Title: Complexity and entropy in quantum many-body systems

Speakers: Philippe Faist

Series: Perimeter Institute Quantum Discussions

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Abstract: Quantifying quantum states' complexity is a key problem in various subfields of science, from quantum computing to black-hole physics. Motivated by the expected behavior of wormholes in quantum gravity, Brown and Susskind conjectured that the quantum complexity of the state output by a random circuit on n qubits grows linearly as more and more random gates are applied, until saturating after a number of gates exponential in n. We prove this conjecture by studying the dimension of the set of all unitaries that can be accessed with a given arrangement of two-qubit gates. Our core technical contribution is a lower bound on this dimension, using techniques from algebraic geometry and considerations based on Clifford circuits. In the second part of my talk, I'll discuss some thermodynamic and effective information-theoretic aspects of the complexity of quantum states and its growth in quantum many-body systems, establishing a resource theory to capture a notion of quantum complexity and drawing a connection between the concepts of complexity and entropy.

Joint work with: Jonas Haferkamp, Teja Naga Bhavia Kothakonda, Anthony Munson, Jens Eisert, Nicole Yunger Halpern

Zoom Link: https://pitp.zoom.us/j/94288479163?pwd=Nm8wOUdReGhreDErdUpJTzFETlBUUT09


# Complexity and entropy in quantum many-body systems

Philippe Faist *Freie Universität Berlin*

*with: Jonas Haferkamp, Teja Naga Bhavia Kothakonda, Anthony Munson, Jens Eisert, Nicole Yunger Halpern*

Perimeter Institute (virtual), Jan 2022

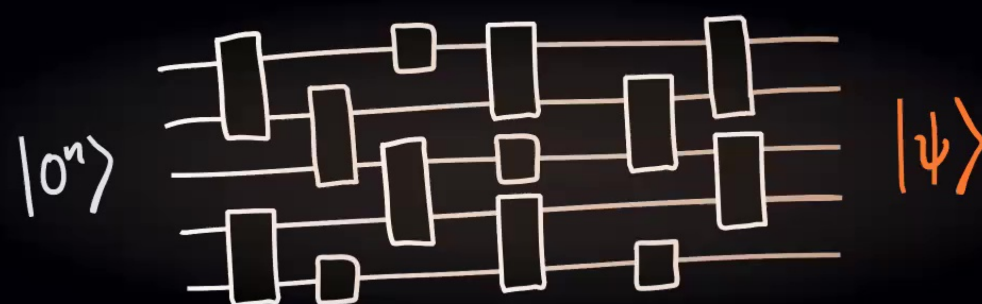
Teja N. B. Kothakonda
Freie Universität Berlin
Master's project: Complexity/work trade-offs
Jonas Haferkamp
Freie
Linear growth of quantum complexity
Jens Eisert
Universität Berlin
Anthony Munson
University of Maryland
Nicole Yunger Halpern
University of Maryland
Philippe Faist

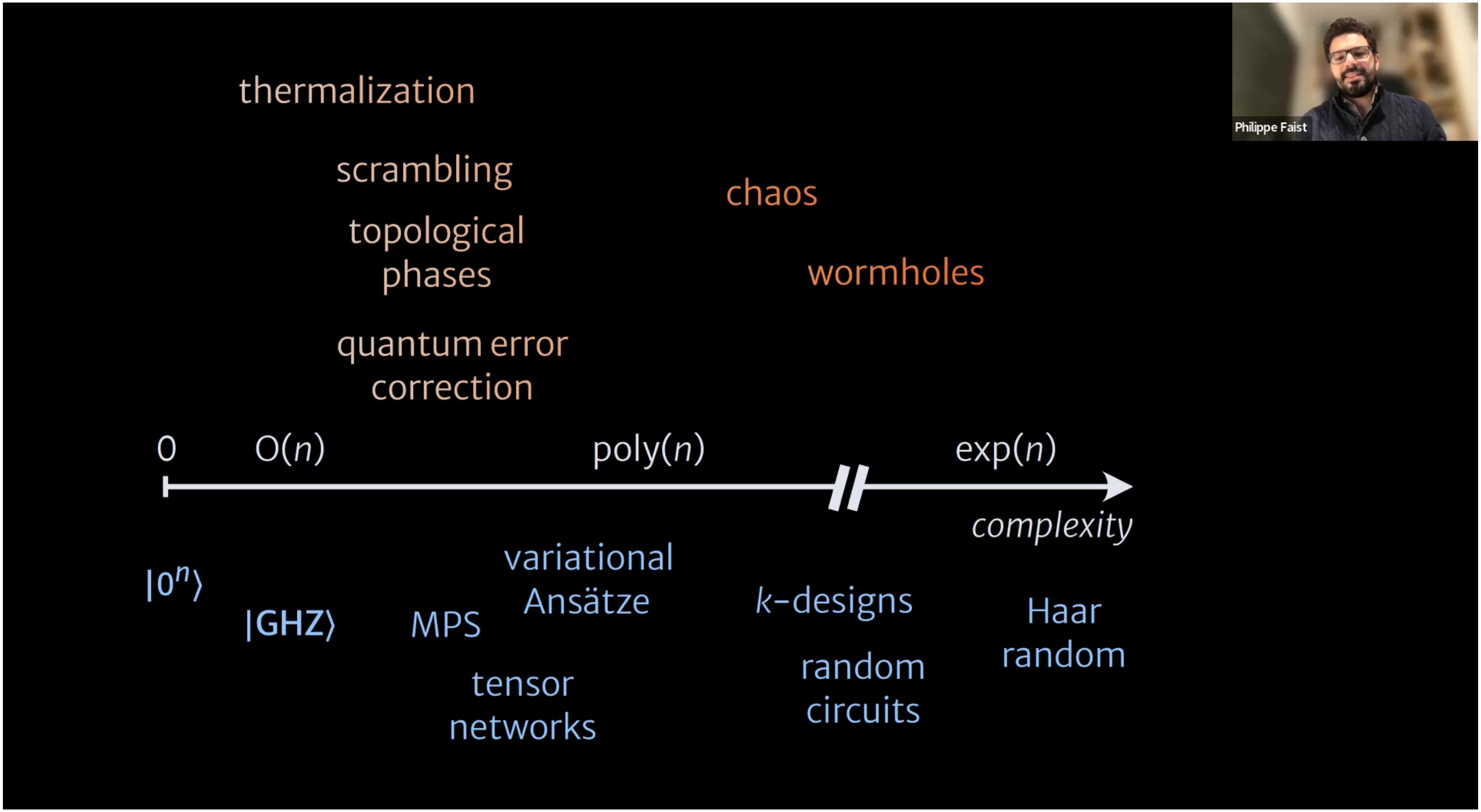

# Physics at the complexity frontier

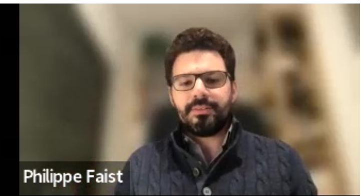
e.g. Preskill
Quantum 2018

How can we make sense of the immensity of the Hilbert space of *n* qubits?

- Description of a state in terms of a quantum circuit that prepares that state. The size of the circuit is the complexity.




thermalization
scrambling
topological phases
quantum error correction
chaos
wormholes
0
O(n)
poly(n)
exp(n)
complexity
|0^n⟩
|GHZ⟩
MPS
variational Ansätze
tensor networks
k−designs
random circuits
Haar random
Philippe Faist




- Can we understand how quantum complexity grows in time in quantum chaotic systems?

- How does quantum complexity manifest itself in physical/operational aspects of the system?

Part I: Linear Growth of Quantum Circuit Complexity

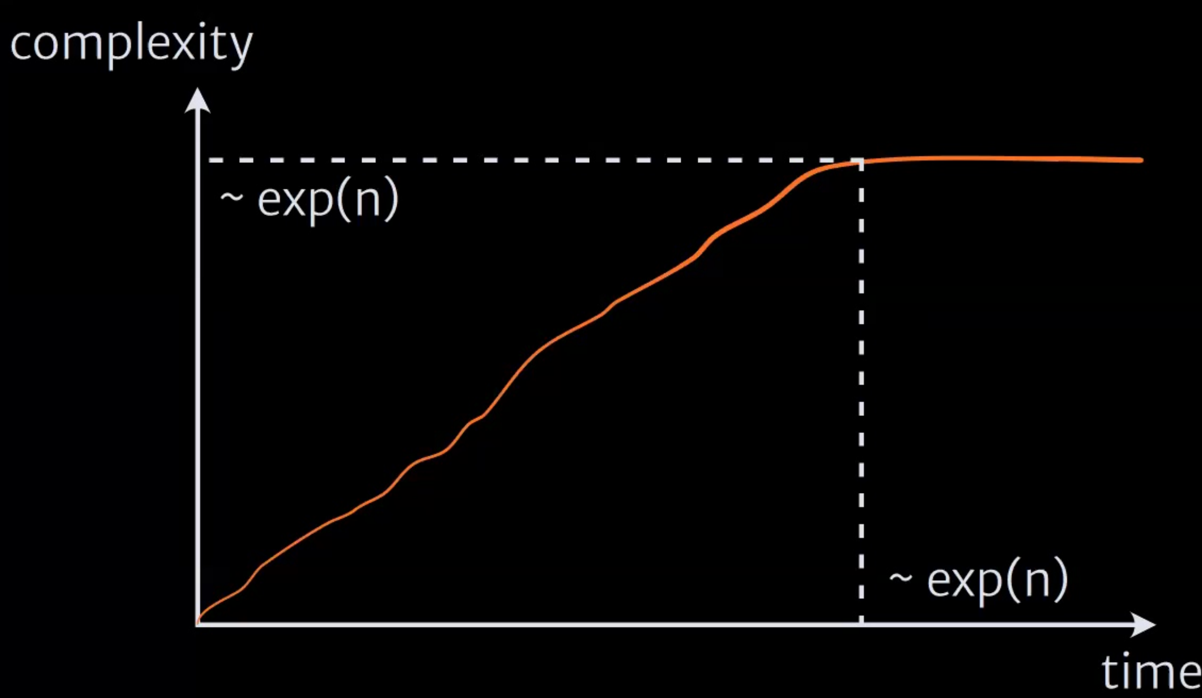
(random circuits, pure states)

Part II: Complexity & Entropy

(operational tasks, resource theory)

Part III: Discussion & Outlook

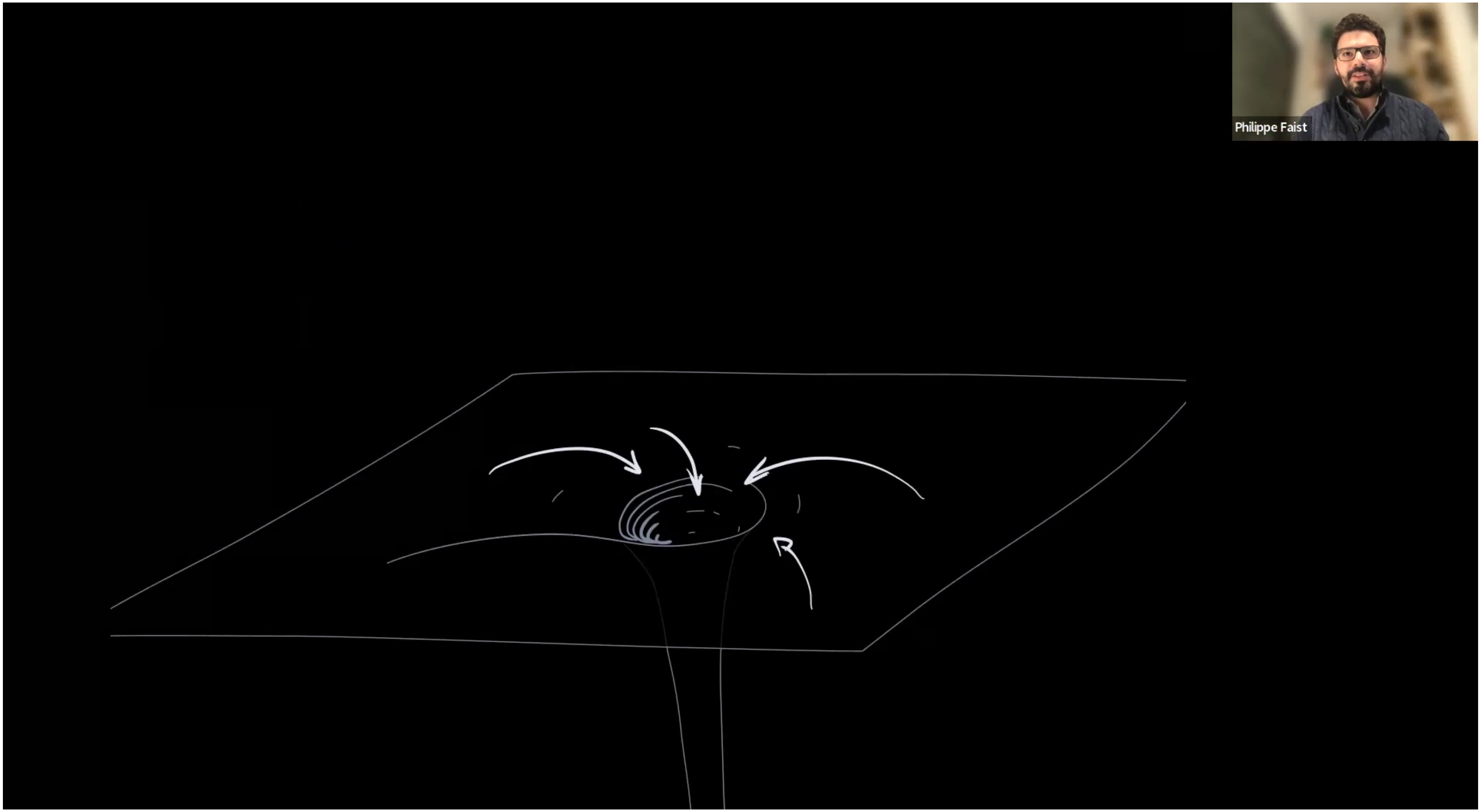
Complexity is believed to grow linearly in time in chaotic quantum systems.

