Title: The Page-Wootters formalism: Where are we now?

Speakers: Alexander Smith

Series: Quantum Foundations

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Abstract: General relativity does not distinguish a preferred reference frame, and conservatively one ought to expect that its quantization does not necessitate such background structure. However, this desire stands in contrast to orthodox formulations of quantum theory which rely on a background time parameter external to the theory, and in the case of quantum field theory a spacetime foliation. Such considerations have led to the development of the Page-Wootters formalism, which seeks to describe motion relative to a reference frame internal to a quantum theory that encompasses both the system of interest and employed reference frame. I will begin by reviewing a modern formulation of the Page-Wootters formalism in terms of Hamiltonian constraints, generalized coherent states, and covariant time observables. I will then present Kuchar's criticisms of the Page-Wootters formalism, and discuss their resolution by showing the equivalence between the formalism and relational Dirac observables. These Dirac observables will then be used to introduce a gauge-invariant, relational notion of subsystems and entanglement. Finally, a field-theoretic extension of the Page-Wootters formalism will be introduced and used to recover the Schwinger-Tomonaga equation.

Zoom Link: https://pitp.zoom.us/j/91420728439?pwd=cXlGZ21tTGZEUjFDVjRKMWxaVFlVZz09

The Page-Wootters formalism: Where are we now?

Alexander R. H. Smith

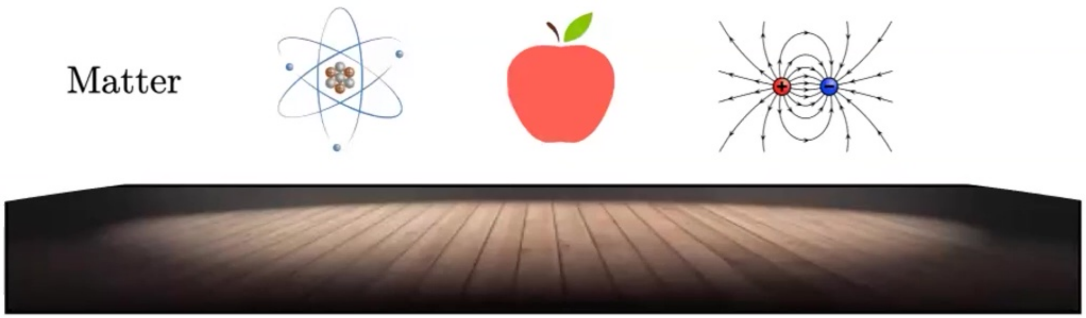
SAINT ANSELM COLLEGE 1889

Quantum Information Science at Dartmouth

Based on work with Mehdi Ahmadi, Shadi Ali Ahmad, Thomas Galley, Philipp Höhn, Maximilian Lock, and Andrea Russo

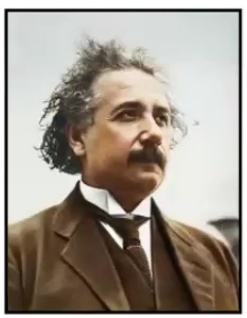
Quantum gravity requires a new dynamical paradigm

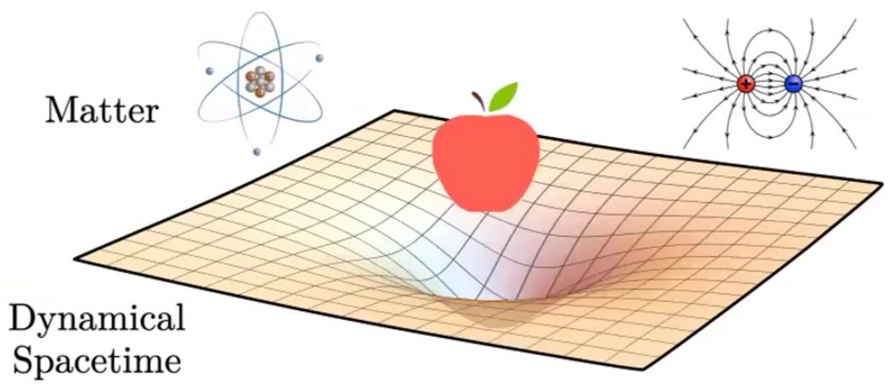
Space and time are fixed background structures on which dynamics unfolds



Absolute Space and Time

Spacetime is dynamic and interacts with matter

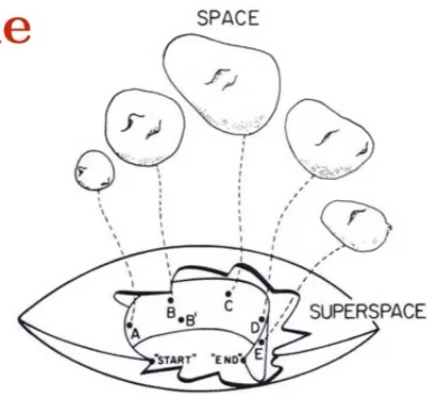




The problem of time

We need to choose canonical variables:

Choose the 3-metric γ_{ab} and its conjugate momentum P^{ab}



J. A. Wheeler, *Time Today* (1994)

Then we can express general relativity in the Hamiltonian form

$$H_{\mathrm{GR}}[\gamma, P] = 16\pi G G_{abcd}[\gamma] P^{ab} P^{cd} + V[\gamma] + \sqrt{\gamma}\rho = 0$$

and three momentum constraints

Dirac/canonical quantization

1. Constraint

$$H_{\mathrm{GR}} |\Psi\rangle = 0$$

2. Dynamics

$$i\frac{d}{dt}|\Psi\rangle = H_{\mathrm{GR}}|\Psi\rangle = 0$$

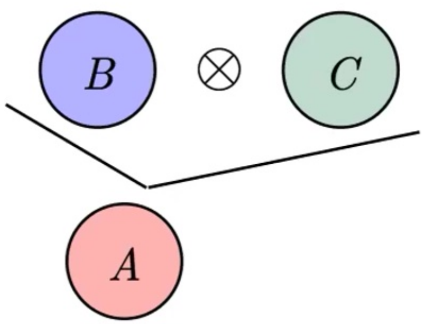
$|\Psi\rangle$ is frozen!

