Title: De Sitter scattering amplitudes in the Born approximation

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Abstract: A basic calculation in QFT is the construction of the Yukawa potential from a tree-level scattering amplitude. In the massless limit, this reproduces the 1/r potential. For gravity, scattering mediated by a massless graviton is thus consistent with the Newtonian potential.

In de Sitter spacetime, the cosmological constant gives rise to a mass-like term in the graviton propagator. This raises the question what the classical potential looks like when taking into account curvature effects.

In this talk, I will introduce an operator-based formalism to compute scattering amplitudes in curved spacetime, and I will show how to construct the Newtonian potential in a dS background. Remarkably, the potential gives rise to an additional repulsive force, and encodes the de Sitter horizon in a novel and non-trivial way.

De Sitter scattering amplitudes in the Born approximation

Chris Ripken

based on: Renata Ferrero & CR, arXiv:2112.03766
Perimeter Institute, 20-01-2022

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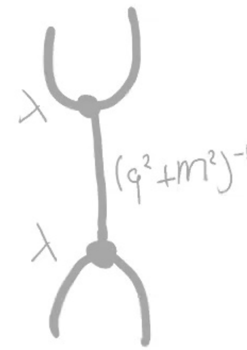
What is the Newtonian potential in dS?

- Tree-level scattering amplitude:
  - Born approximation: "1st quantized"
  - Flat, non-relativistic limit: Yukawa/Coulomb potential
- Expectations:
  - Expanding universe → repulsive force
  - Geodesic equation: $V(r) = -\frac{G_N m}{r} - \frac{1}{6}\Lambda r^2$
- Questions:
  - Graviton propagator $\sim (\Box + 2\Lambda)^{-1}$
    → massless or massive?
  - Effects from spin-2 particle?
  - Horizon effects?

Warmup exercise: the Yukawa potential

- ϕ^3-theory 2-2 scattering amplitude:

$$\mathcal{M} \propto \frac{\lambda^2}{q^2+m^2} \overset{m \gg \vec{q}}{\propto} \frac{-\lambda^2}{|\vec{q}|+m^2} \equiv \tilde{V}(\vec{q})$$



- Classical potential: take Fourier transform

$$V(\vec{x}) = \int \frac{\mathrm{d}^3\vec{q}}{(2\pi)^3} \tilde{V}(\vec{q}) \mathrm{e}^{i\vec{q}\cdot\vec{x}} = -\frac{\lambda^2}{2\pi^2 r} \int_0^\infty \mathrm{d}q \;\, q \sin(qr) \frac{1}{q^2+m^2}$$

$$\propto -\lambda^2 \frac{\mathrm{e}^{-m\,r}}{r}$$

Curved-space amplitudes (1)

- Action
  $$S = S_{\text{grav}} + S_{\text{gf}} + S_\phi + S_\chi$$
- Gravitational action:
  $$S_{\text{grav}} = \frac{1}{16\pi\, G_N}\int \sqrt{-g}\,(R - 2\Lambda)$$
- Scalar action:
  $$S_\phi = -\frac{1}{2}\int \sqrt{-g}\,\phi\left(\Box + m_\phi^2\right)\phi$$

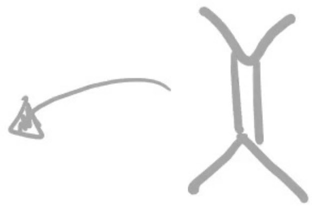
(similar for S_χ)

- Vertices:
  - $T^{(\phi\phi)} = \left.\frac{\delta^3 S}{\delta g \delta\phi\delta\phi}\right|_{g=\bar{g},\phi=\bar{\phi},\chi=\bar{\chi}}$
  - $T^{(\chi\chi)} = \left.\frac{\delta^3 S}{\delta g \delta\chi\delta\chi}\right|_{g=\bar{g},\phi=\bar{\phi},\chi=\bar{\chi}}$
- Propagator:
  - $\mathcal{G}^{(hh)} = \left.\left(\frac{\delta^2 S}{\delta g \delta g}\right)^{-1}\right|_{g=\bar{g},\phi=\bar{\phi},\chi=\bar{\chi}}$