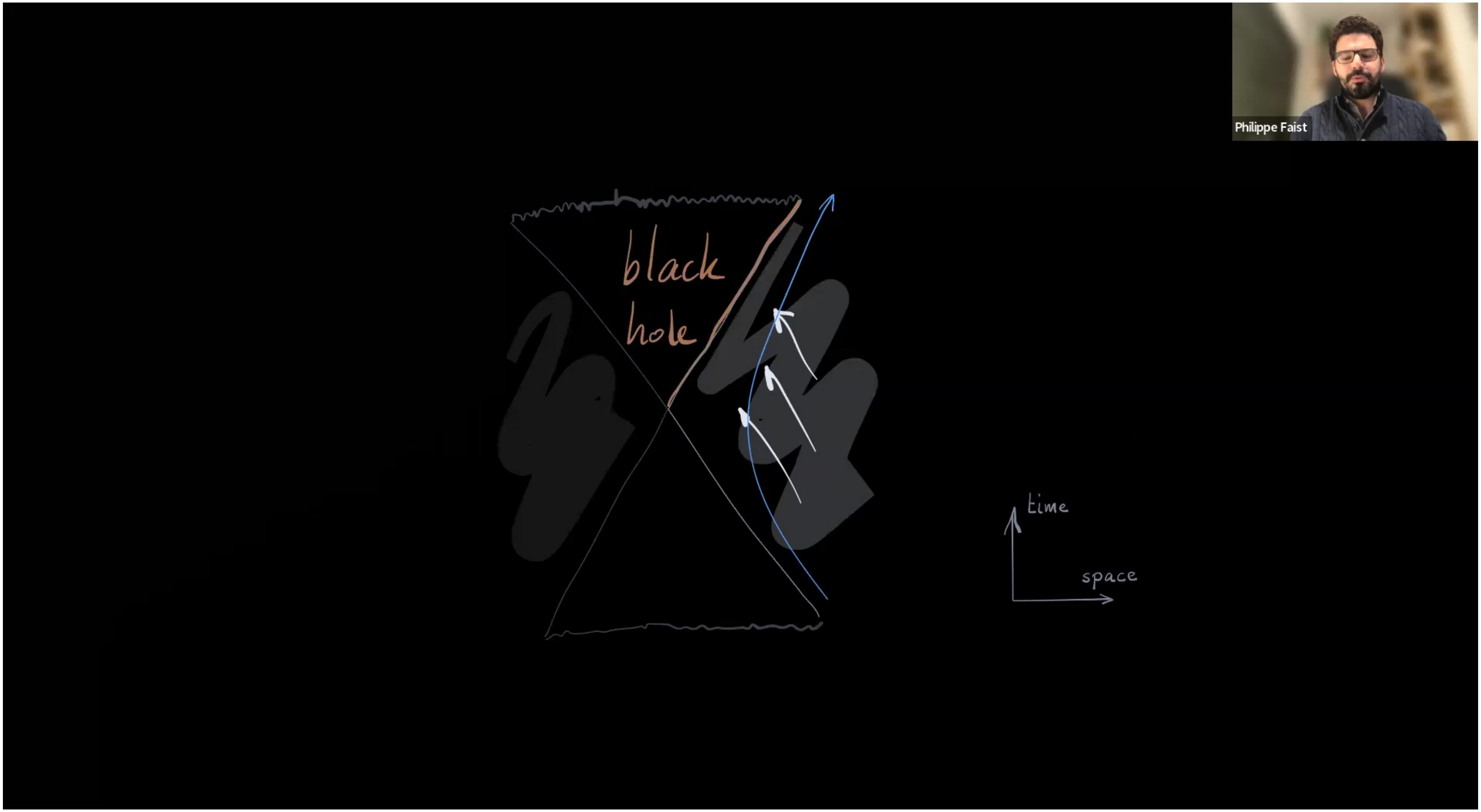




Brown & Susskind conjecture

Brown & Susskind PRD 2018


Philippe Faist



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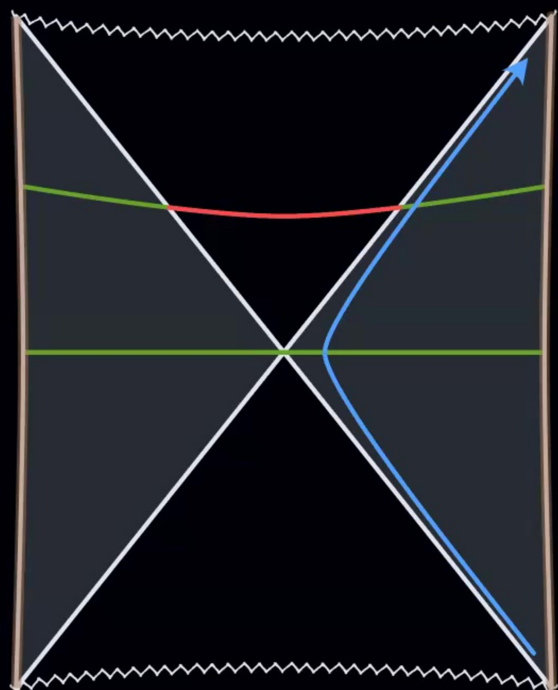
black hole
time
space

In AdS/CFT, physics in the bulk (AdS) has a dual description on the boundary (CFT).


*Two entangled sides of a black hole connected by a "wormhole"*

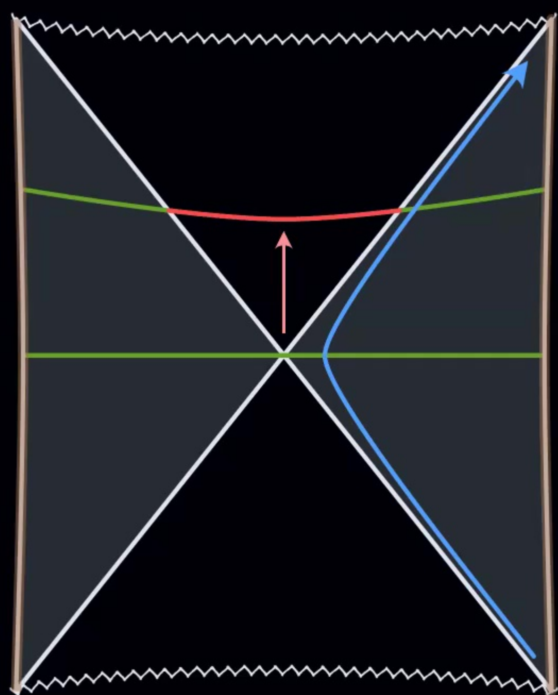


$$|\psi_0\rangle \propto \sum_k e^{-\beta E_k/2} |E_k\rangle_L |E_k\rangle_R$$

*thermofield double*

Susskind, PiTP (2018) 1810.11563; …

In AdS/CFT, physics in the bulk (AdS) has a dual description on the boundary (CFT).

*Two entangled sides of a black hole connected by a "wormhole"*

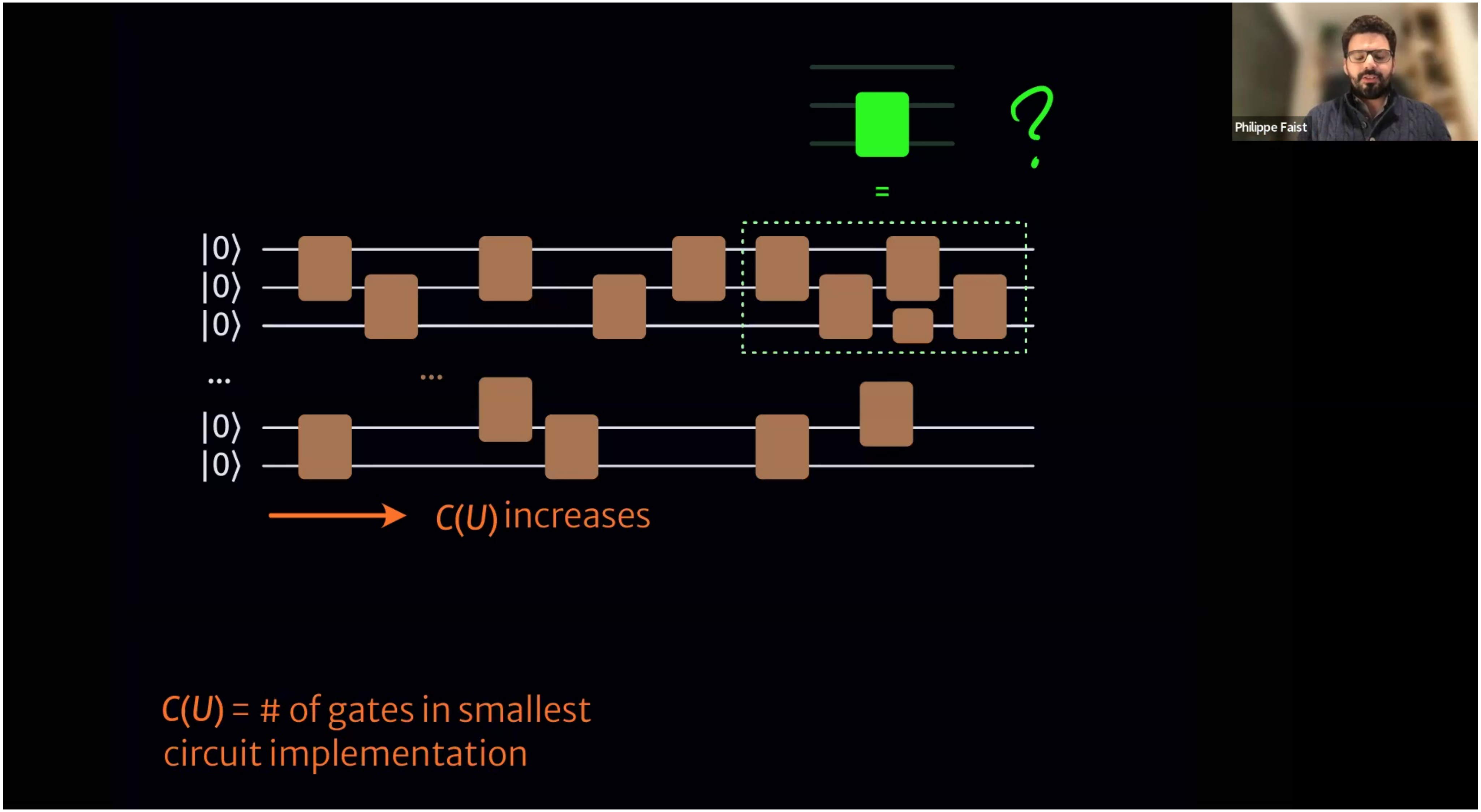
$$|\psi_t\rangle = U_L(t)\, U_R(t)\, |\psi_0\rangle$$

$$|\psi_0\rangle \propto \sum_k e^{-\beta E_k/2} |E_k\rangle_L |E_k\rangle_R$$

*thermofield double*

Complexity expected to be the CFT dual quantity to the wormhole length & to grow linearly in time

Susskind, PiTP (2018) 1810.11563; …



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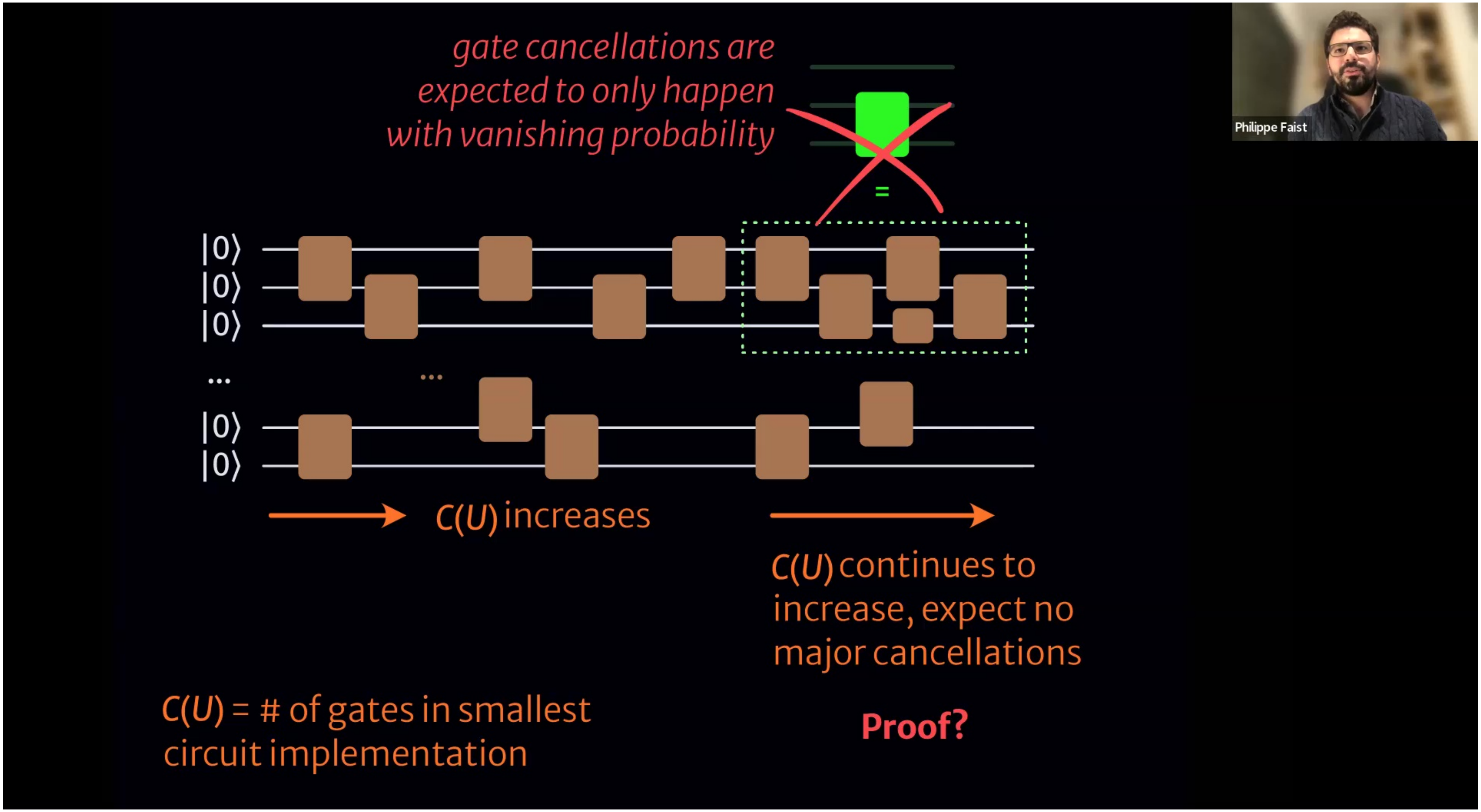
=
?
C(U) increases
C(U) = # of gates in smallest circuit implementation
Philippe Faist


gate cancellations are
expected to only happen
with vanishing probability
=
|0⟩
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|0⟩
...
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|0⟩
C(U) increases
C(U) continues to
increase, expect no
major cancellations
C(U) = # of gates in smallest
circuit implementation
Proof?
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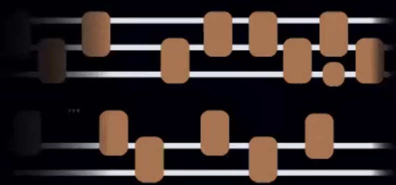
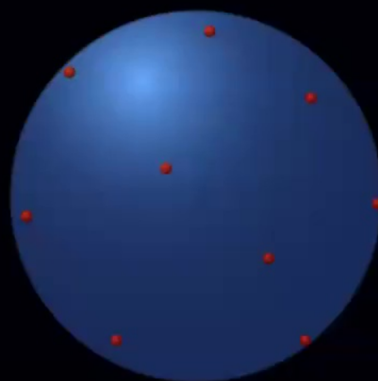
# Complexity growth from $k$-designs

$\approx$



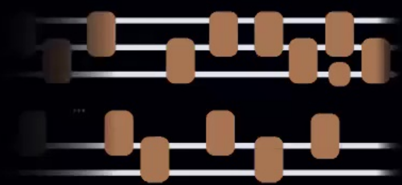
*"evenly distributed" in unitary space*

$SU(2^n)$

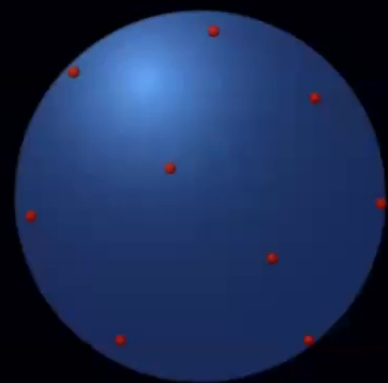
Brandão *et al.* CMP 2016;
Roberts & Yoshida JHEP 2017;
Hunter-Jones 1905.12053; …