Talk outline

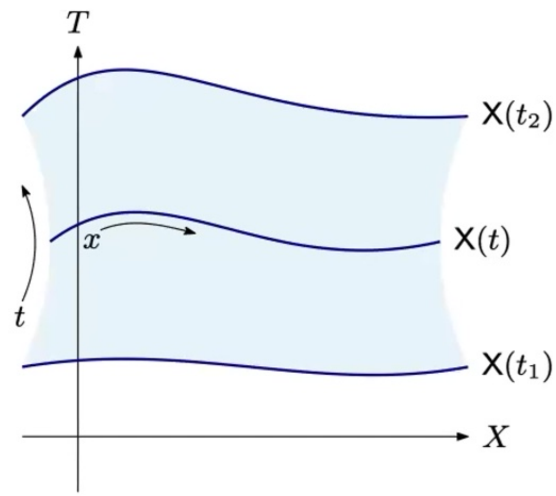
1. A quantum description of clocks

2. The Page-Wootters formalism

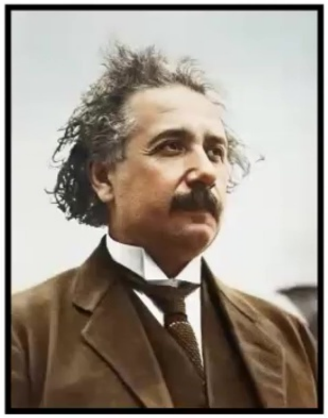
3. Gauge-invariant, reference frame-dependent subsystems



4. Field-theoretic extension of the Page-Wootters formalism



An operational notion of time



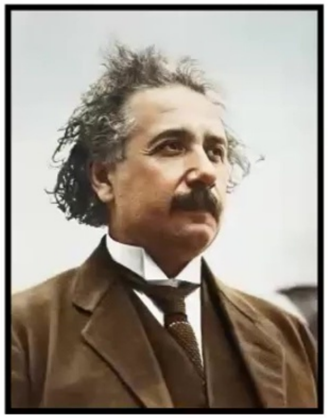
"[Time is] considered measurable by a clock (ideal periodic process) of negligible spatial extent. The time of an event taking place at a point is then defined as the time shown on the clock simultaneous with the event."

A. Einstein, The Special Theory of Relativity (1949)

"Einstein, in seizing on the act of the observer as the essence of the situation, is actually adopting a new point of view as to what the concepts of physics should be, namely, the operational view."

P.W. Bridgman, *The Logic of Modern Physics* (1927)

An operational notion of time



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A. Einstein, The Special Theory of Relativity (1949)

"In general, we mean by any concept nothing more than a set of operations; the concept is synonymous with the corresponding set of operations."

P.W. Bridgman, *The Logic of Modern Physics* (1927)

What is a clock?

A clock (aka temporal reference frame) is defined by the quadruple

Hilbert space — Fiducial state

$$= \{\mathcal{H}_C,\, H_C,\, \rho_C,\, T_C\}$$

Hamiltonian — Time observable

Elapsed time is unitarily encoded in the state fiducial state

$$\rho_C(\tau) = e^{-iH_C\tau}\rho_C e^{iH_C\tau} \quad \text{for all } \tau \in G \subseteq \mathbb{R}$$

A time observable T_C should satisfy the following properties:

1. On average T_C should estimate the elapsed time τ

2. The variance in a measurement of T_C should be independent of the elapsed time τ

Covariant time observables

T_C is a POVM that best estimates the elapsed time:

$$T_C : G \to \mathcal{E}\left(\mathcal{H}_C\right)$$
$$\tau \mapsto E_C(\tau) = |\tau\rangle\langle\tau|$$

Clock states

1. $E_C(\tau) \geq 0$
2. $E_C(G_1 + G_2) = E_C(G_1) + E_C(G_2)$
3. $\int_G d\tau\, E(\tau) = \mu \int_G d\tau\ |\tau\rangle\langle\tau| = I_C$

$$\text{Prob}[T_C = \tau] = \text{tr}\left[\rho E(\tau)\right]$$

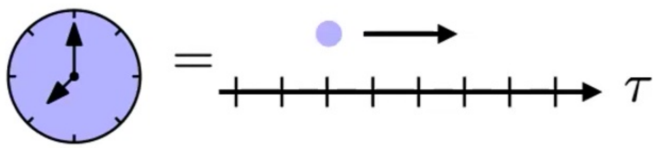
These properties are satisfied if T_C is **covariant** with respect to the group G generated by H_C, which is equivalent to the following relation between clocks states

$$|\tau + \tau'\rangle = e^{-iH_C\tau'}\,|\tau\rangle$$

A. S. Holevo, *Probabilistic and Statistical Aspects of quantum Theory*, (North-Holland, 1982)
A. Perelomov, *Generalized Coherent States and their Applications*, (North-Holland, 1986)
S. L. Braunstein and C. Caves, *Statistical distance and the geometry of quantum states*, Phys. Rev. Lett. 72 3439 (1994)
P. Busch, M Grabowski, and P. J. Lahti, *Operational Quantum Physics*, (Springer-Verlag, 1995)

Examples of quantum clocks

The ideal clock



Clock Hilbert space

$$\mathcal{H}_C \simeq L^2(\mathbb{R})$$

Clock Hamiltonian

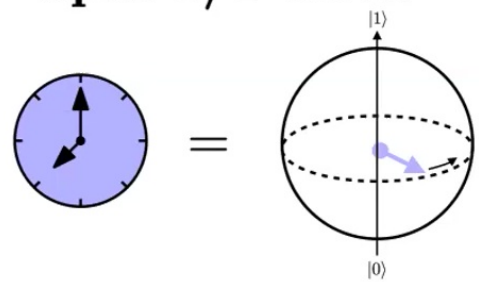
$$H_C = P_C$$

Clock states

$$|\tau\rangle = e^{-iP_C\tau}\,|\tau_0\rangle$$

Time observable associated with a self-adjoint operator $T_C = X_C$

Spin-1/2 clock



Clock Hilbert space

$$\mathcal{H}_C \simeq \mathbb{C}^2$$

Clock Hamiltonian

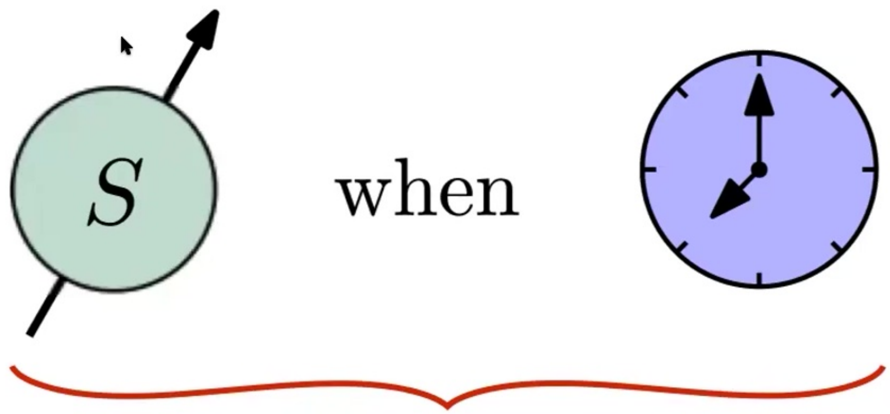
$$H_C = \Omega\sigma_z$$

Clock states

$$|\tau\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{-i\Omega\tau}\,|1\rangle\right)$$

Time observable T_C associated with a POVM, $E_C(\tau) = |\tau\rangle\langle\tau|$

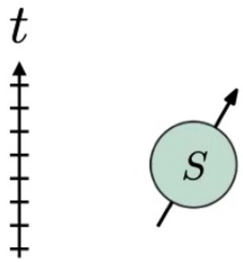
2. The Page-Wootters formalism



$$|\Psi\rangle \quad \text{where} \quad C_H |\Psi\rangle = 0$$

No background time, but instead recover a relational unitary evolution between the clock and system