$\bar{\phi} = 0$, $\bar{\chi} = 0$, $\bar{g}$ dS metric: solutions to EoM (on-shell)

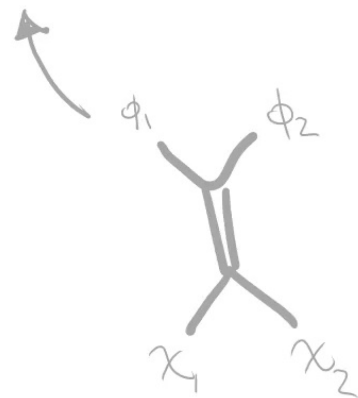
Curved space amplitudes (2)

- Amplitude:
  $A = T^{(\chi\chi)} \mathcal{G}^{(hh)} T^{(\phi\phi)}$



- Amplitude **functional**:

$$\mathcal{A}[\chi_1, \chi_2, \phi_1, \phi_2] = \int T^{(\chi\chi)}[\chi_1, \chi_2] \mathcal{G}^{(hh)} T^{(\phi\phi)}[\phi_1, \phi_2]$$



- Example: scalar ϕ^3-theory
  $\mathcal{A} = \lambda^2 \int \chi_1 \chi_2 (\Box + m^2)^{-1} \phi_1 \phi_2$

Essentials of dS

- Maximally symmetric, constantly curved space
  - $C_{\mu\nu\rho\sigma} = 0$
  - $R_{\mu\nu} = \frac{1}{4} R \, g_{\mu\nu}$
  - $R = 12H^2$

- FLRW universe:
  $ds^2 = -dt^2 + e^{2Ht} d\vec{x}^2$
- Conformally flat:
  $ds^2 = \frac{1}{H^2\eta^2}(-d\eta^2 + d\,\vec{x}^2)$

- Solution to Einstein equation (EoM):
  $$R = 4\Lambda \Rightarrow H^2 = \frac{1}{3}\Lambda$$
- On-shell background

Feynman rules in dS

- Propagator:
  - Defining equation:
    $$\left[\frac{\delta^2 S}{\delta g \delta g}\right]_{\mu\nu}^{\rho\sigma}\left[\mathcal{G}^{(hh)}\right]_{\rho\sigma}^{\alpha\beta} = \delta^{\alpha}_{(\mu}\delta^{\beta}_{\nu)}$$
  - Ansatz:
    $$\left[\mathcal{G}^{(hh)}\right]_{\rho\sigma}^{\alpha\beta} = \sum_i \mathcal{G}_i(H,\Box)\,[\mathcal{T}_i]_{\rho\sigma}^{\alpha\beta}$$

- Vertices:
  $$T_{\rho\sigma} \simeq g_{\rho\sigma}\phi\Box\phi + \nabla_\rho\phi\nabla_\sigma\phi + \cdots$$

- Contraction:
  $$\mathcal{A} = \int \nabla\chi\nabla\chi\mathcal{G}(H,\Box)\nabla\phi\nabla\phi + \cdots$$

→ how to commute ∇ and $\mathcal{G}(H,\Box)$?

$$[\mathcal{T}_i]_{\rho\sigma}^{\alpha\beta} \in \left\{\delta^{\alpha}_{(\rho}\delta^{\beta}_{\sigma)}, \bar{g}_{\rho\sigma}\bar{g}^{\alpha\beta}, \bar{g}_{\rho\sigma}\overline{\nabla}^{(\alpha}\overline{\nabla}^{\beta)}, \bar{g}^{\alpha\beta}\overline{\nabla}_{(\rho}\overline{\nabla}_{\sigma)}, \overline{\nabla}_{(\rho}\delta^{(\alpha}_{\sigma)}\overline{\nabla}^{\beta)}, \overline{\nabla}_{(\rho}\overline{\nabla}_{\sigma)}\overline{\nabla}^{(\alpha}\overline{\nabla}^{\beta)}\right\}$$

Ordering of derivatives

- Want: commute derivative $f(\Box)\nabla_\mu \boldsymbol{X}$
- Strategy:
  1. expand f by "inverse Laplace transform"
  2. Use Baker-Campbell-Hausdorff formula