$k$-design = reproduces the first $k$ moments of the Haar measure

# Complexity growth from $k$-designs

$\approx$



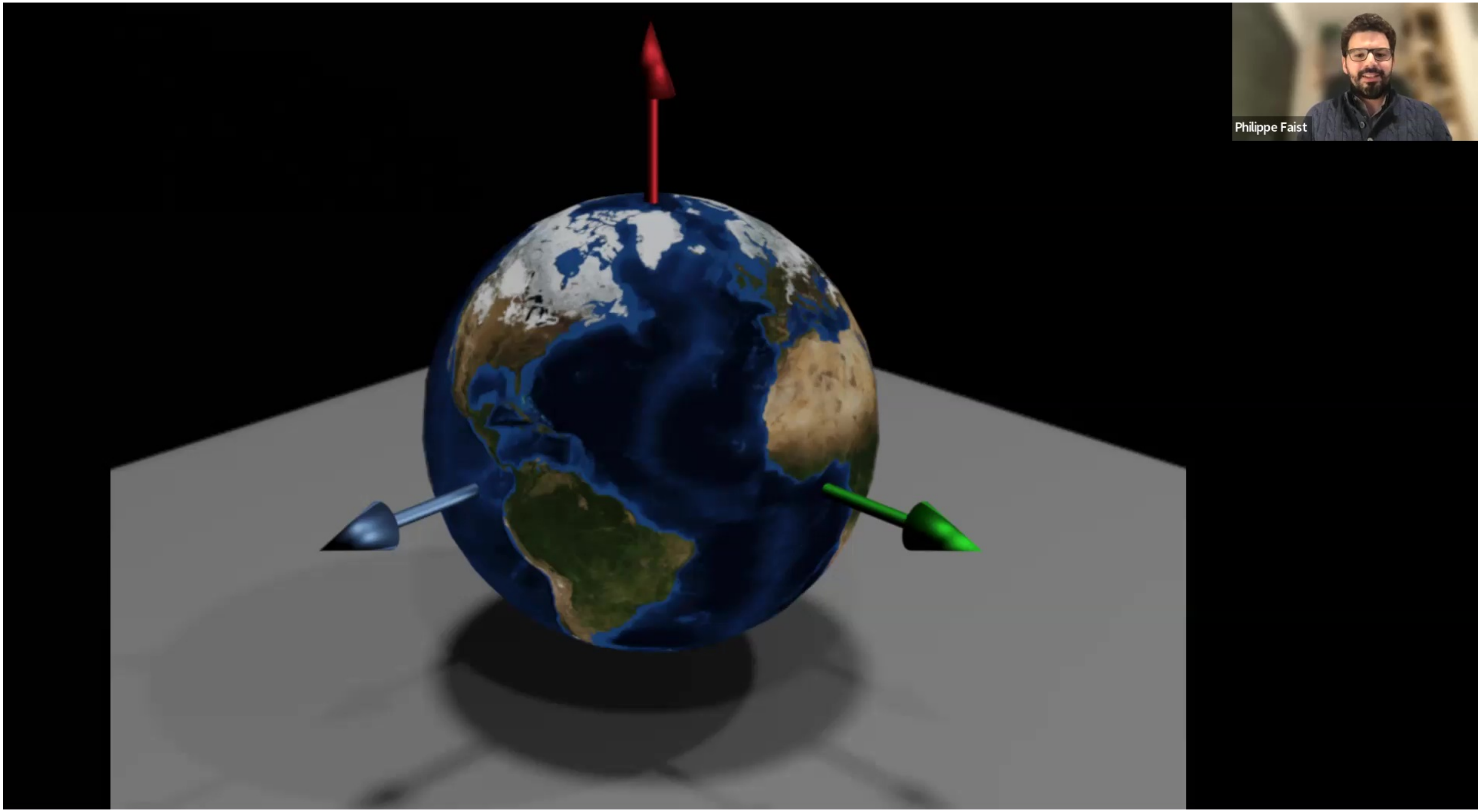

Brandão *et al.* CMP 2016;
Roberts & Yoshida JHEP 2017;
Hunter-Jones 1905.12053; …

$k$-design = reproduces the first $k$ moments of the Haar measure

- $k$-designs have complexity $\propto k$

  polynomial growth of complexity for random circuits (linear in some regimes)

Brandão *et al.*
PRXQ 2021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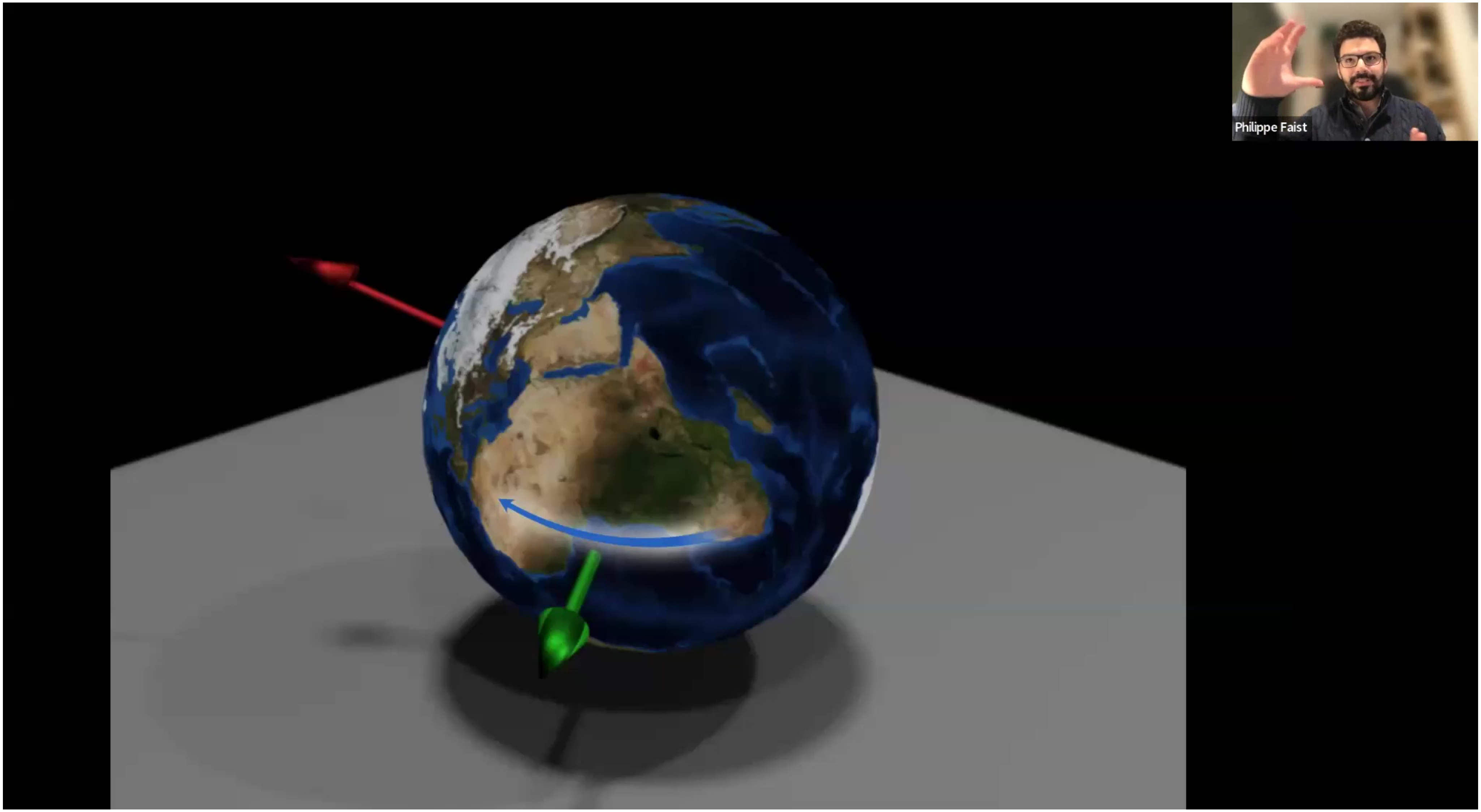
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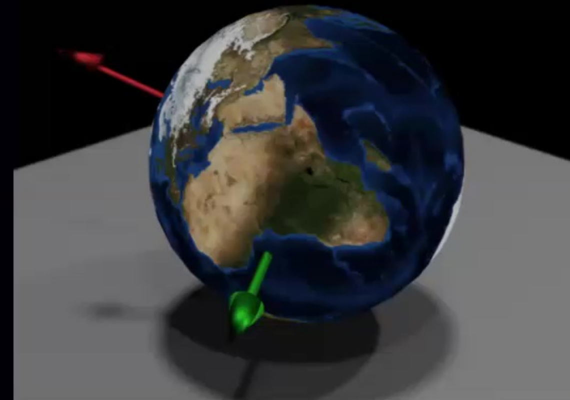
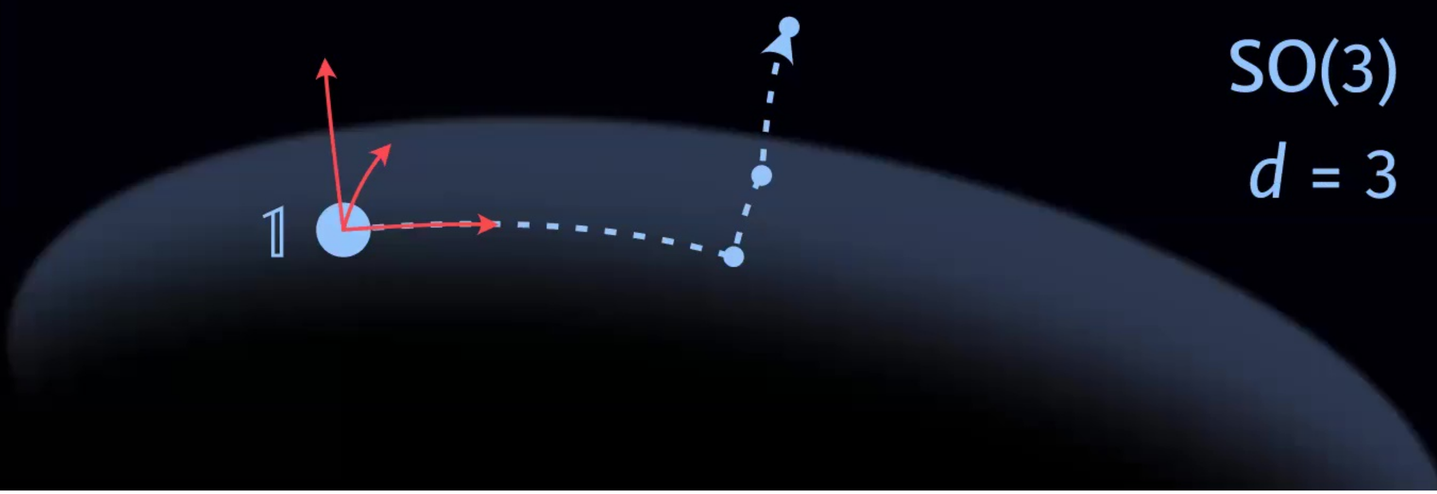
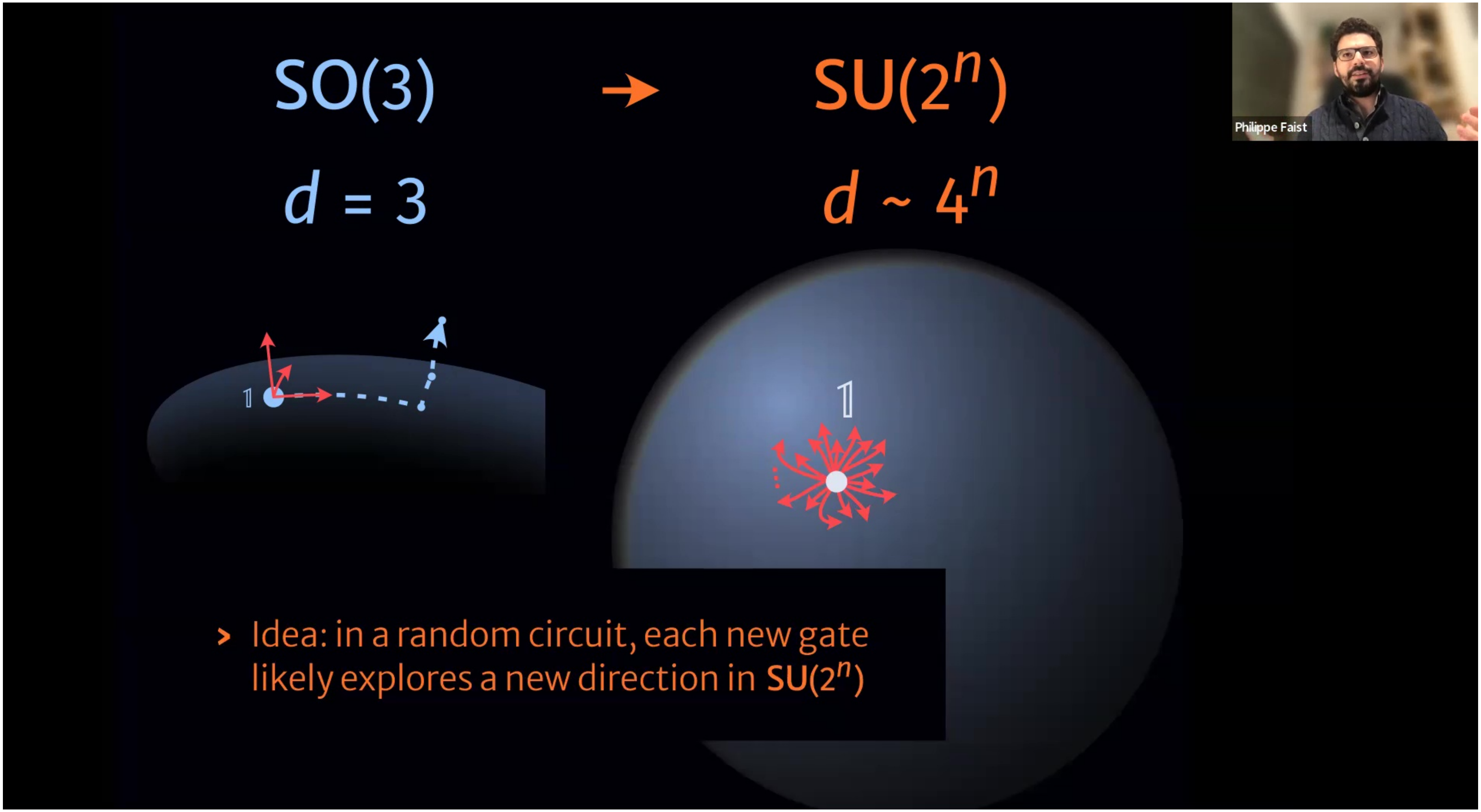
- the first few rotations explore "new directions in rotation space"
- Any rotation of the 2-sphere can be decomposed into at most 3 rotations around $X$, $Z$




SO(3)
d = 3
SU(2^n)
d ~ 4^n
𝟙
𝟙
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› Idea: in a random circuit, each new gate likely explores a new direction in SU(2^n)


*Theorem:* With unit probability, a random $n$-qubit circuit with $R \leq 4^n$ 2-local gates has (exact) complexity

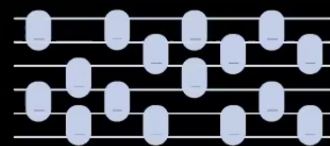
Result #1

$$C(U) \geq \frac{R}{9L} - \frac{n}{3}.$$

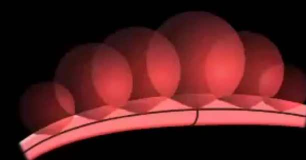
$L$ is related to the connectedness of the circuit layout