Theory 1



External time + System

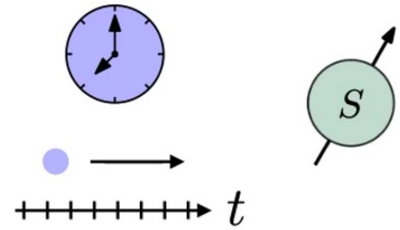
$$\mathcal{S} = \int_{t_1}^{t_2} dt \, L_S(q, q')$$

Hamiltonian analysis

$$q' = \frac{\partial H_S}{\partial p} \qquad p' = -\frac{\partial H_S}{\partial q}$$

Theory 2

Extended phase space



Clock + System

$$\mathcal{S}' = \int_{\tau_1}^{\tau_2} d\tau \, L(t, \dot{t}, q, \dot{q})$$

$$= \int_{\tau_1}^{\tau_2} d\tau \; \dot{t} L_S(q, \dot{q}/\dot{t}) = \mathcal{S}$$

Hamiltonian analysis

$$H \propto C_H := P_C + H_S = 0$$

which implies $\dot{t} = \dot{P}_C = \dot{q} = \dot{p} = 0$

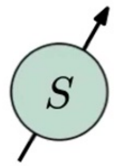
The Page-Wootters formalism

(a.k.a. The conditional probability interpretation)

$$C_H |\Psi\rangle = (H_C + H_S) |\Psi\rangle = 0 \quad \text{where} \quad |\Psi\rangle \in \mathcal{H}_{\text{phys}} \subset \mathcal{H}_{\text{kin}}$$

Time observable

$$T_C = \{ |t\rangle\langle t| \; \forall t \in G \}$$

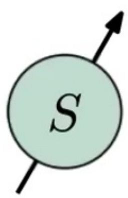


System observable

$$A = \{ |a\rangle\langle a| \; \forall a \in \Omega \}$$

Dynamics arises from conditional probabilities of the sort

$$\text{Prob}(a \text{ when } t) = \frac{\langle \Psi | \left(|t\rangle\langle t| \otimes |a\rangle\langle a| \right) |\Psi\rangle}{\langle \Psi | \left(|t\rangle\langle t| \otimes I \right) |\Psi\rangle} = |\langle a | \psi_S(t) \rangle|^2$$



Conditional state of the system

$$|\psi_S(t)\rangle := \left(\langle t | \otimes I \right) |\Psi\rangle$$

$$i \frac{d}{dt} |\psi_S(t)\rangle = H_S |\psi_S(t)\rangle$$

Page and Wootters, *Evolution without evolution*, Phys. Rev. D 27, 2885 (1983); W. K. Wootters, *"Time" Replaced by Quantum Correlations* (1984)

Why formulate quantum theory this way?

Relational quantum dynamics can be formulated with just the Born rule and the conditional probability

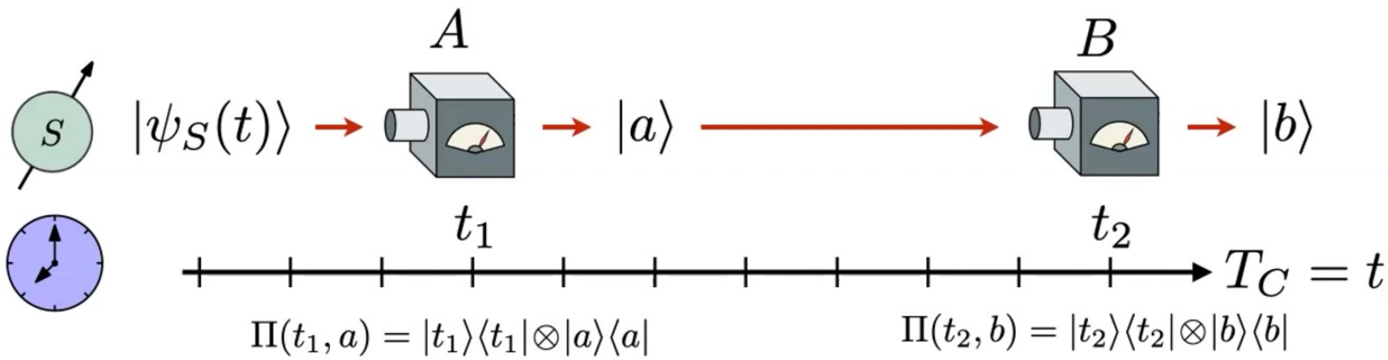
$$\mathrm{Prob}(a \text{ when } t) = \frac{\langle\Psi|\left(|t\rangle\langle t| \otimes |a\rangle\langle a|\right)|\Psi\rangle}{\langle\Psi|\left(|t\rangle\langle t| \otimes I\right)|\Psi\rangle} = |\langle a|\psi_S(t)\rangle|^2$$

‘One motivation for considering such a "condensation" of history is the desire for economy as regards the number of basic elements of the theory: quantum correlations are an integral part of quantum theory already; so one is not adding a new element to the theory. And yet an old element, time, is being eliminated, becoming a secondary and even approximate concept.’

W. K. Wootters, *"Time" Replaced by Quantum Correlations* (1984)

Kuchar's criticism

You never apply conditional probability formula to answering the fundamental DYNAMICAL question of the internal Schrödinger interpretation, namely, "If one finds the particle at a at the time t_1, what is the probability of finding it at b at the time t_2?" By your formula, that conditional probability differs from zero only if $t_1 = t_2$ and $<a|b> \neq 0$.



$$P(b \text{ when } t_2 \,|\, a \text{ when } t_1) = \frac{\langle\Psi|\,\Pi(t_1,a)\Pi(t_2,b)\Pi(t_1,a)\,|\Psi\rangle}{\langle\Psi|\,\Pi(t_1,a)\,|\Psi\rangle}$$

$$= |\delta(t_2 - t_1)|^2\, |\langle a|b\rangle|^2 \neq \underline{|\langle b|U_S(t_2,t_1)|a\rangle|^2}$$

Kuchar's Criticism

In brief, your interpretation prohibits the time to flow and the system to move! For me, this virtually amounts to a reductio ad absurdum of the conditional probability proposal.