- Example: scalar

$$f(\Box)\nabla_\mu\phi = \int_0^\infty \mathrm{d}s\, \tilde{f}(s)\mathrm{e}^{-s\Box}\nabla_\mu\phi = \int_0^\infty \mathrm{d}s\, \tilde{f}(s)\sum_{i\geq 0}\frac{(-s)^i}{i!}[\Box,\nabla_\mu]_i\, \mathrm{e}^{-s\Box}\phi$$

$$= \int_0^\infty \mathrm{d}s\, \tilde{f}(s)\sum_{i\geq 0}\frac{(-s)^i}{i!}\nabla_\mu[3H^2]^i\, \mathrm{e}^{-s\Box}\phi$$

$$= \nabla_\mu f(\Box + 3H^2)\phi$$

B. Knorr, CR: Phys. Rev. D. **103**, 105019

Feynman rules in dS

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  - Defining equation:
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- Vertices:
$$T_{\rho\sigma} \simeq g_{\rho\sigma}\phi\Box\phi + \nabla_\rho\phi\nabla_\sigma\phi + \cdots$$

- Contraction:
$$\mathcal{A} = \int \nabla\chi\nabla\chi\mathcal{G}(H,\Box)\nabla\phi\nabla\phi + \cdots$$

→ how to commute ∇ and $\mathcal{G}(H,\Box)$?

$$[\mathcal{T}_i]_{\rho\sigma}^{\alpha\beta} \in \left\{\delta^{\alpha}_{(\rho}\delta^{\beta}_{\sigma)}, \bar{g}_{\rho\sigma}\bar{g}^{\alpha\beta}, \bar{g}_{\rho\sigma}\overline{\nabla}^{(\alpha}\overline{\nabla}^{\beta)}, \bar{g}^{\alpha\beta}\overline{\nabla}_{(\rho}\overline{\nabla}_{\sigma)}, \overline{\nabla}_{(\rho}\delta^{(\alpha}_{\sigma)}\overline{\nabla}^{\beta)}, \overline{\nabla}_{(\rho}\overline{\nabla}_{\sigma)}\overline{\nabla}^{(\alpha}\overline{\nabla}^{\beta)}\right\}$$

dS amplitude functional

- Use commutation formula to
  - Compute $\mathcal{G}_i(H,\Box)$
  - Contract $T^{(\chi\chi)}\mathcal{G}^{(hh)}T^{(\phi\phi)}$
- Use on-shell relation $\Box\phi = -m_\phi^2\phi$
- Result:

$$T_{\mu\nu}[\phi_1,\phi_2] = \frac{1}{2}(\nabla_\mu\nabla_\nu\phi_1)\phi_2 + \frac{1}{2}\phi_1(\nabla_\mu\nabla_\nu\phi_2)$$

$$\mathcal{A}[\chi_1,\chi_2,\phi_1,\phi_2] = -16\pi G_N \int T_{\mu\nu}[\chi_1,\chi_2](\Box + 2H^2)^{-1}T^{\mu\nu}[\phi_1,\phi_2]$$

$$-\frac{1}{4}\chi_1\chi_2\left(\frac{1}{2}\Box + \frac{m_\chi^2 m_\phi^2}{\Box + 2H^2} + \frac{m_\chi^2 m_\phi^2}{\Box - 6H^2}\right)\phi_1\phi_2$$

- Gauge invariant iff fields+background metric are on-shell
- "massive propagators:" non-local operators on dS → hard to recognize tachyons

Compute amplitude: the very massive limit

- Want to compute $f(\square)\phi_1\phi_2$
- How to distribute $\square$?