Haferkamp+ 2106.05305


1) Architecture $A$



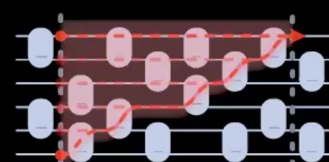
2) Accessible dimension $d_A$



3) Lower bound on $d_A$

$$d_A \geq \Omega(\#\text{ gates})$$

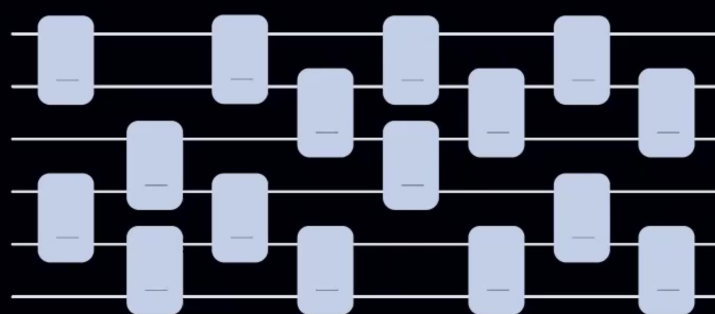
↳ backwards light cone



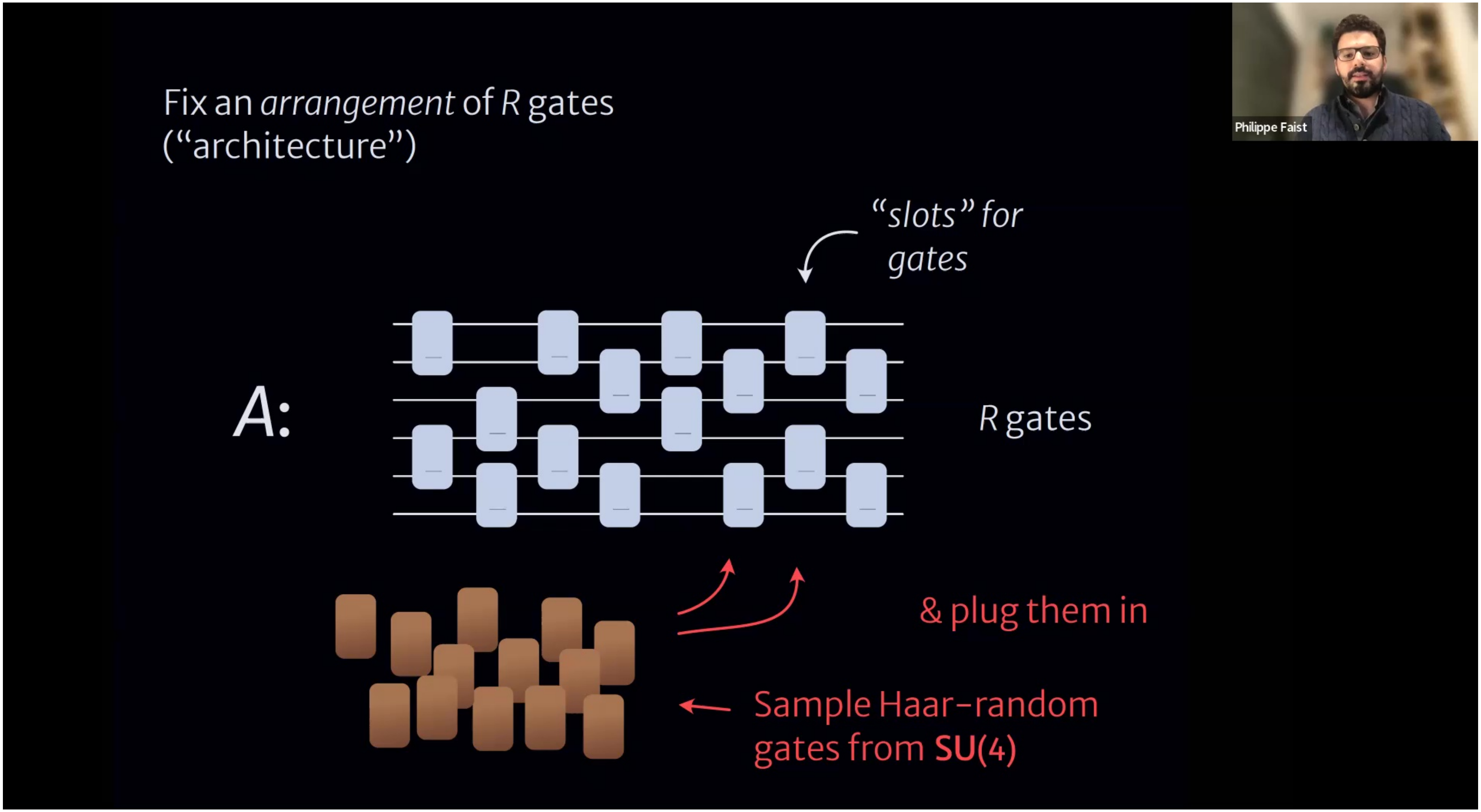
Fix an *arrangement* of $R$ gates
(“architecture”)

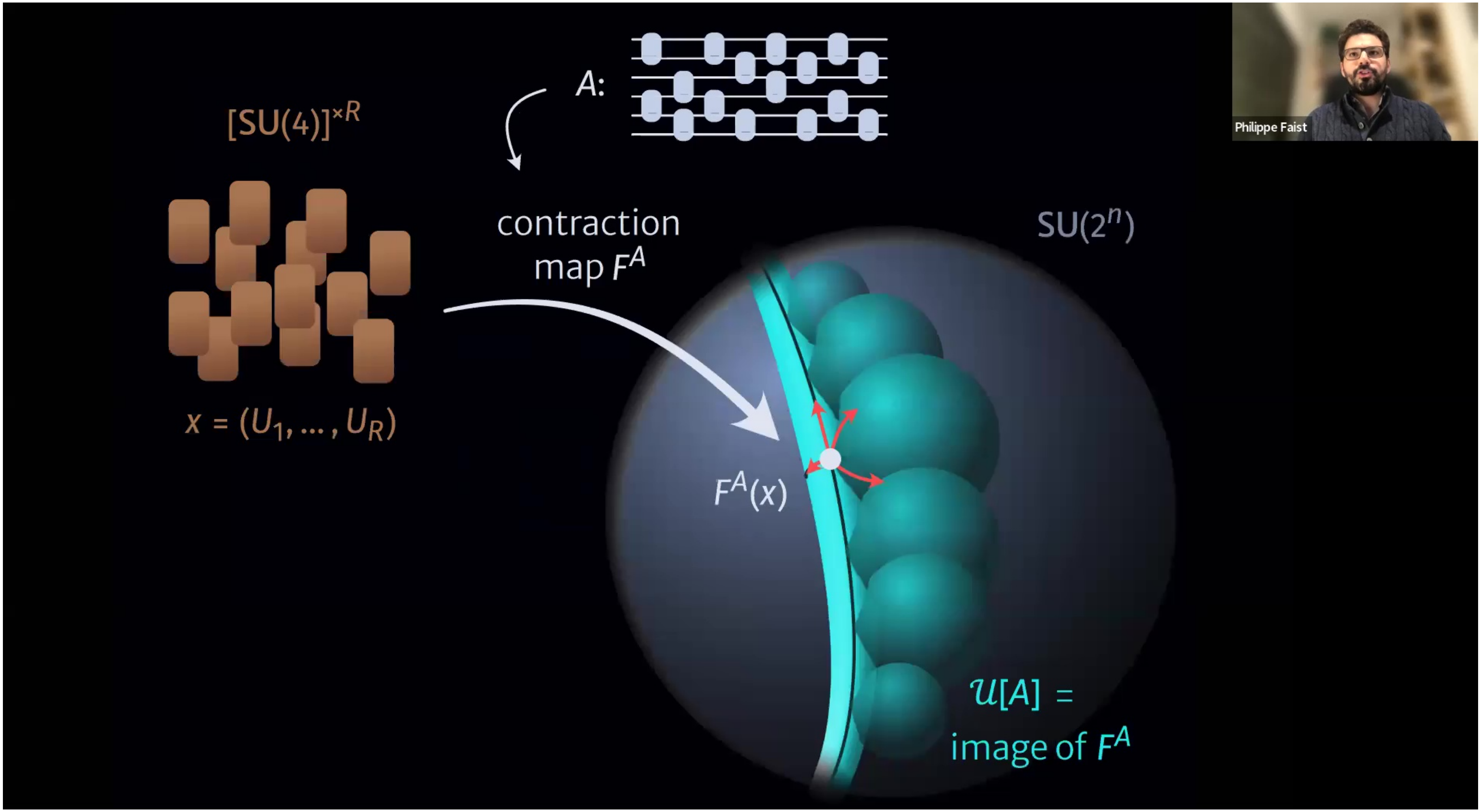
*“slots” for gates*

$A$:

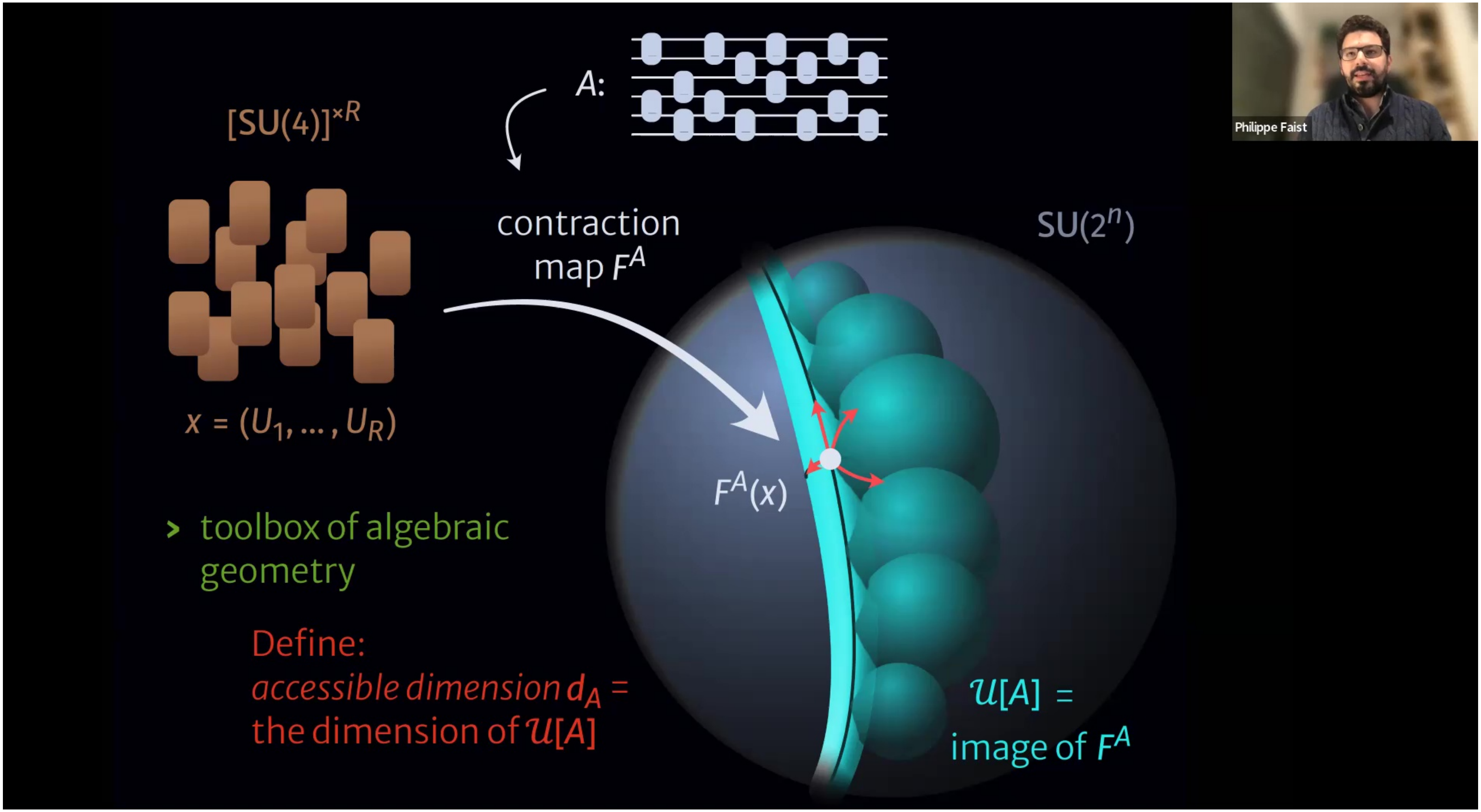


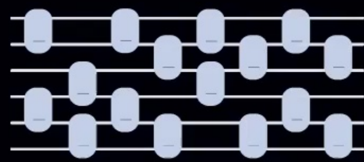
$R$ gates


Fix an *arrangement* of $R$ gates
("architecture")
"slots" for gates
A:
$R$ gates
& plug them in
Sample Haar-random gates from **SU(4)**
Philippe Faist



A:
$[SU(4)]^{\times R}$
contraction
map $F^A$
$SU(2^n)$
$x = (U_1, \dots, U_R)$
$F^A(x)$
$\mathcal{U}[A]$ =
image of $F^A$
Philippe Faist


A:
[SU(4)]^{×R}
contraction
map F^A
SU(2^n)
x = (U_1, … , U_R)
F^A(x)
toolbox of algebraic
geometry
Define:
accessible dimension d_A =
the dimension of 𝒰[A]
𝒰[A] =
image of F^A
Philippe Faist

$A$: 

$R$ gates $n$ qubits

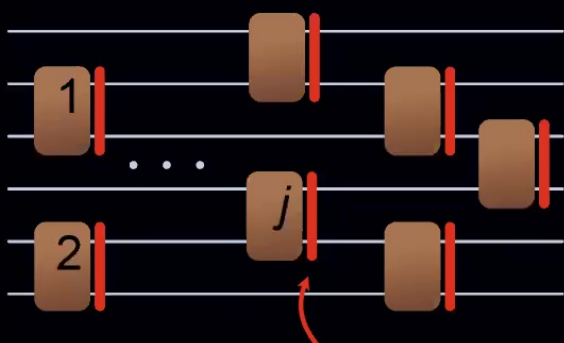
$d_A$ = dimension of $\mathcal{U}[A]$


- $d_A \leqslant 9R + 3n$ for any architecture $A$ of $R$ gates (parameter counting)

- $d_A \geqslant \frac{1}{L} R$ for any architecture $A$ that is sufficiently connected ($L$)

CORE TECHNICAL RESULT

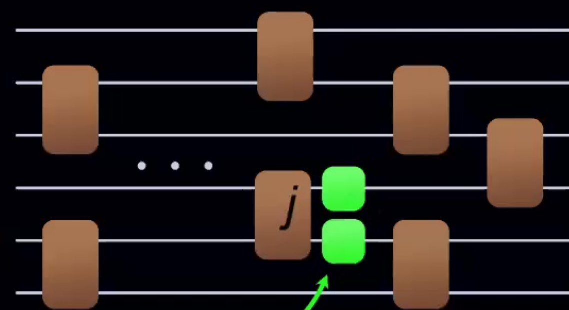

- To lower bound $d_A$, it suffices to find a single point $x$ such that $F^A(x)$ can be perturbed in many different directions
- Strategy: Pick $x$ as a set of Clifford gates
- Perturb gates to identify independent degrees of freedom in $\mathcal{U}[A]$