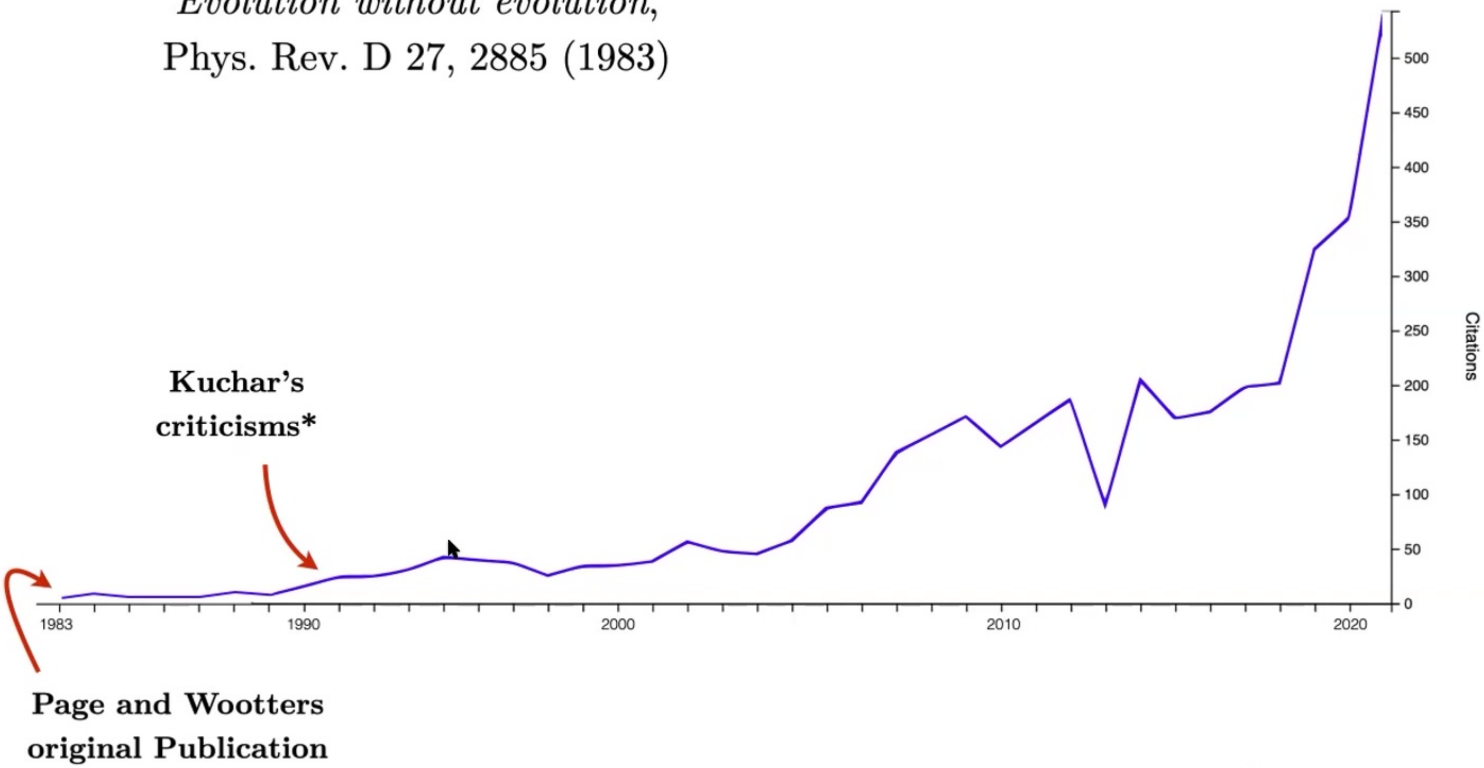
[...] **only quantities at a single instant of time are directly accessible, and so one cannot directly test the two-time probability you discuss.** One could instead at one time test the conditional probability that the particle is at b, given that the time is t_2 and that at this time there is a **record** indicating that the particle was at a at the time t_1.

Don, you are the first person I met who simultaneously believes in the existence of many worlds and is a solipsist of an instant.

I believe that different instants, i.e., different clock times, are actually examples of the different worlds. They all exist, but each observation, and its associated conditional probability, occurs at one single time [...]. We can only directly observe and be aware of the world, and the time, in and at which we exist, though the **correlations in memories** and other structures we observe in one world give indirect evidence of other worlds, and other clock times, in the full quantum state of the universe. [...]

So what happened next to the PW formalism?

Citations to D. Page and W. Wootters
Evolution without evolution,
Phys. Rev. D 27, 2885 (1983)



*K. Kuchar, *Time and interpretations of quantum gravity,* Proceedings of the 4th Canadian Conference on General Relativity and Relativistic Astrophysics, (1992); and in the discussion section of D. Page, *Clock Time and Entropy*, in Physical Origins of Time Asymmetry, NATO Advanced Research workshop (1991)

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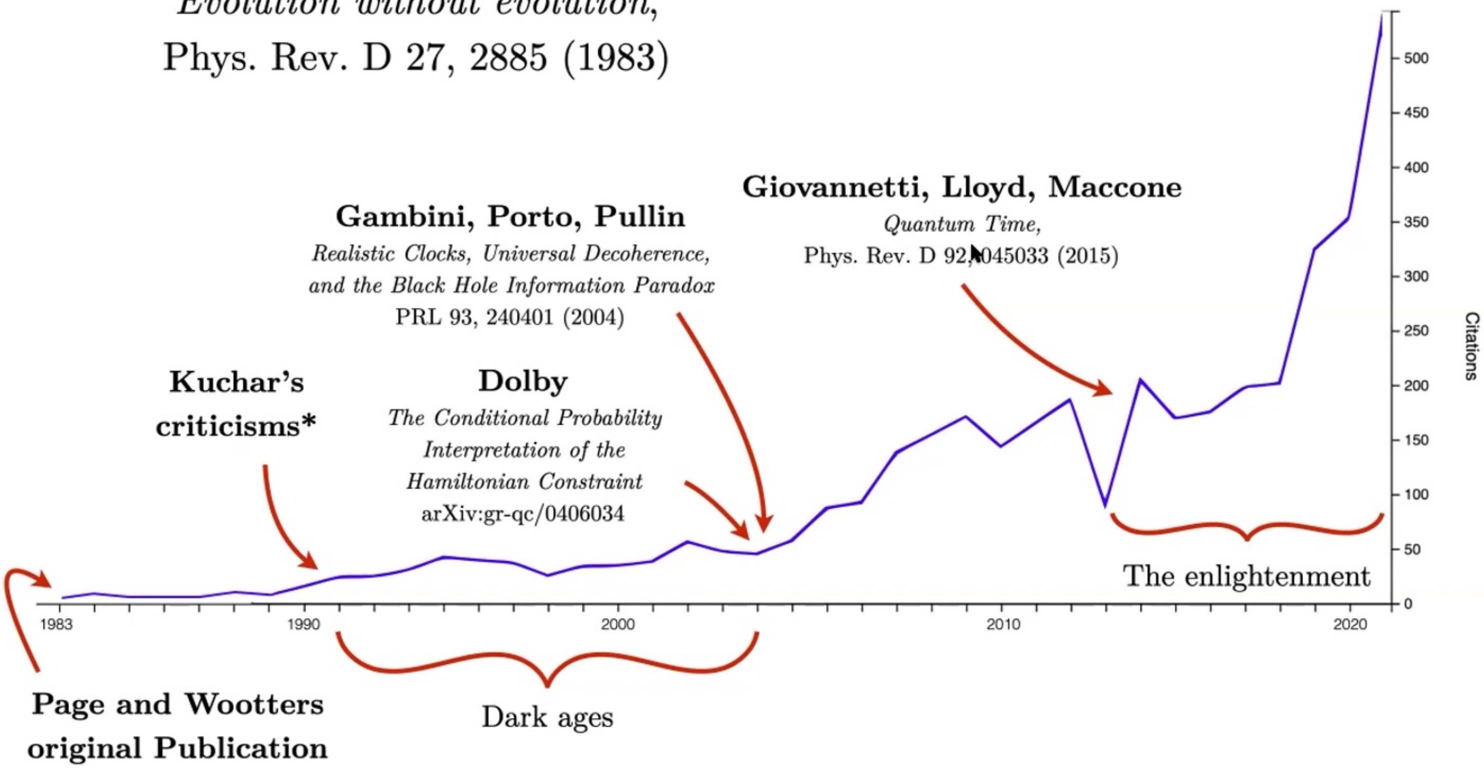