$\rightarrow$ nonrelativistic limit $\mu_\phi = \frac{m_\phi}{H} \gg 1$

- Solve EoM:

Hankel function (Bunch-Davies vacuum)

$$\phi(\eta, \vec{x}) = \eta^{3/2} H_{i\sqrt{\mu^2 - \frac{9}{4}}}(-p\eta) \mathrm{e}^{-i\,\vec{p}\cdot\vec{x}}$$

- Distribution of $\square$:

$$\square\,\phi_1\phi_2 = \left(-2m_\phi^2 + 2\,\vec{p}_1\cdot\vec{p}_2\right)\phi_1\phi_2 - 2H^2\eta^2\left(\partial_\eta\phi_1\right)\left(\partial_\eta\phi_2\right)$$

$$= \left(H^2\eta^2\,\vec{q}^{\,2} + \mathcal{O}\left(\frac{1}{\mu^0}\right)\right)\phi_1\phi_2$$

$\vec{q} = \vec{p}_1 - \vec{p}_2$

Compute amplitude: resolve the propagator

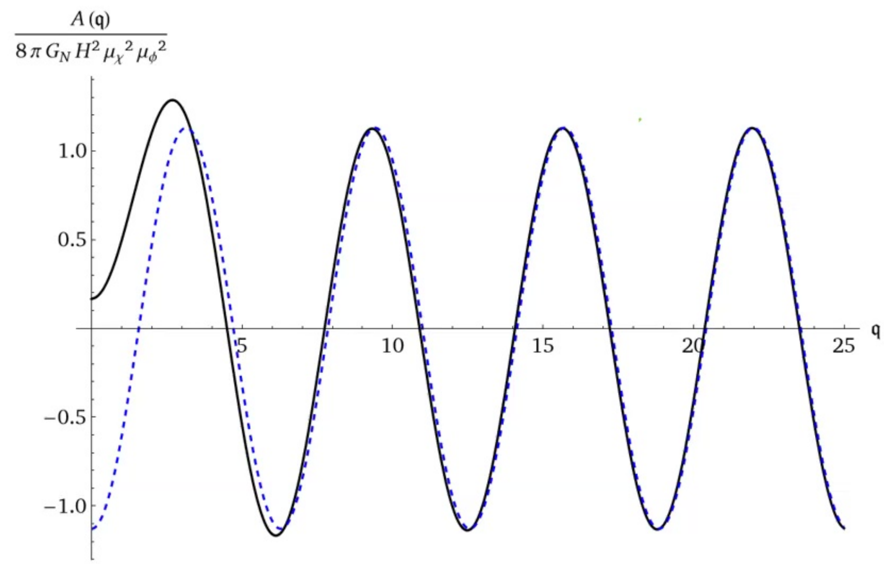
- Compute propagator with mass parameter $z = \zeta H^2$
- Ansatz:
  - $(\Box - \zeta H^2)^{-1}\phi_1\phi_2 = \left(G_0(\eta;\zeta) + \mathcal{O}\left(\frac{1}{\mu}\right)\right)\phi_1\phi_2$
  - $(\Box - \zeta H^2)^{-1}T_{00}[\phi_1,\phi_2] = \left(G_{00}(\eta;\zeta)\mu^2 + \mathcal{O}\left(\frac{1}{\mu^0}\right)\right)\phi_1\phi_2$
  - $(\Box - \zeta H^2)^{-1}T_{i0}[\phi_1,\phi_2] = \left([G_1(\eta;\zeta)\vec{p}_{1i} + G_2(\eta;\zeta)\vec{p}_{2i}]\,\mu^2 + \mathcal{O}\left(\frac{1}{\mu^0}\right)\right)\phi_1\phi_2$
  - 5 functions for $T_{ij}[\phi_1,\phi_2]$
- Strategy:
  - Act with $(\Box - \zeta H^2)$ on $G_i(\eta)\phi_1\phi_2$
  - Distribute $\Box$ over $G_i(\eta)$ and $\phi_1\phi_2$
  - Use $(\Box - \zeta H^2)(\Box - \zeta H^2)^{-1}\phi_1\phi_2 = \phi_1\phi_2$

→ result: system of coupled, 2nd order inhomogeneous differential equations

$$T_{\mu\nu}[\phi_1,\phi_2] = \frac{1}{2}(\nabla_\mu\nabla_\nu\phi_1)\phi_2 + \frac{1}{2}\phi_1(\nabla_\mu\nabla_\nu\phi_2)$$