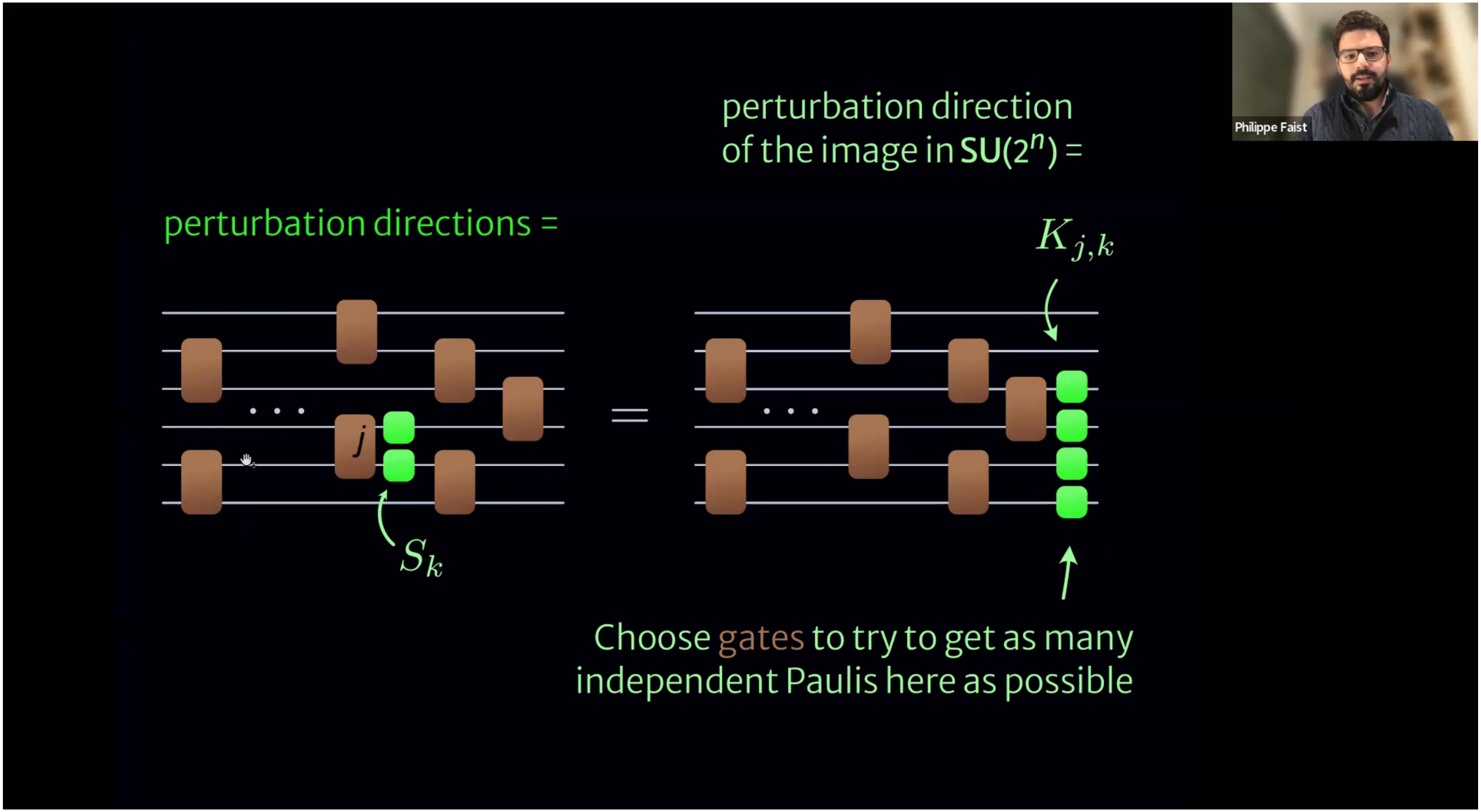

$e^{i \sum_k \epsilon_{j,k} S_k}$

perturbation generated by Pauli operators

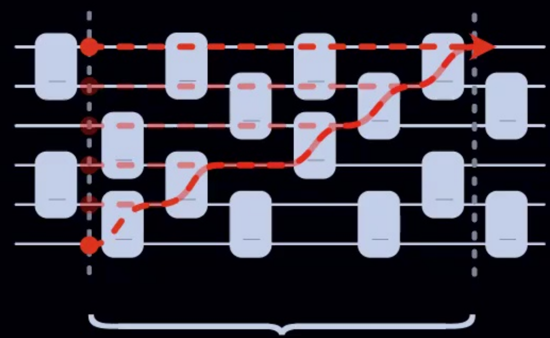
perturbation directions =




$S_k$ Pauli operator


perturbation direction
of the image in SU(2^n) =
perturbation directions =
j
S_k
K_{j,k}
Choose gates to try to get as many
independent Paulis here as possible
Philippe Faist

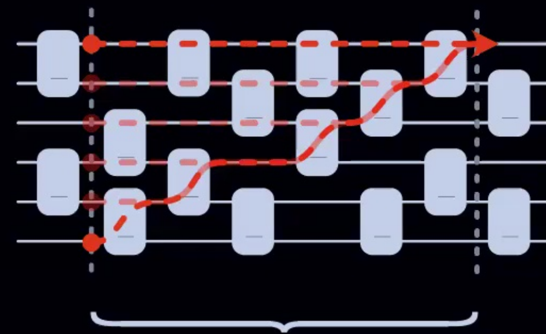

## The architecture must be "sufficiently connected"



$L$ gates

*backwards light cone:* all qubits can reach a common qubit

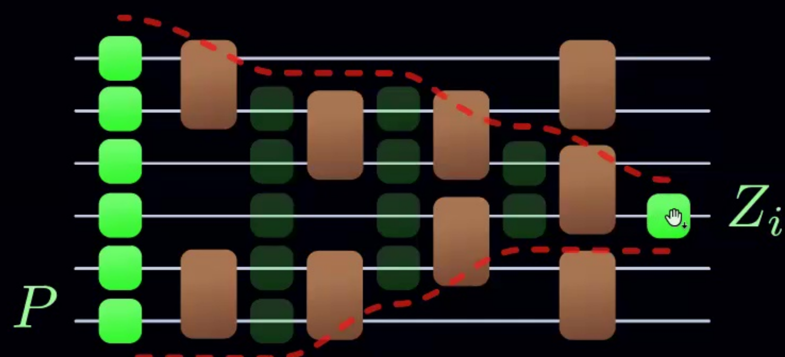
- The architecture must be "sufficiently connected"

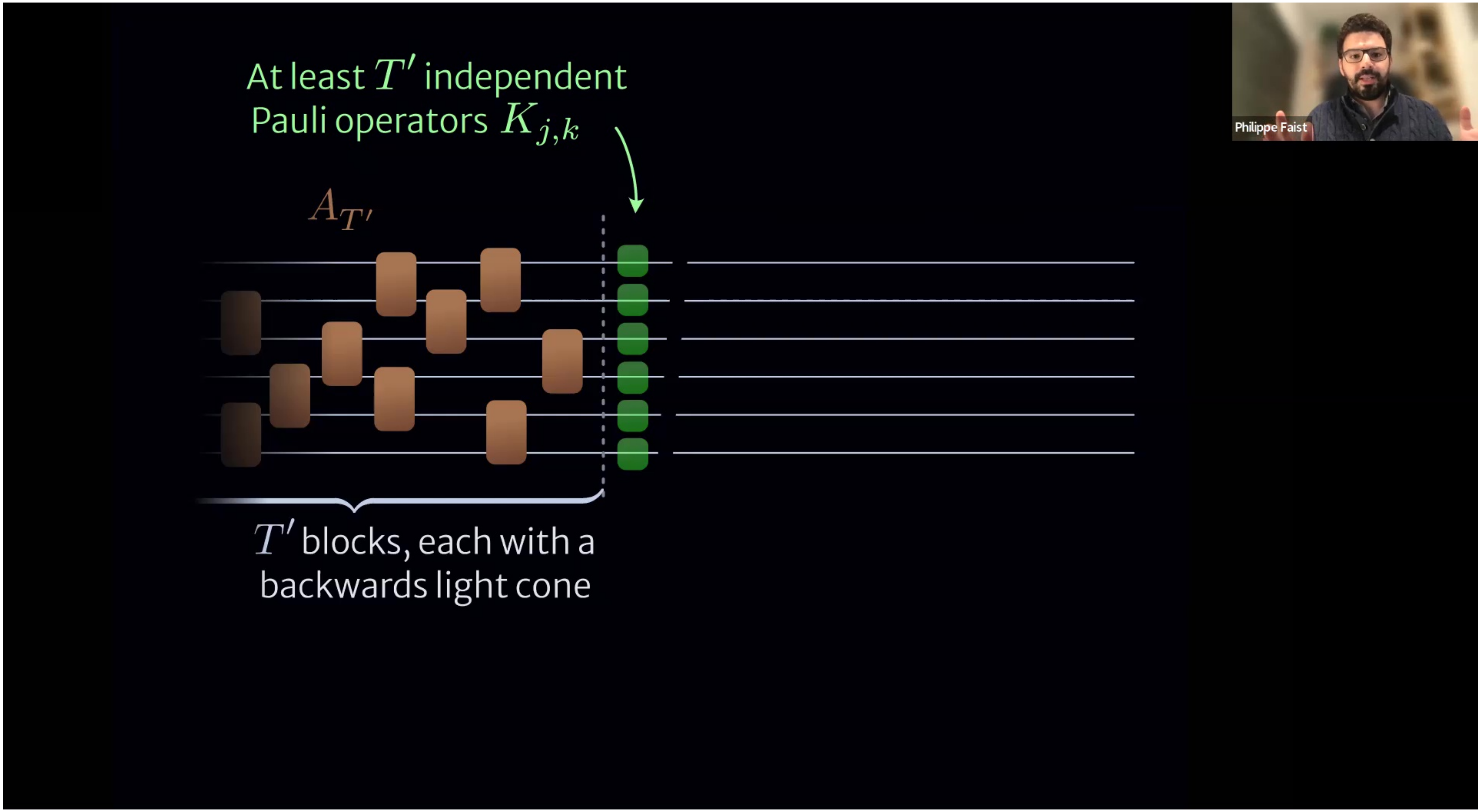


*L* gates

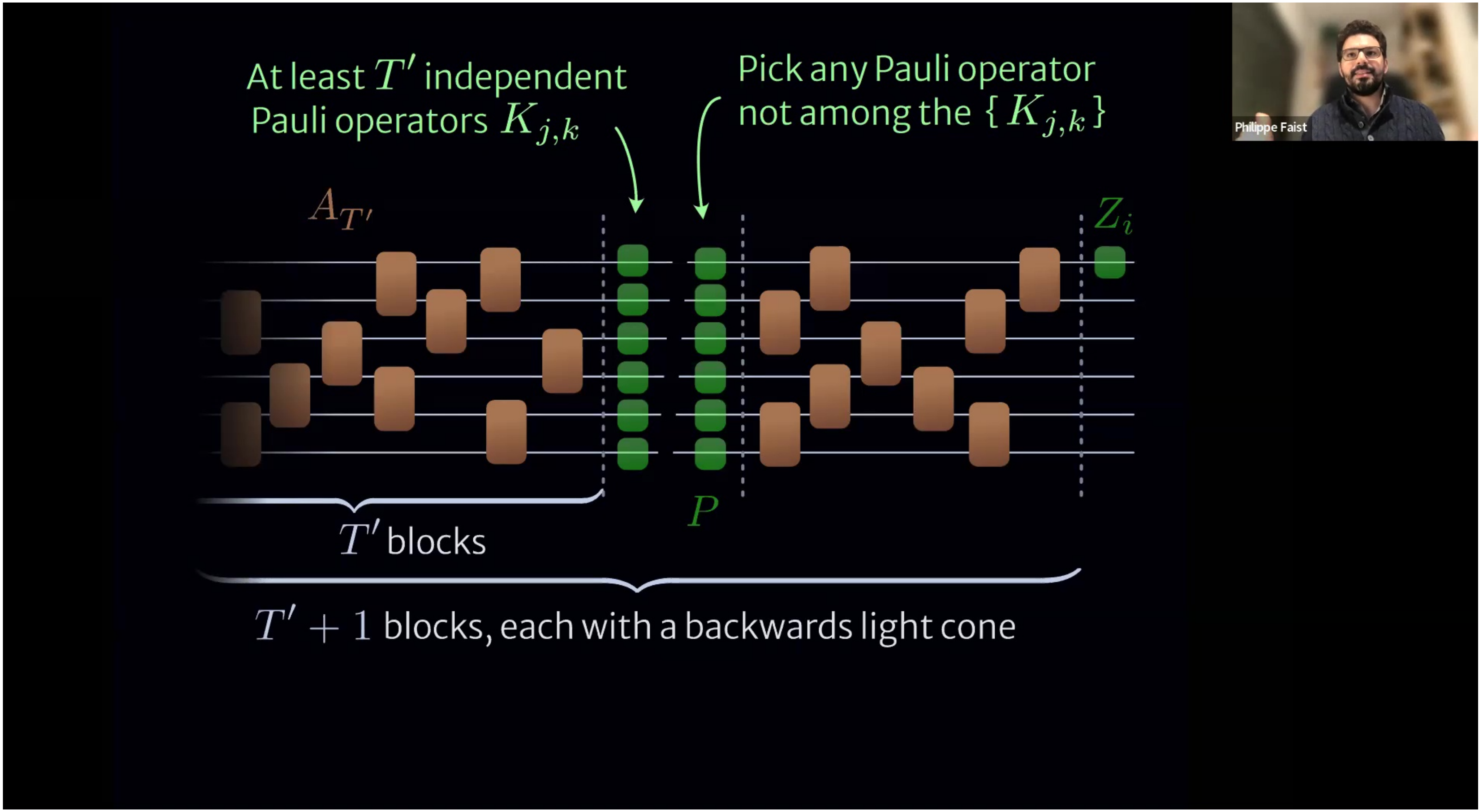
*backwards light cone:* all qubits can reach a common qubit

- For any Pauli operator *P*, we can pick Cliffords in the block to map *P* to a single-site *Z*



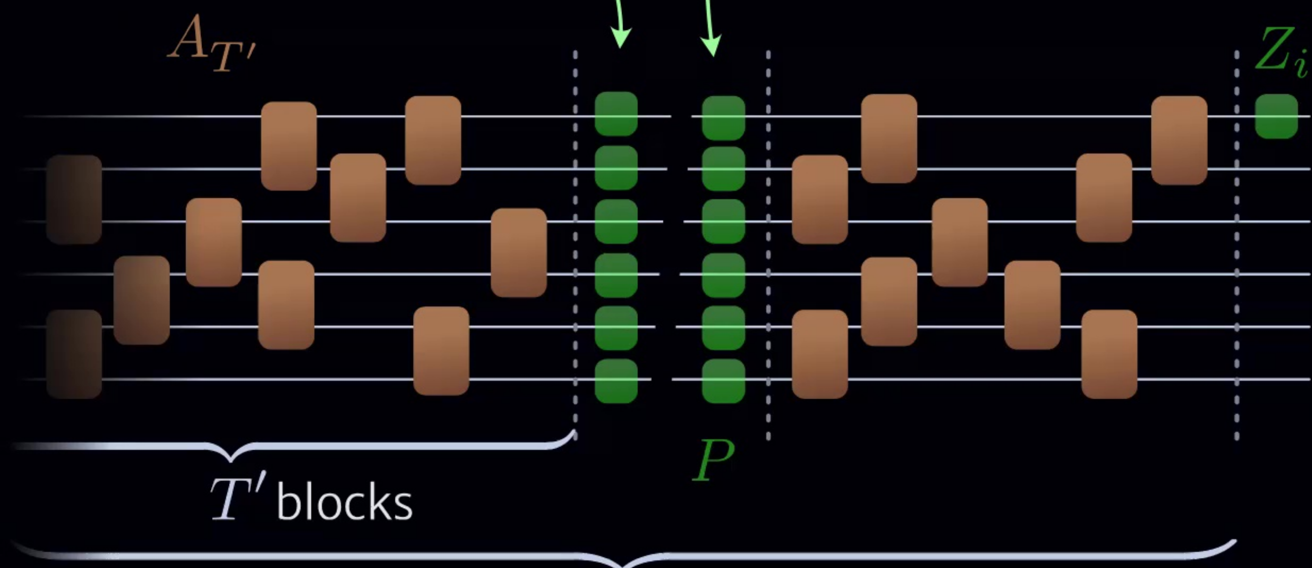
At least $T'$ independent
Pauli operators $K_{j,k}$
$A_{T'}$
$T'$ blocks, each with a
backwards light cone
Philippe Faist



At least $T'$ independent Pauli operators $K_{j,k}$
Pick any Pauli operator not among the $\{ K_{j,k} \}$
$A_{T'}$
$Z_i$
$P$
$T'$ blocks
$T' + 1$ blocks, each with a backwards light cone
Philippe Faist

At least $T'$ independent Pauli operators $K_{j,k}$

Pick any Pauli operator not among the $\{K_{j,k}\}$





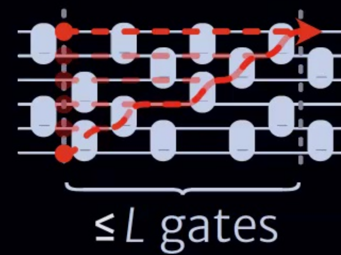
$T' + 1$ blocks, each with a backwards light cone

→ There are now $T' + 1$ independent perturbation directions of the image.