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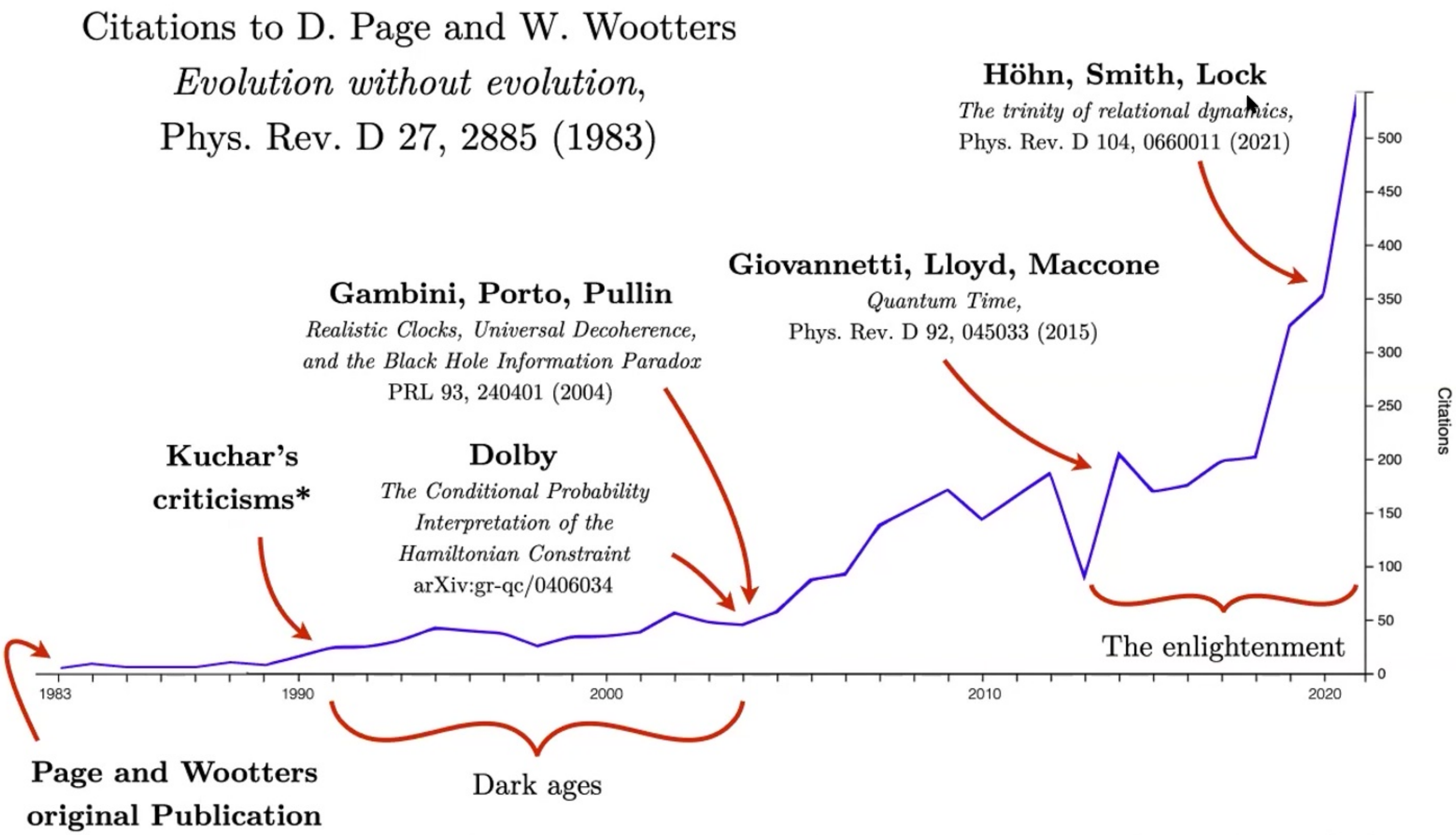

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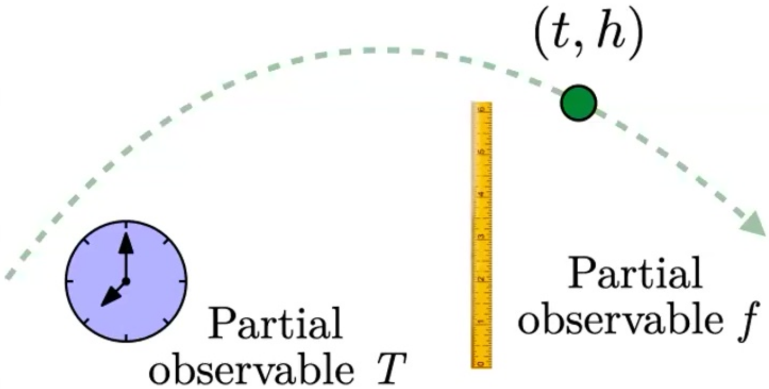
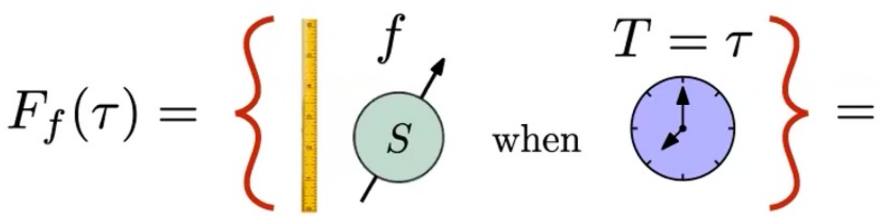
Partial and complete/relational observables



Correlations between observables encoded in the physical state $|\Psi\rangle$.