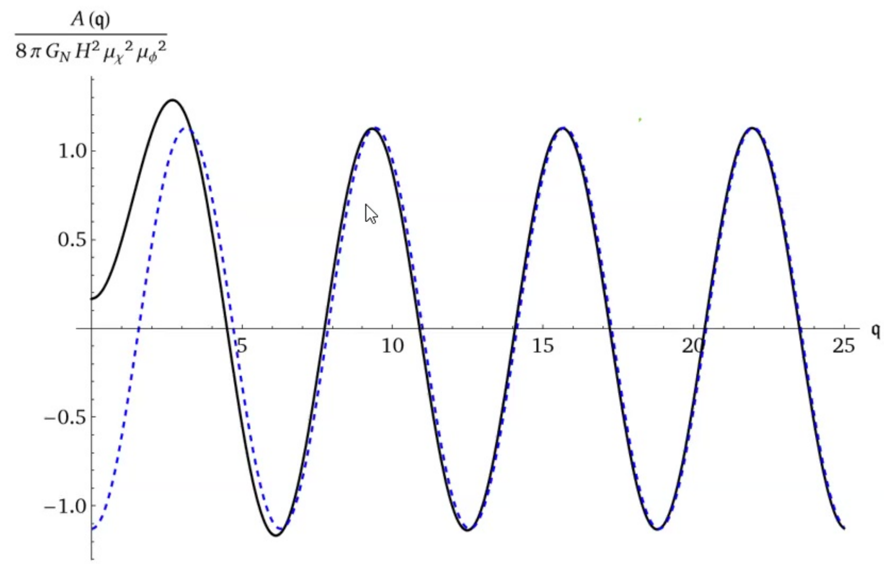
Scattering amplitude (1)

- Solve differential equations using power series ansatz (Frobenius method)
- Require analyticity
- function of $\mathfrak{q} = |\eta\vec{q}|$ only: central potential
- Solution: $\mathrm{A}(\mathfrak{q}) = [\mathrm{many}\ \ {}_1F_2\mathrm{s}]$



Scattering amplitude (2)

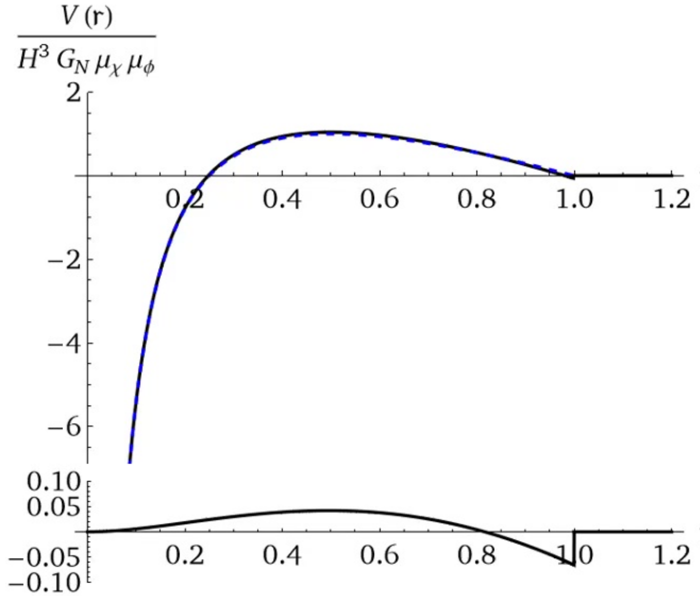
- finite limit $\mathfrak{q} \to 0$: no IR divergence
- $A(\mathfrak{q}) \xrightarrow{\mathfrak{q}\to\infty} -1.13 \cos \mathfrak{q}$:
  - Discrete forbidden momentum transfer
  - “particle in box” bounded by dS horizon
  - Period $\sim$ Hubble energy flat limit: continuum



Scattering potential (1)