$d_{A_{T'+1}} \geq T' + 1$

$$d_A \geqslant \frac{1}{L} R$$

if $A$ has $T \geq R/L$ blocks of gates, each with a backwards light cone



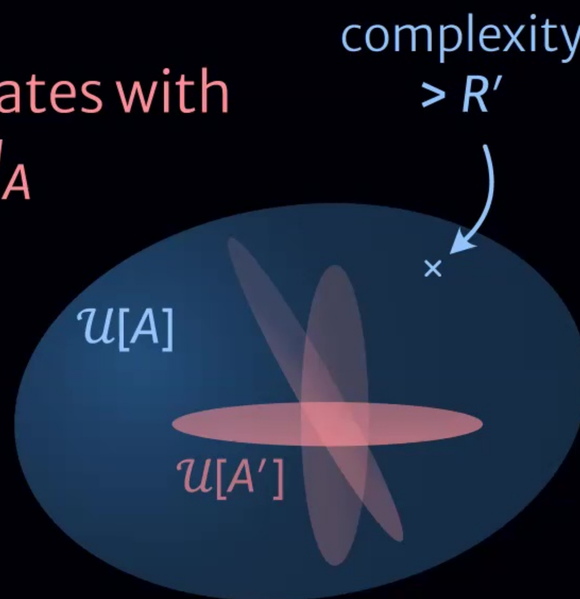


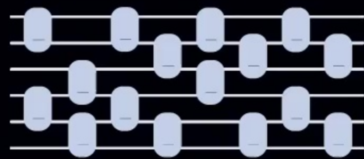
$$d_{A'} \leqslant 9R' + 3n$$

from parameter counting

For any architecture $A'$ of $R'$ gates with $R' < R/(9L) - n/3$, then $d_{A'} < d_A$

$\Rightarrow$ A random circuit from $A$ has zero chance of landing in $\mathcal{U}[A']$



$A$: 

$R$ gates $n$ qubits

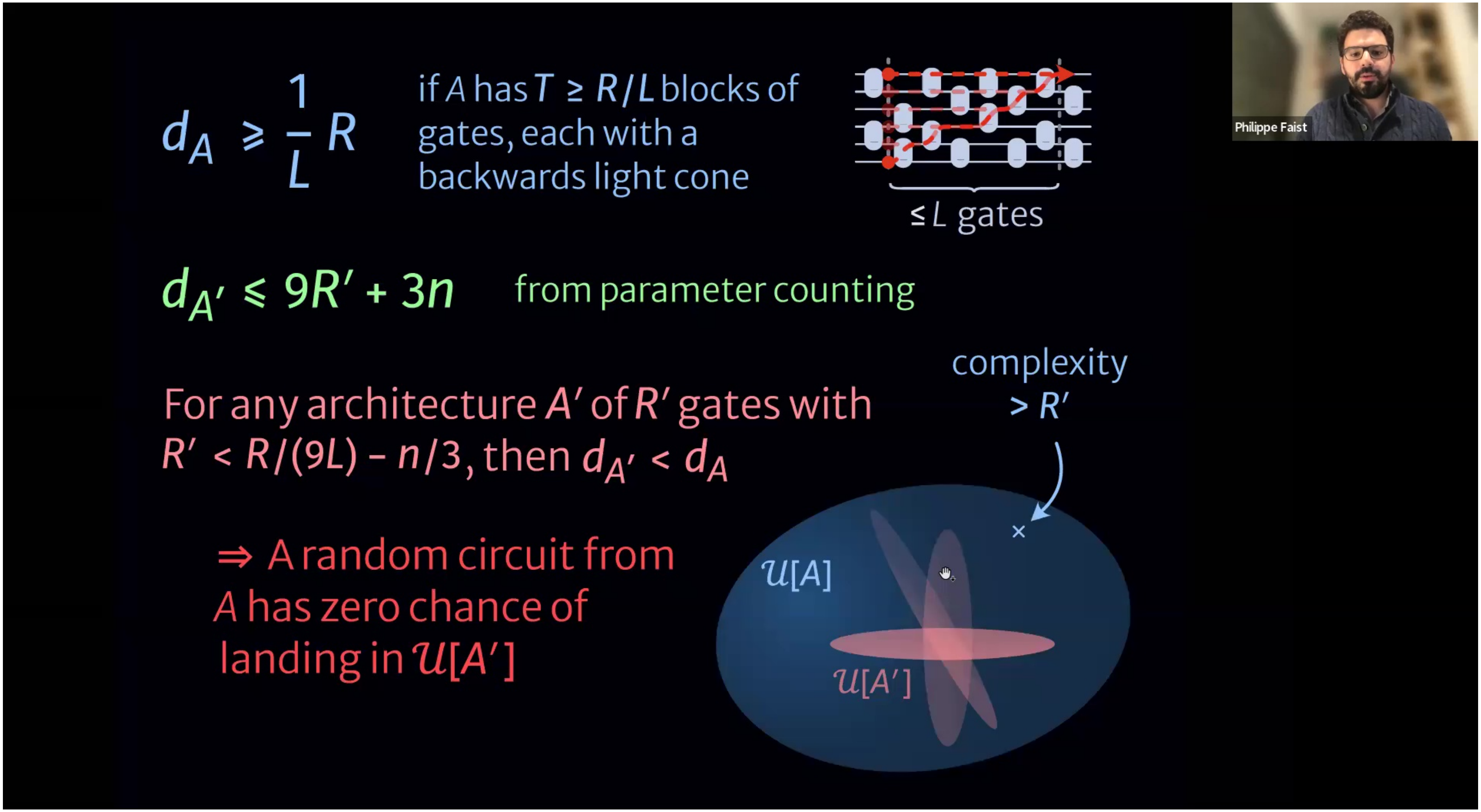
$d_A$ = dimension of $\mathcal{U}[A]$


*Theorem:* With unit probability, a random circuit sampled using an architecture $A$ with $R \leq 4^n$ gates has (exact) complexity

$$C(U) \geq \frac{R}{9L} - \frac{n}{3}.$$

(for $C(U)$ to be lower than this value, the sampled $U$ would have to land in a lower-dimensional set)

Haferkamp+ 2106.05305


$d_A \geqslant \frac{1}{L} R$
if $A$ has $T \geq R/L$ blocks of gates, each with a backwards light cone
$\leq L$ gates
$d_{A'} \leqslant 9R' + 3n$
from parameter counting
For any architecture $A'$ of $R'$ gates with $R' < R/(9L) - n/3$, then $d_{A'} < d_A$
⇒ A random circuit from $A$ has zero chance of landing in $\mathcal{U}[A']$
complexity $> R'$
$\mathcal{U}[A]$
$\mathcal{U}[A']$
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Proof works for any measure on the 2-qubit gates that is absolutely continuous w.r.t. Haar

Architecture can be chosen randomly

Notion of backwards light cone related to the switchback effect?

Stanford & Susskind PRD 2014

Robustness to small errors / approximate complexity?

Brandão *et al.* PRXQ 2021

Part I: Linear Growth of Quantum Circuit Complexity

(random circuits, pure states)

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- To lower bound $d_A$, it suffices to find a single point $x$ such that $F^A(x)$ can be perturbed in many different directions
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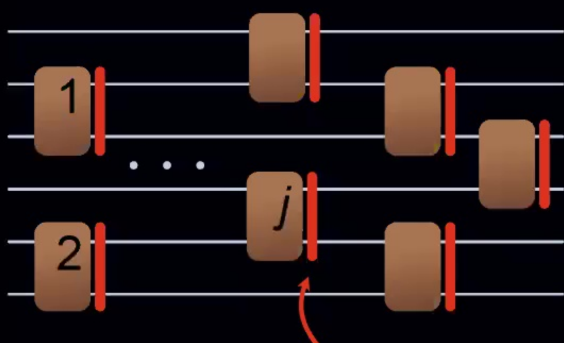
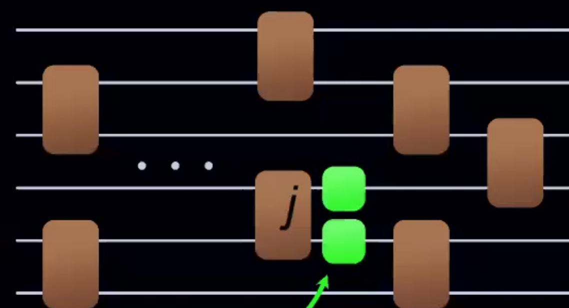