$T = t$	$f = h$
t_1	h_1
t_2	h_2
$\vdots$	$\vdots$

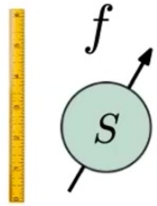
Given $C_H |\Psi\rangle = 0$ need to construct a complete observable



Complete/relational observable on an extended phase space

C. Rovelli, *Quantum Gravity*, (Cambridge University Press, 2004)
B. Dittrich, *Partial and complete observables for Hamiltonian constrained systems*, Gen. Relativ. Gravit. 39 1891 (2007)
T. Thiemann, *Modern Canonical Quantum General Relativity*, (2008)

Partial and complete/relational observables

$F_f(\tau) = \Big\{$ f 

when $T = \tau$  $\Big\}$ = **Complete/relational observable on an extended phase space**

The relational observable $F_f(\tau)$ can be obtained by solving

$$\alpha^s_{C_H} \cdot T = \tau \quad \Longrightarrow \quad s = s_T(\tau)$$

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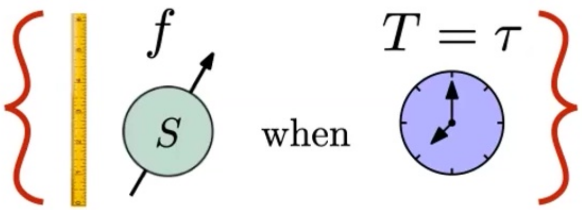

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which can then be used to construct the relational observable

$$F_f(\tau) := \alpha^s_{C_H} \cdot f \,\Big|_{s = s_T(\tau)} \approx \sum_{n=0}^{\infty} \frac{(T-\tau)^n}{n!} \{f, C_H\}_n$$

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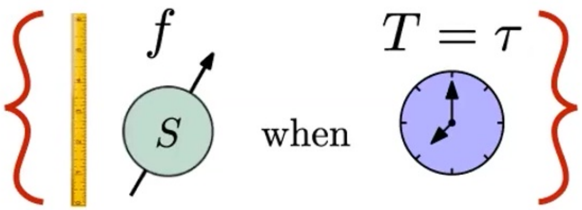

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$\{f, C_H\}_{n+1} := \{\{f, C_H\}_n, C_H\}$

Nested Poisson bracket

C. Rovelli, *Quantum Gravity*, (Cambridge University Press, 2004)
B. Dittrich, *Partial and complete observables for Hamiltonian constrained systems*, Gen. Relativ. Gravit. 39 1891 (2007)
T. Thiemann, *Modern Canonical Quantum General Relativity*, (2008)

Quantizing complete/relational observables

$$F_f(\tau) \approx \sum_{n=0}^{\infty} \frac{(T-\tau)^n}{n!} \{f, C_H\}_n$$

Quantization using a covariant time (partial) observable T

$$F_f(\tau) = \frac{1}{2\pi} \int_{\mathbb{R}} dt \; |t\rangle\langle t| \otimes \sum_{n=0}^{\infty} \frac{i^n}{n!} (t-\tau)^n \, [f, C_H]_n$$

$$= \mathcal{G}\left(|\tau\rangle\langle\tau| \otimes f\right)$$

Where we have introduced the so-called ***G*-twirl**

$$\mathcal{G}\left(|t\rangle\langle t| \otimes f\right) := \int dt \, e^{-itC_H} \left(|\tau\rangle\langle\tau| \otimes f\right) e^{itC_H}$$

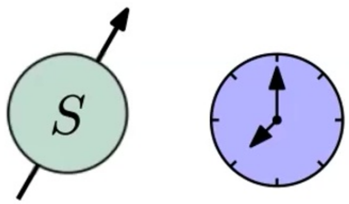
P. A. Höhn, A. R. H. Smith and M. P. E. Lock, *The Trinity of Relational Quantum Dynamics*, PRD 104, 0660011 (2021)

S. D. Bartlett, T. Rudolph, R. W. Spekkens, *Reference frames, superselection rules, and quantum information*, Rev. Mod. Phys. 79, 55 (2007)

A. R. H. Smith, M. Piani, R. B. Mann, *Quantum reference frames associated with noncompact groups: The case of translations and boosts and the role of mass,* Phys. Rev. A 94 012333 (2016)

Resolving Kuchar's criticism

Define **relational Dirac observables** associated with the proposition



$$F_a(t) := \int_G dg\, e^{-iC_H g} \left(|t\rangle\langle t| \otimes |a\rangle\langle a| \right) e^{iC_H g}$$

$A = a$ when $T_C = t$

One time probabilities

$$\text{Prob}(a \text{ when } t) = \frac{\langle\Psi| \left(|t\rangle\langle t| \otimes |a\rangle\langle a|\right) |\Psi\rangle}{\langle\Psi| \left(|t\rangle\langle t| \otimes I\right) |\Psi\rangle} = \langle\Psi|F_a(t)|\Psi\rangle_{\text{phys}} = |\langle a|\psi_S(t)\rangle|^2$$

Two time probabilities

$$\text{Prob}(b \text{ when } t_2 \,|\, a \text{ when } t_1) = \frac{\langle\Psi|\, F_a(t_1) \cdot F_b(t_2) \cdot F_a(t_1)\, |\Psi\rangle_{\text{phys}}}{\langle\Psi|\, F_a(t_1)\, |\Psi\rangle_{\text{phys}}} = |\langle b|U_S(t_2, t_1)|a\rangle|^2$$

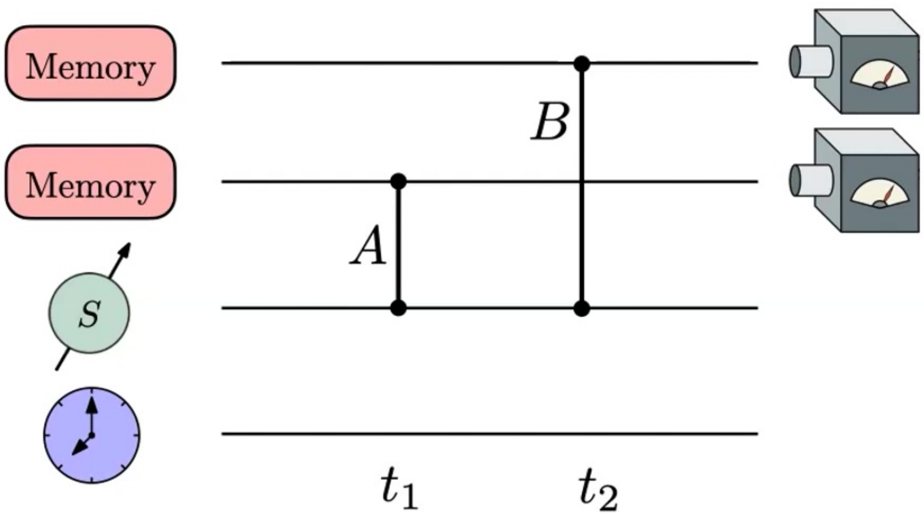
P. A. Höhn, A. R. H. Smith and M. P. E. Lock, *The Trinity of Relational Quantum Dynamics,* PRD 104, 0660011 (2021)

Resolving Kuchar's criticism

Here I simply wish to argue that all of the testable predictions of ordinary quantum mechanics appear to arise from one-time conditional probabilities. [...] **We can never directly test what happened yesterday, but we can check the consequences that a hypothetical scenario for yesterday has on the situation today.**

D. N. Page, *Time as an Inaccessible Observable*, NSF-ITP-89-18 (1989)

$$C_H\,|\Psi\rangle = \Big(H_S + \delta(t-t_1)H_{SA} + \delta(t-t_2)H_{SB}\Big)\,|\Psi\rangle = 0$$



$$\mathrm{Prob}(b \text{ when } t_2 \,|\, a \text{ when } t_1)$$

$$= \frac{\mathrm{Prob}(a \text{ and } b \text{ when } t_2)}{\mathrm{Prob}(b \text{ when } t_2)}$$

$$= |\langle b|U_S(t_2,t_1)|a\rangle|^2$$

V. Giovannetti, S. Lloyd, and L. Maccone, *Quantum time*, Phys. Rev. D 92, 045022 (2015)

Where are we now? (Not exhaustive)

What more general dynamics is possible?