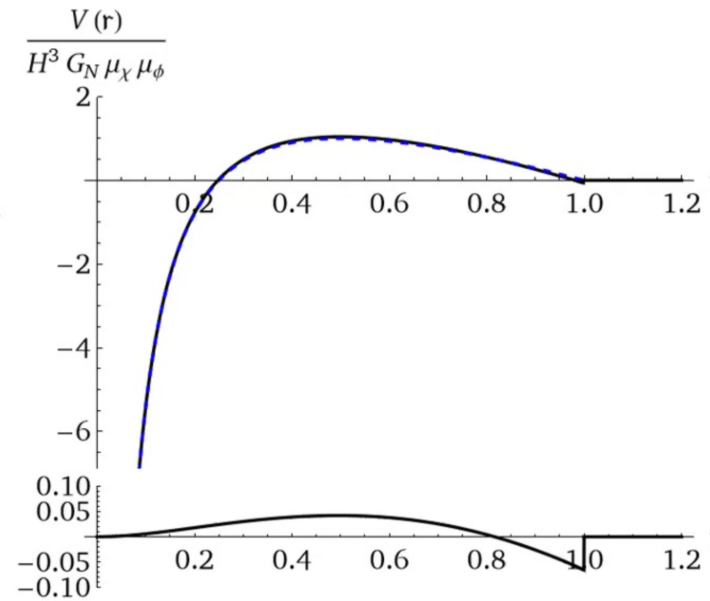
- Fourier transform:
- $\mathfrak{r} = |\eta^{-1} r|$
- $V(\mathfrak{r}) \sim \frac{1}{2\pi^2 \mathfrak{r}} \int_0^\infty \mathrm{d}\mathfrak{q}\, \mathfrak{q} \sin(\mathfrak{q}\, \mathfrak{r})\, A(\mathfrak{q})$

$$= \begin{cases} [\text{rational}] + [\ {}_2F_1\ \ \text{s}] & \mathfrak{r}^2 < 1 \\ 0 & \mathfrak{r}^2 > 1 \end{cases}$$



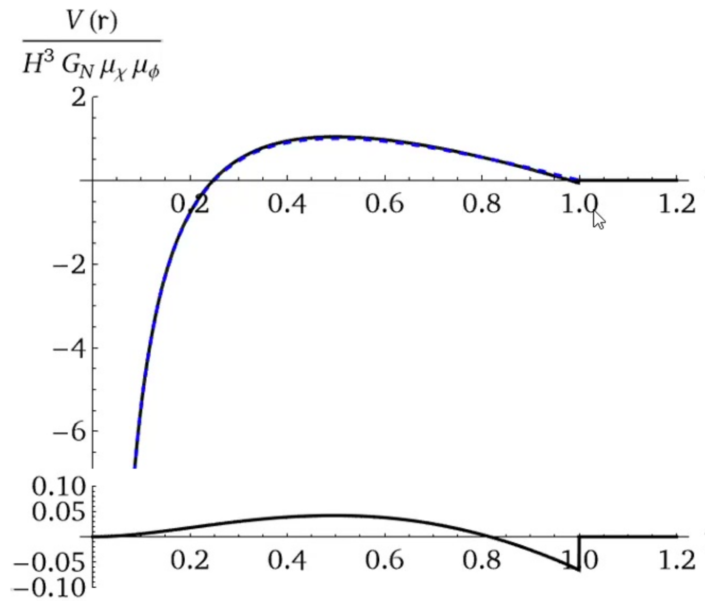
Scattering potential (2)

- $V(\mathrm{r}) \sim G_N m_\phi m_\chi \left(-\frac{1}{\mathrm{r}} + 5 - 4\mathrm{r}\right)$
- Good approximation
- 1/r term: classical regime
- Linear term: repulsive force → expansion of the universe
- $\neq V_{\mathrm{geodesic}}(r)$:
  - Heavy-mass ≠ light test particle
  - Double-BH geometry?



Scattering potential (3)

- $V(\mathfrak{r} > 1) = 0$:
- No causal interaction beyond dS horizon
- Bounded support at large distances: no IR divergence
- Discontinuity:
  - Analogy with ED: surface energy density?
  - Tunneling?
  - Thermodynamic description?



Summary

- Amplitude functional
  - dS background
  - Functional commutator relations
- Amplitude & potential
  - Large-mass limit
  - Propagator acting on products of fields
- Amplitude properties
  - No IR divergence
  - Oscillations
- Potential properties
  - Repulsive force at large distances
  - Vanishing outside dS horizon