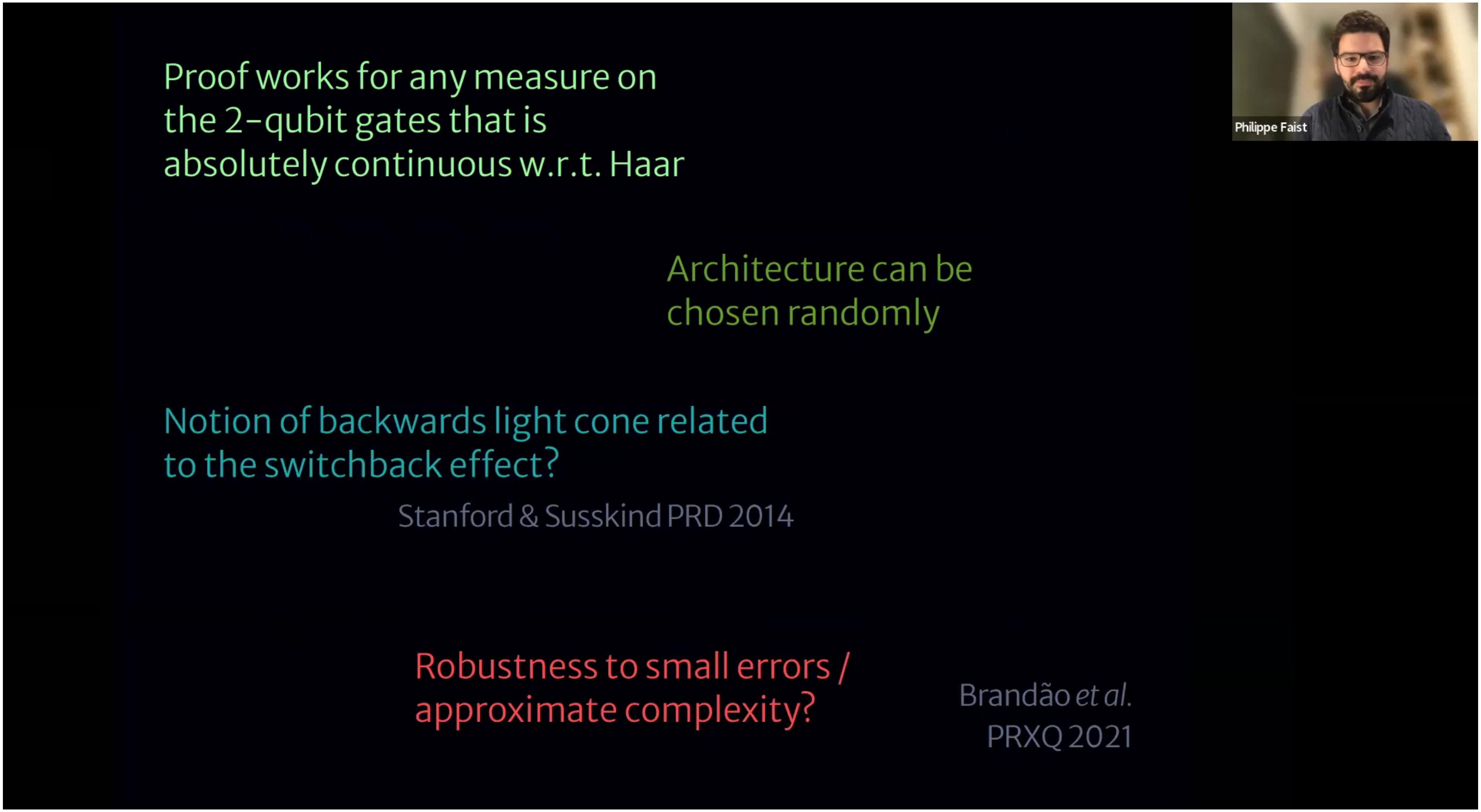

$e^{i \sum_k \epsilon_{j,k} S_k}$

perturbation generated by Pauli operators

perturbation directions =




$S_k$ Pauli operator



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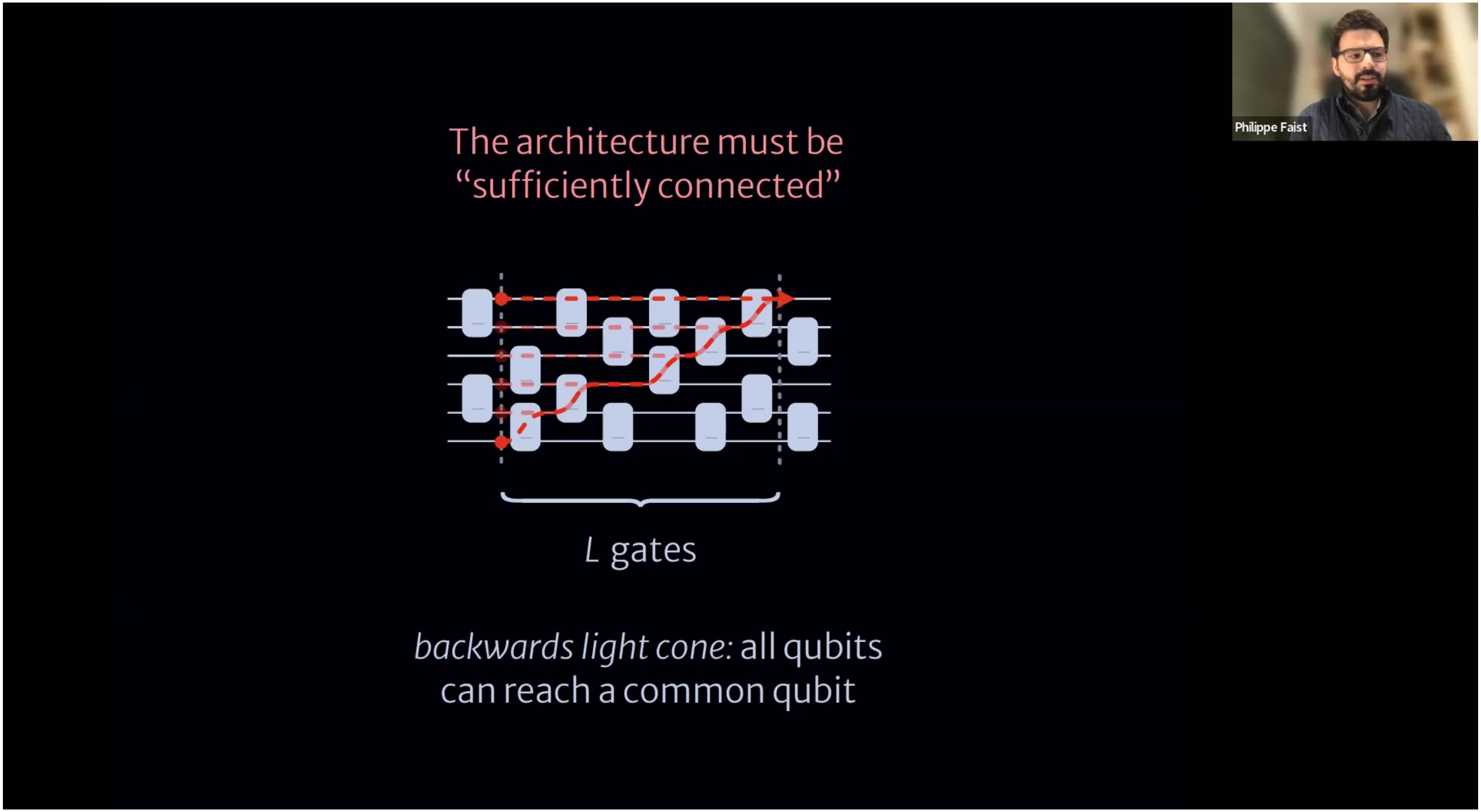
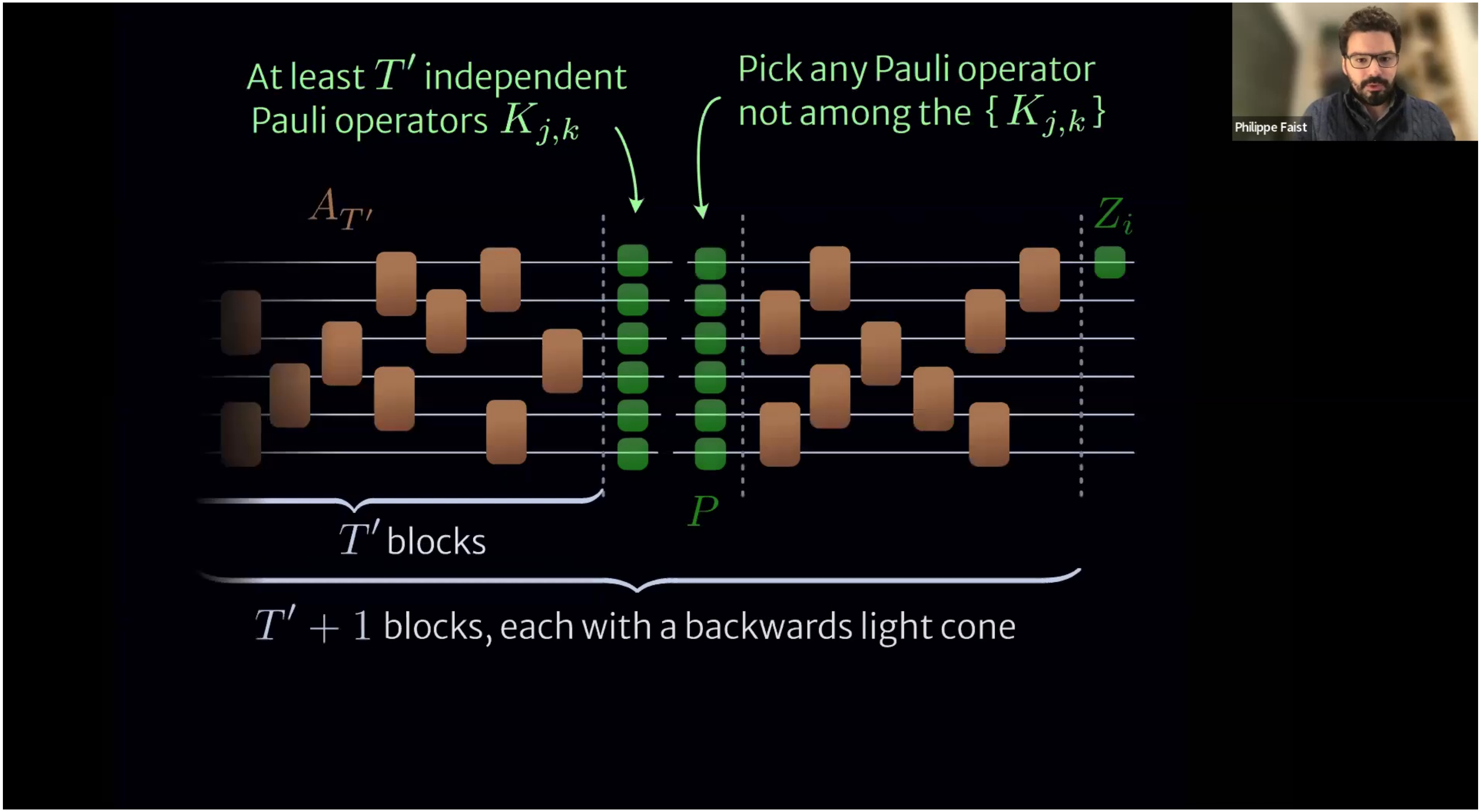

The architecture must be
“sufficiently connected”
L gates
backwards light cone: all qubits
can reach a common qubit
Philippe Faist



At least $T'$ independent Pauli operators $K_{j,k}$
Pick any Pauli operator not among the $\{K_{j,k}\}$
$A_{T'}$
$Z_i$
$P$
$T'$ blocks
$T' + 1$ blocks, each with a backwards light cone
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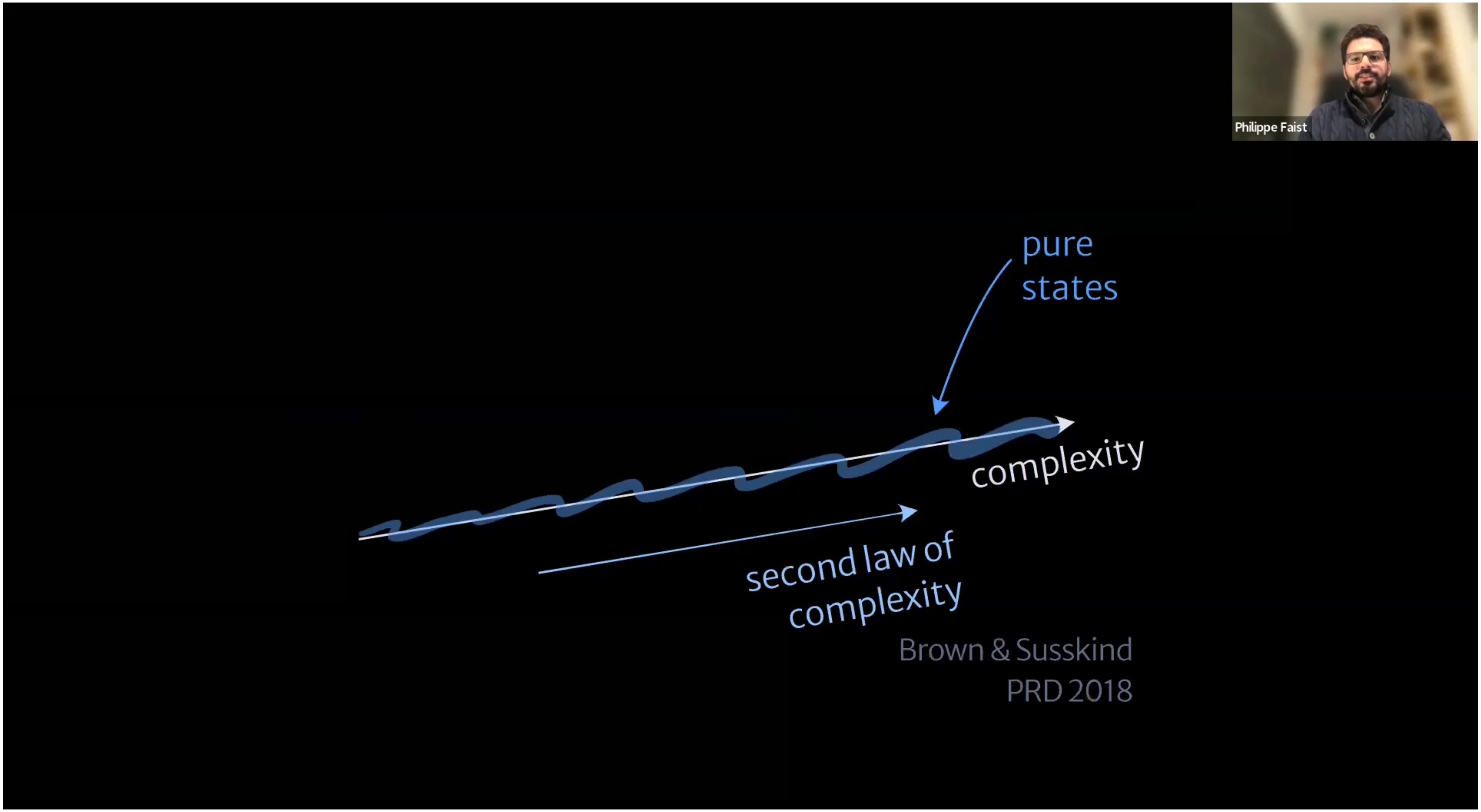
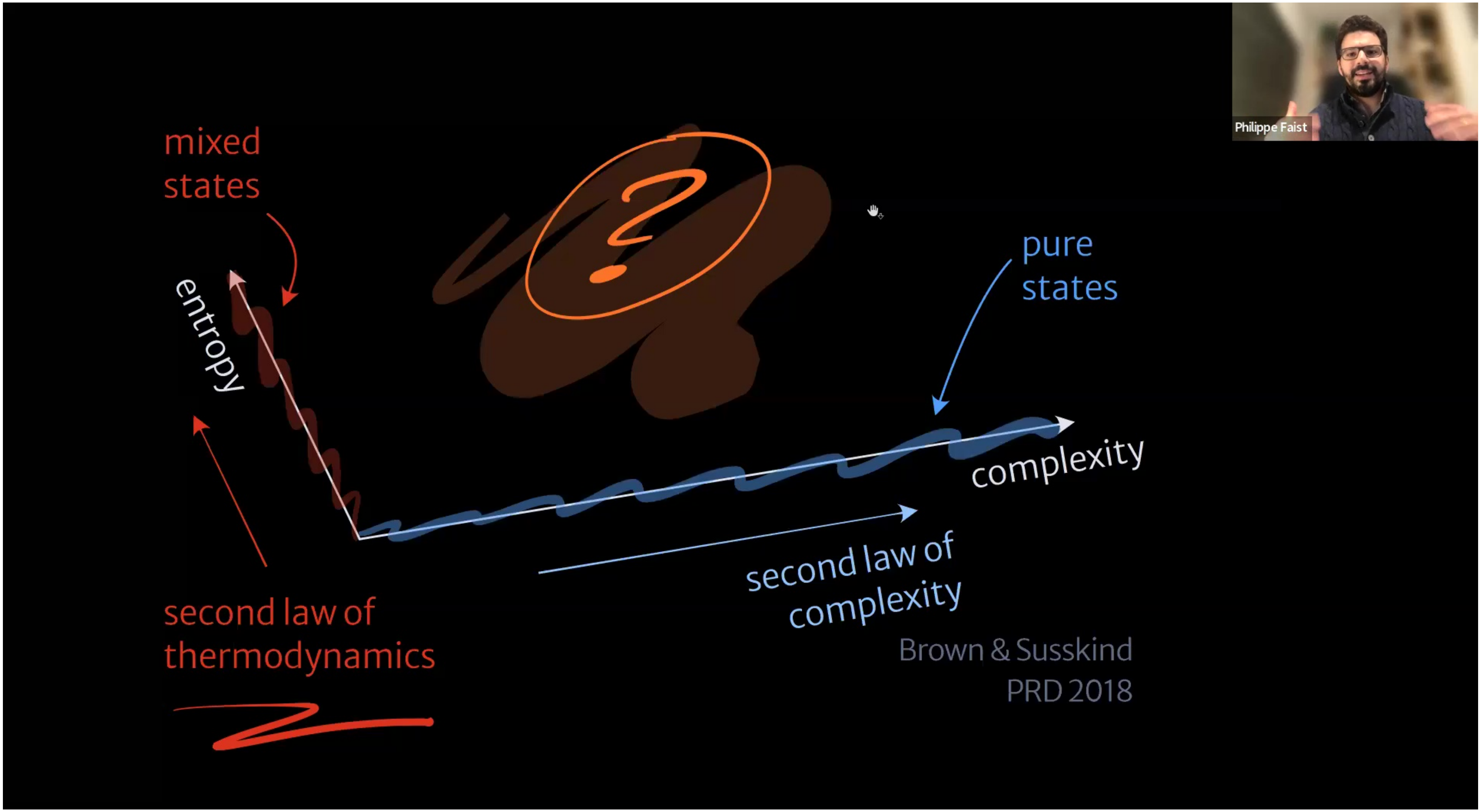

Philippe Faist
pure
states
complexity
second law of
complexity
Brown & Susskind
PRD 2018


Philippe Faist
mixed
states
entropy
pure
states
complexity
second law of
complexity
second law of
thermodynamics
Brown & Susskind
PRD 2018

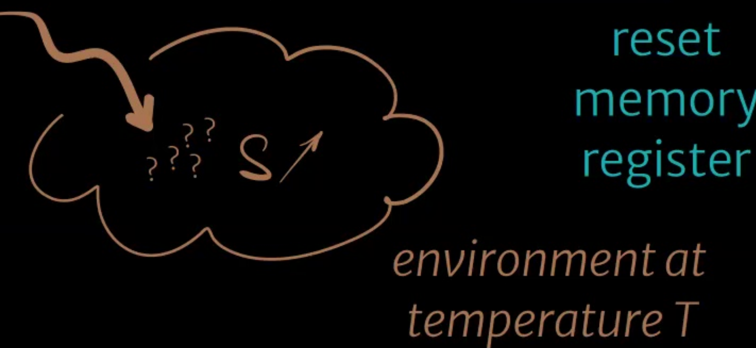
# Information & thermodynamics: Landauer erasure


