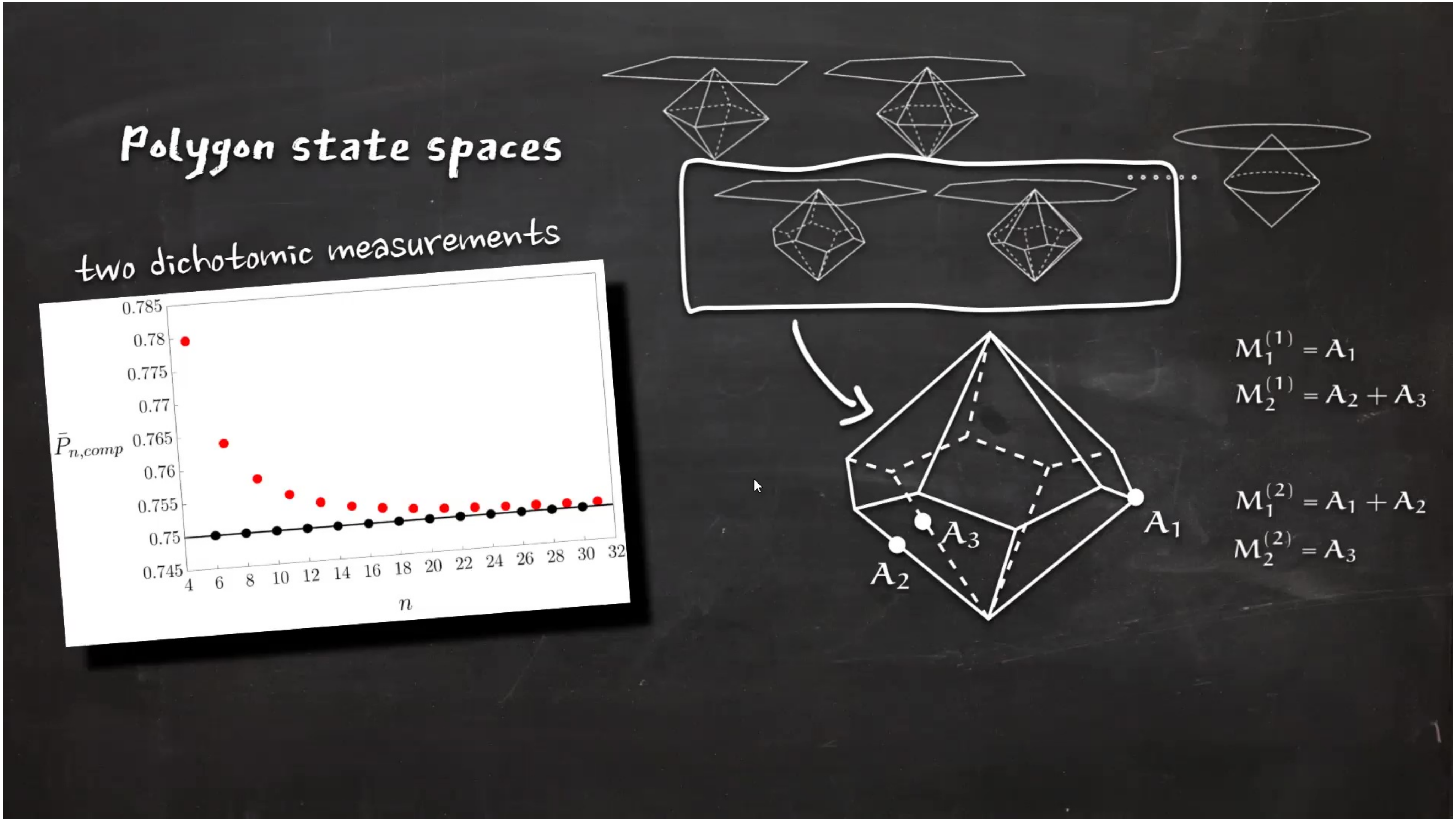
Let $\mathrm{M}^{(1)}, \mathrm{M}^{(2)} \in \tilde{\mathcal{O}}$ be two observables with $\mathrm{m}_1 = \mathrm{m}_2 = \mathrm{d}$ outcomes.