- Nonideal clocks and epistemic uncertainty lead to decoherence
- Clock-system interactions leads to a modified Schrödinger equation
- Temporally delocalized evolution
- Arbitrary reference frames
- Causal structure

Gambini, Porto, Pullin, *Realistic Clocks, Universal Decoherence, and the Black Hole Information Paradox* PRL 93, 240401 (2004)

Gambini, Porto, Pullin, Conditional probabilities with Dirac observables and the problem of time in quantum gravity, PRD 79, 041501

Mendes, Brito, Soares-Pinto, *Non-linear equation of motion for Page-Wootters mechanism with interaction and quasi-ideal clocks*, arXiv:2107.11452 (2021)

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A.-C. de la Hamette, S. L. Ludescher, M. P. Muller, *Entanglement/Asymmetry correspondence for internal quantum reference frames* arXiv:2112.00046 (2021)

V. Baumann, F. Del Santo, A. R. H. Smith, F. Giacomini, E. Castro-Ruiz, C. Brukner, *Generalized probability rules from a timeless formulation of Wigner's friend scenarios*, Quantum 5, 524 (2021)

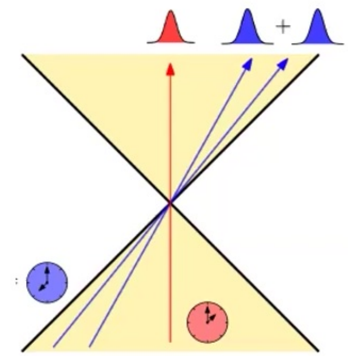
V. Baumann, M. Krumm, P. A. Guérin, Č. Brukner, P*age-Wootters formulation of indefinite causal order*, arXiv:2105.02304 (2021)

Where are we now? (Not exhaustive)

What more general dynamics is possible?

- Nonideal clocks and epistemic uncertainty lead to decoherence
- Clock-system interactions leads to a modified Schrödinger equation
- Temporally delocalized evolution
- Arbitrary reference frames
- Causal structure

Relativistic generalizations



- Probabilistic notion of time dilation between quantum clocks
- Generalization to quadratic constraints
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A. R. H. Smith and M. Ahmadi, *Quantum clocks observe classical and quantum time dilation* Nat. Commun. 11, 5360 (2020)

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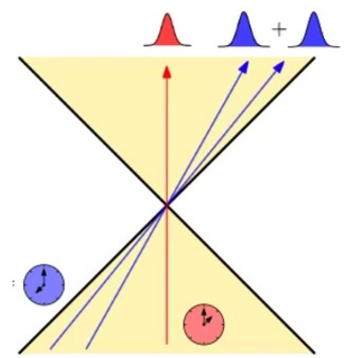

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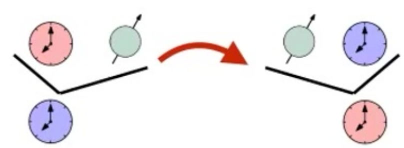
Quantum reference frame transformations

- Changes of quantum reference frames
- Perspective-neutral interpretation
- Changes of temporal reference frames within the PW formalism



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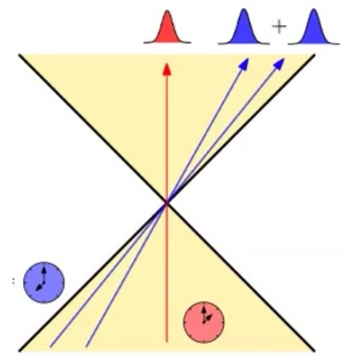

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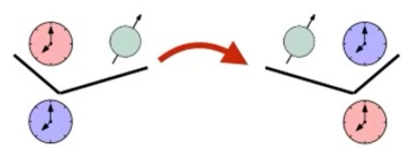

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Equivalence with other formalisms

- The trinity of relational dynamics: Relational Dirac observables (aka evolving constants of motions), the PW formalism, and quantum symmetry reduction
- Connection with integral representations of Dirac observables

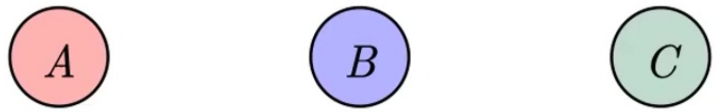
P. A. Höhn, A. R. H. Smith and M. P. E. Lock, *The Trinity of Relational Quantum Dynamics,* PRD 104, 0660011 (2021)

L. Chataignier, *Relational observables, reference frames and conditional probabilities*, PRD 103, 026013 (2021)

A.-C. de la Hamette, T. D. Galley, P. A. Hoehn, L. Loveridge, M. P. Muller,
Perspective-neutral approach to quantum frame covariance for general symmetry groups, arXiv:2110.13824 (2021)

3. Physical notion of subsystem and entanglement

State spaces involved in Dirac quantization



Classical phase space $\qquad \Gamma_{\text{tot}} = \Gamma_A \times \Gamma_B \times \Gamma_C$

Kinematical Hilbert space $\qquad \mathcal{H}_{\text{kin}} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$

Physical Hilbert space $\qquad \mathcal{H}_{\text{phys}} = \{|\Psi\rangle \text{ such that } C\,|\Psi\rangle = 0\}$

How can we construct a physical notion of subsystems?

P. A. Höhn, M. P. E. Lock, S. Ali Ahmad, A. R. H. Smith, and T. D. Galley,
Quantum relativity of subsystems, arXiv:2103.01232 (2021)

Is this state entangled?

$$|\psi\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle \big)$$

Entanglement is defined relative to a tensor product partition

$$\mathcal{H} \simeq \mathcal{H}_A \otimes \mathcal{H}_B$$

$$\{A \otimes I \mid A = A^\dagger\}$$

$$\{I \otimes B \mid B = B^\dagger\}$$

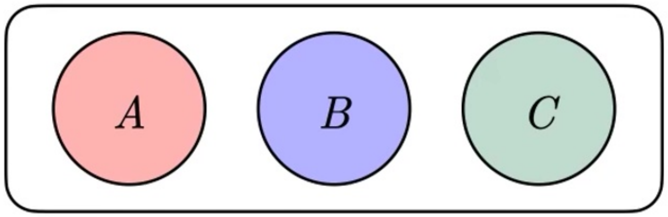
$$\{\sigma_x \otimes \sigma_x,\ \sigma_z \otimes \sigma_z,\ \sigma_x \otimes \sigma_y,\ \sigma_y \otimes \sigma_z\}$$

P. Zanardi, D. A. Lidar, and S. Lloyd, *Quantum Tensor Product Structures are Observable Induced*, PRL 92, 6 (2004)

H. Barnum, E. Knill, G. Ortiz, R. Somma, L. Viola, *A subsystem-independent generalization of entanglement*, PRL 92, 107902 (2004)

Algebraically induced tensor products

Consider a system characterized by the Hilbert space $\mathcal{H}$ and the associative algebra of observables $\mathcal{A} \simeq \mathcal{B}(\mathcal{H})$.