If $\bar{\mathrm{P}}\left(\mathrm{M}^{(1)}, \mathrm{M}^{(2)}\right) > \bar{\mathrm{P}}_c^{(2,\mathrm{d})}$, then

super information storability holds, i.e.,

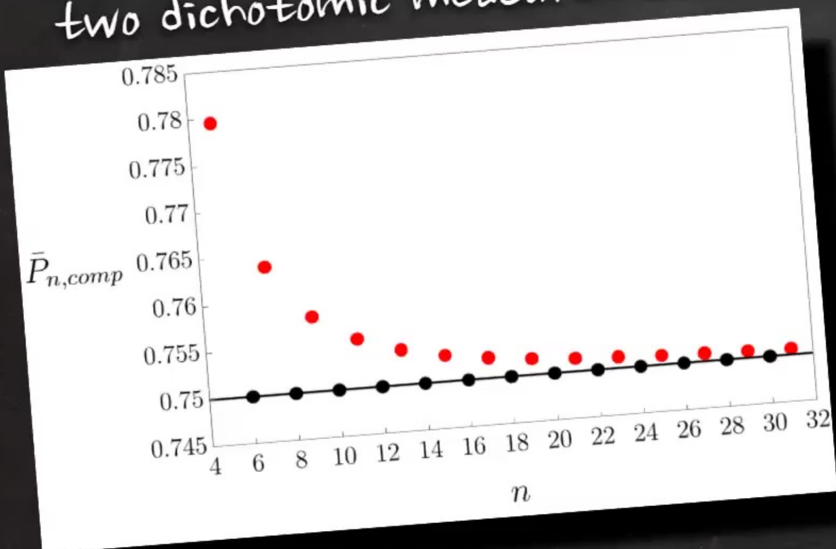
$$\lambda_{max}(\tilde{\mathcal{O}}) > \mathrm{d},$$

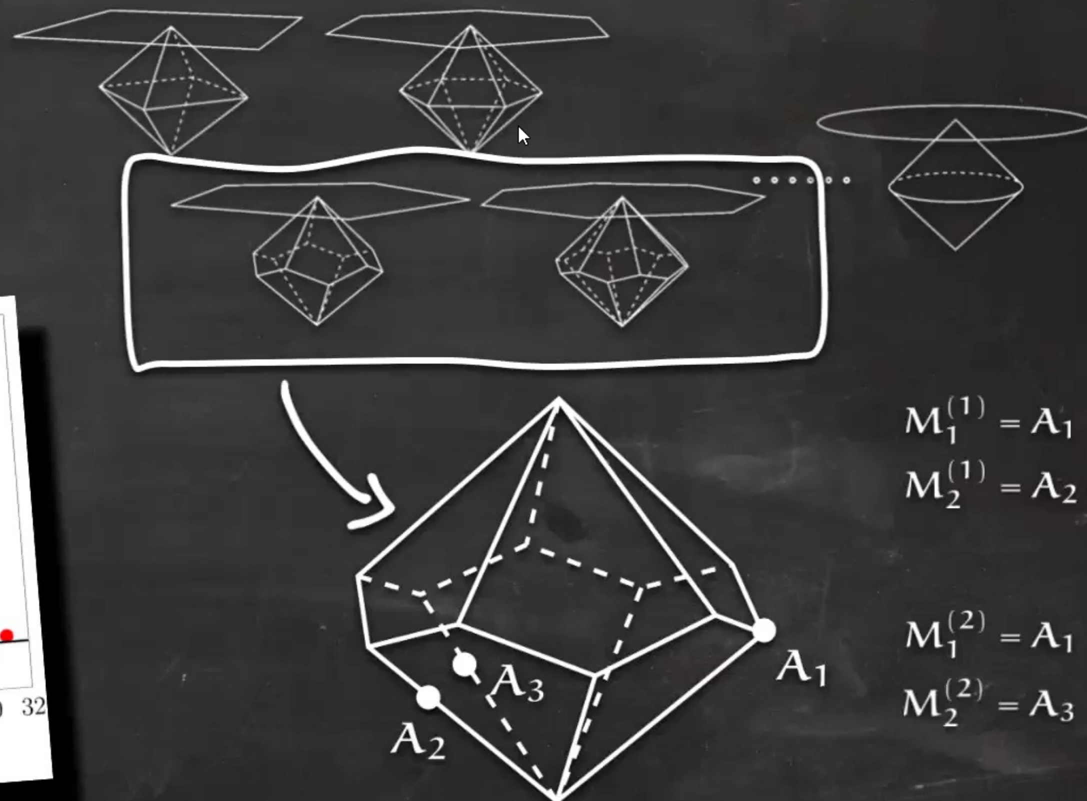
and/or $\mathrm{M}^{(1)}$ and $\mathrm{M}^{(2)}$ are $\tilde{\mathcal{O}}$-incompatible.


Polygon state spaces
two dichotomic measurements
$\bar{P}_{n,comp}$
0.785
0.78
0.775
0.77
0.765
0.76
0.755
0.75
0.745
4 6 8 10 12 14 16 18 20 22 24 26 28 30 32
n
A_1
A_2
A_3
$M_1^{(1)} = A_1$
$M_2^{(1)} = A_2 + A_3$
$M_1^{(2)} = A_1 + A_2$
$M_2^{(2)} = A_3$

Polygon state spaces

two dichotomic measurements

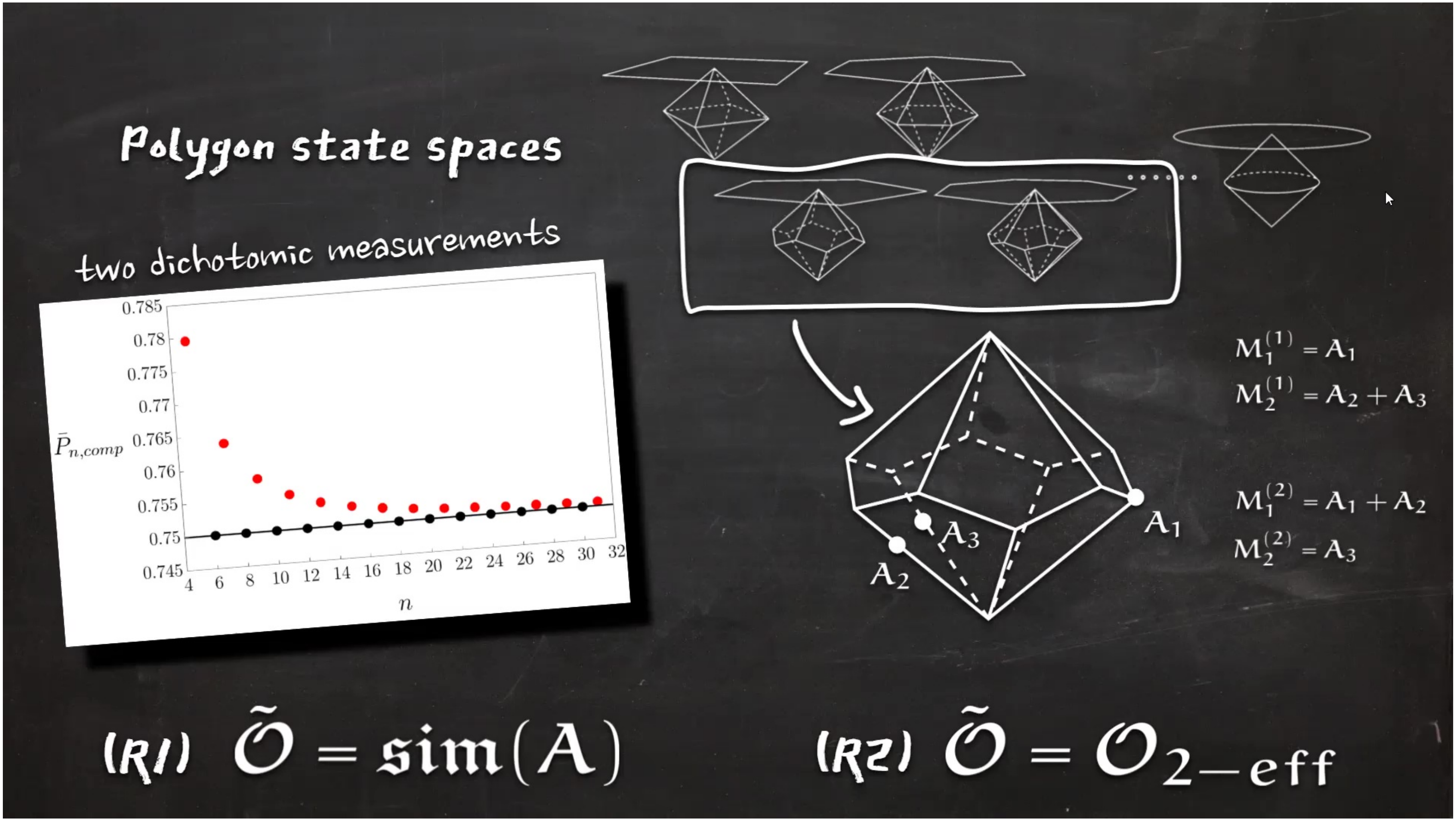




$$M_1^{(1)} = A_1$$
$$M_2^{(1)} = A_2 + A_3$$

$$M_1^{(2)} = A_1 + A_2$$
$$M_2^{(2)} = A_3$$

$$(R1)\quad \tilde{\mathcal{O}} = \mathbf{sim}(A)$$


Polygon state spaces
two dichotomic measurements
$\bar{P}_{n,comp}$
0.785
0.78
0.775
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0.765
0.76
0.755
0.75
0.745
4 6 8 10 12 14 16 18 20 22 24 26 28 30 32
n
$M_1^{(1)} = A_1$
$M_2^{(1)} = A_2 + A_3$
$M_1^{(2)} = A_1 + A_2$
$M_2^{(2)} = A_3$
A_1
A_2
A_3
(R1) $\tilde{\mathcal{O}} = \mathrm{sim}(A)$
(R2) $\tilde{\mathcal{O}} = \mathcal{O}_{2-\mathrm{eff}}$


THANK YOU!