Now suppose that the algebra of observables partitions into a collection of independently accessible, mutually commuting associative sub-algebras $\{\mathcal{A}_A, \mathcal{A}_B, \mathcal{A}_C\}$, such that

$$\mathcal{A} \simeq \mathcal{A}_A \otimes \mathcal{A}_B \otimes \mathcal{A}_C \quad \text{and} \quad [\mathcal{A}_i, \mathcal{A}_j] = 0 \quad \forall i \neq j$$

This set of algebras induces a tensor product structure

$$\mathcal{H} \simeq \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$$

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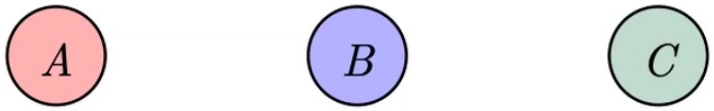
The state of BC relative to A

$$C\,|\Psi\rangle = (C_A + C_B + C_C)\,|\Psi\rangle = 0$$



Define the set of orientations of frame A as $|g\rangle := e^{-igC_A}\,|0\rangle$

And define the conditional state of BC relative to A

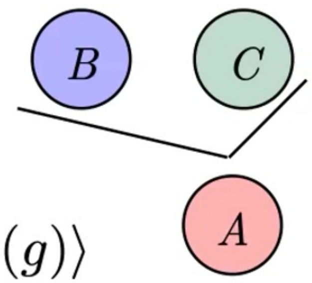
$$\mathcal{R}_A(g) \ : \ \mathcal{H}_{\text{phys}} \to \mathcal{H}_{BC|A}$$

$$|\Psi\rangle \mapsto |\psi_{BC|A}(g)\rangle := \langle g| \otimes I_{BC}\,|\Psi\rangle$$

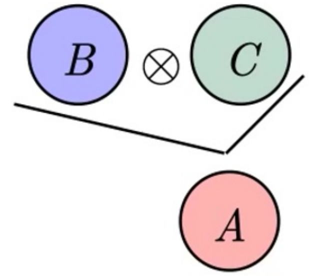
Which is an invertible isometry

$$\mathcal{R}_A^{-1}(g) \ : \ \mathcal{H}_{BC|A} \to \mathcal{H}_{\text{phys}}$$

$$|\psi_{BC|A}(g)\rangle \mapsto |\Psi\rangle = \int dg\ |g\rangle\,|\psi_{BC|A}(g)\rangle$$



Physical subsystems and entanglement



Now suppose that the conditional Hilbert space admits a tensor product partition

$$\mathcal{H}_{BC|A} \simeq \mathcal{H}_{B|A} \otimes \mathcal{H}_{C|A}$$

The joint algebra of observables of BC relative to A

$$\mathcal{A}_{BC|A} = \mathcal{A}_{B|A} \otimes \mathcal{A}_{C|A} = \mathcal{B}(\mathcal{H}_{B|A}) \otimes \mathcal{B}(\mathcal{H}_{C|A})$$

Can be lifted to the physical space using the map between the physical and conditional Hilbert spaces:

$$\mathcal{A}_{i|A}^{\text{phys}} = \mathcal{R}_A^{-1}(g)\mathcal{A}_{i|A}\mathcal{R}_A(g) \quad \text{for} \quad i \in \{B, C\}$$

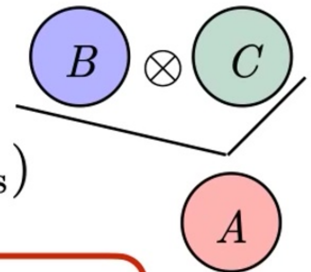
$$= \mathcal{G}\left(|g\rangle\langle g| \otimes \mathcal{A}_{i|A}\right) = F_{\mathcal{A}_{i|A},G}(g) \subset \mathcal{B}\left(\mathcal{H}_{\text{phys}}\right)$$

The algebra of Dirac observables of i relative to A

Physical subsystems and entanglement

$$\mathcal{H}_{BC|A} \simeq \mathcal{H}_{B|A} \otimes \mathcal{H}_{C|A} \implies \text{consider } \mathcal{A}^{\text{phys}}_{B|A} \text{ and } \mathcal{A}^{\text{phys}}_{C|A}$$

It follows that $\left[\mathcal{A}^{\text{phys}}_{B|A}, \mathcal{A}^{\text{phys}}_{C|A}\right] = 0$



And $\mathcal{A}^{\text{phys}}_{B|A} \otimes \mathcal{A}^{\text{phys}}_{C|A}$ is dense in $\mathcal{A}_{\text{phys}} = \mathcal{B}(\mathcal{H}_{\text{phys}})$

And a physical tensor product structure is induced

$$\mathcal{H}_{\text{phys}} \simeq \mathcal{H}^{\text{phys}}_{B|A} \otimes \mathcal{H}^{\text{phys}}_{C|A}$$

Theorem 1: Different frames induce different partitions of the physical Hilbert space.

Theorem 2: When does the physical Hilbert space inherited the kinematical tensor product structure?

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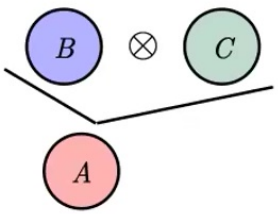
Summary and outlook

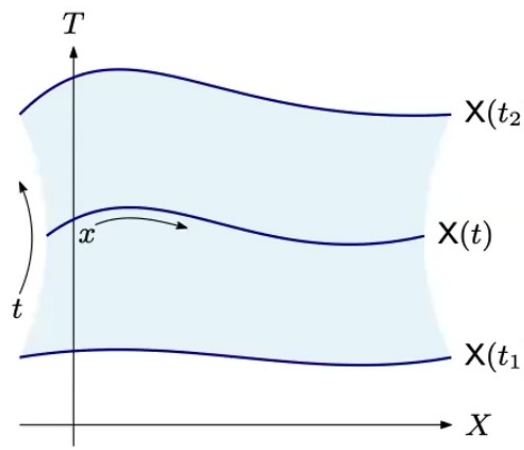
1. A quantum description of clocks

2. The Page-Wootters formalism

3. Gauge-invariant, reference frame-dependent subsystems



4. Field-theoretic extension of the Page-Wootters formalism



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