? ? ? ? ? ? ? ? → 0 0 0 0 0 0 0 0

reset
memory
register

Szilárd 1929;
Landauer 1961;
Bennett 1982; …

# Information & thermodynamics: Landauer erasure

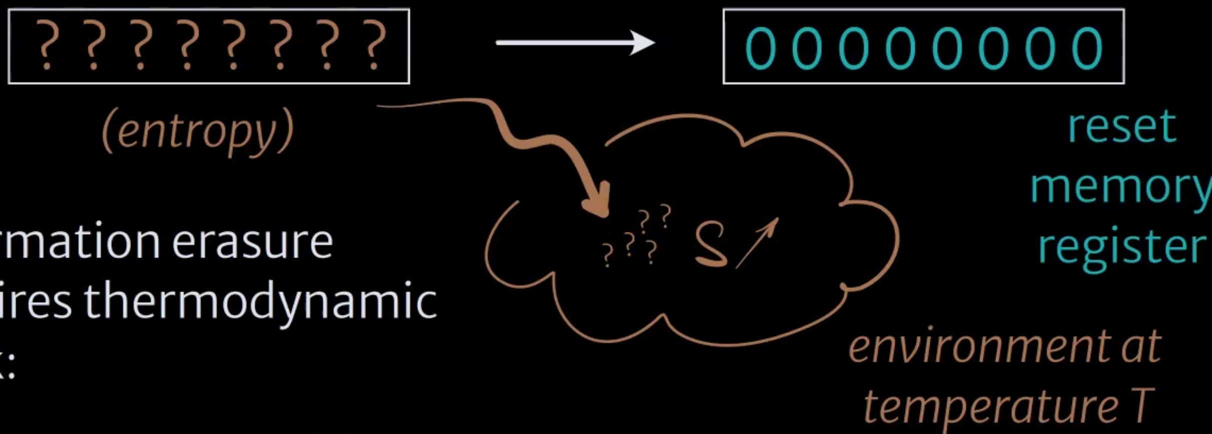




Information erasure requires thermodynamic work:

Work ≥ $kT$ **ln**(2) per bit discarded in the environment

Szilárd 1929;
Landauer 1961;
Bennett 1982; …

# Information & thermodynamics: Landauer erasure





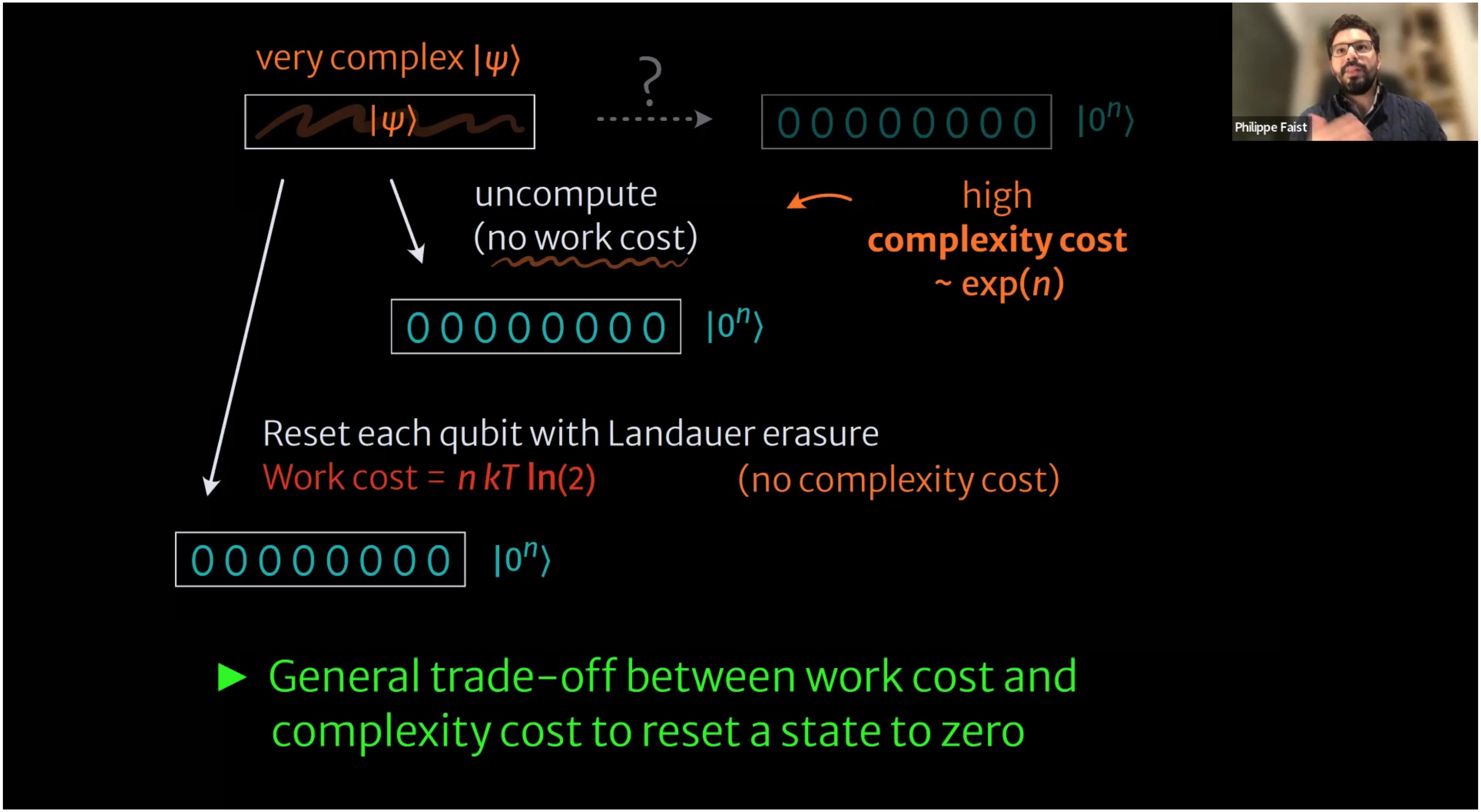
Information erasure requires thermodynamic work:

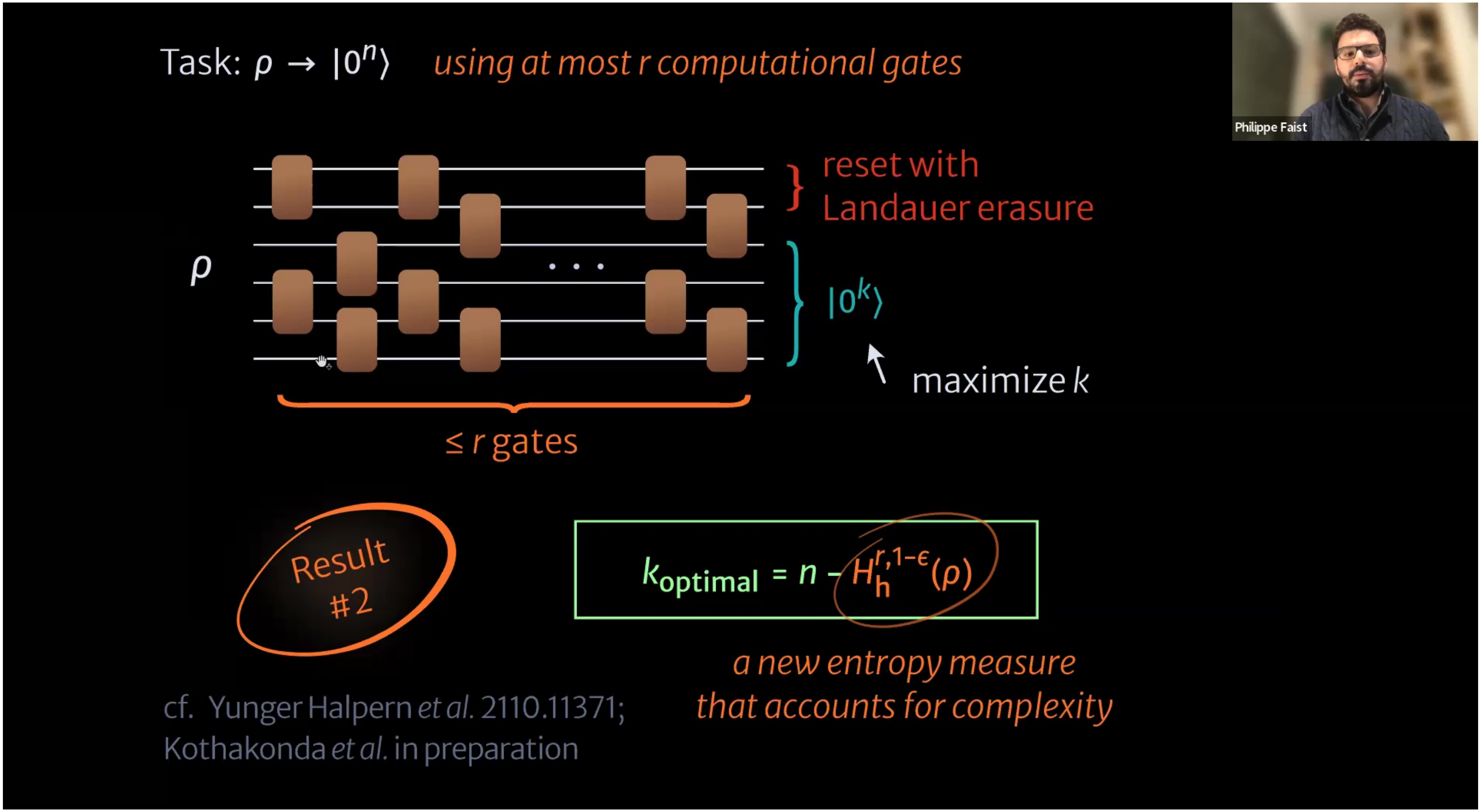
Work ≥ $kT$ **ln**(2) per bit discarded in the environment

Szilárd 1929;
Landauer 1961;
Bennett 1982; …

Formalized in the resource theory of thermodynamics

Brandão *et al* PRL 2013;
PhF *et al* Nat Comm 2015;
Chitambar & Gour RMP 2019; …


very complex |ψ⟩
|ψ⟩
?
0 0 0 0 0 0 0 0 0
|0^n⟩
uncompute
(no work cost)
high
complexity cost
~ exp(n)
0 0 0 0 0 0 0 0 0
|0^n⟩
Reset each qubit with Landauer erasure
Work cost = n kT ln(2)
(no complexity cost)
0 0 0 0 0 0 0 0 0
|0^n⟩
General trade-off between work cost and complexity cost to reset a state to zero
Philippe Faist



Task: ρ → |0ⁿ⟩ using at most r computational gates
reset with Landauer erasure
ρ
|0ᵏ⟩
maximize k
≤ r gates
Result #2
k_optimal = n − H_h^{r,1−ϵ}(ρ)
a new entropy measure that accounts for complexity
cf. Yunger Halpern et al. 2110.11371; Kothakonda et al. in preparation
Philippe Faist


Complexity entropy = amount of entropy the state *appears to have* if only accessed by observables of complexity at most $r$

$$H_h^{r,\eta}(\rho) = \log \min_{\substack{\mathrm{tr}(Q\rho)\geq\eta \\ Q\in M_r}} \mathrm{tr}(Q)$$

… how mixed does $Q$ have to be?

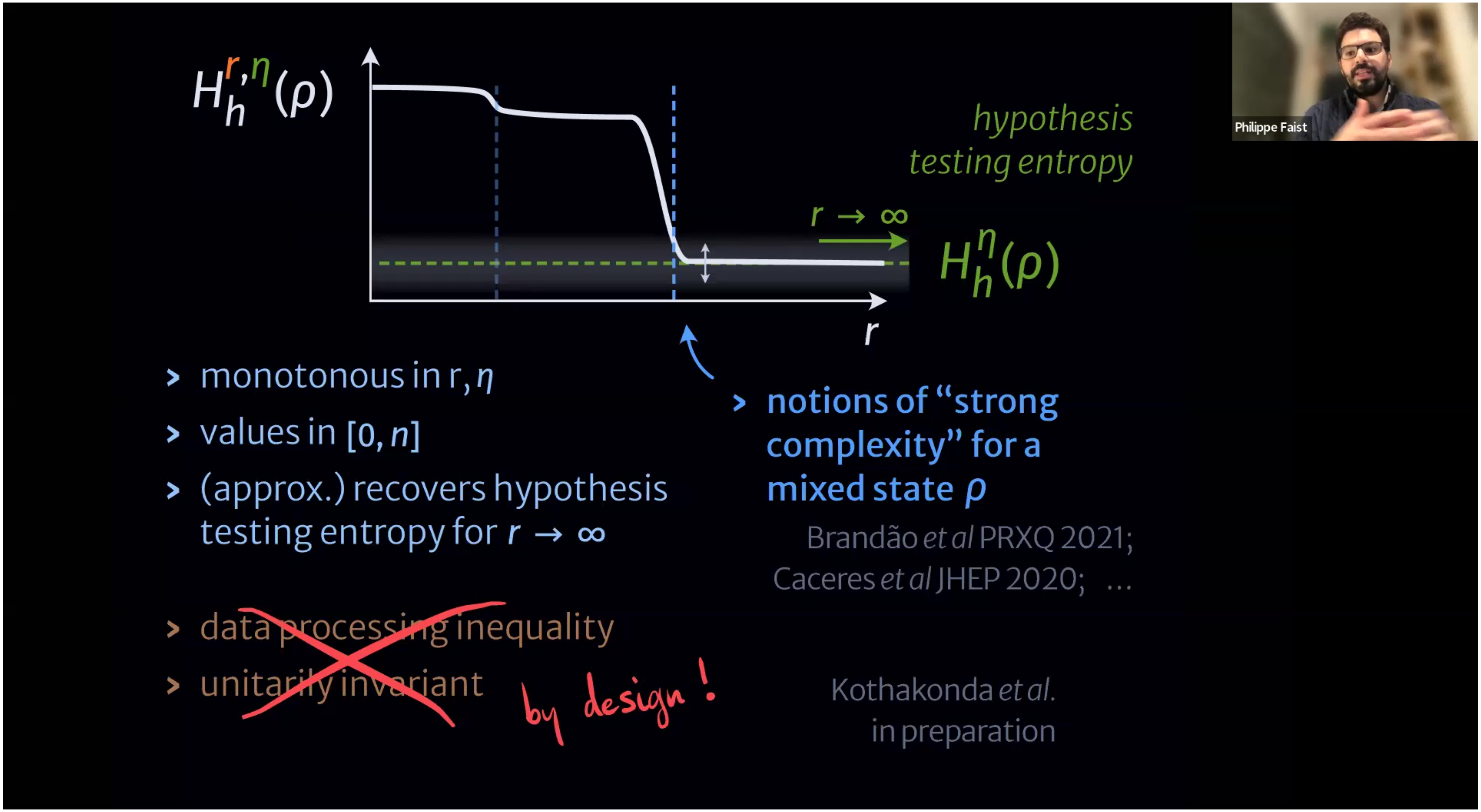
*based on the "hypothesis testing entropy"*

Brandao+ IEEE TIT 2011; Dupuis+ ICMP 2013; …

A POVM effect *$Q$ of complexity at most $r$ …*

… must capture most weight of the state $\rho$; …

*cf. also:* Gell-Mann & Hartle PRA 2007; Chen+ 1704.07309; …


$H_h^{r,\eta}(\rho)$
r
hypothesis testing entropy
$r \to \infty$
$H_h^{\eta}(\rho)$
› monotonous in r, $\eta$
› values in $[0, n]$
› (approx.) recovers hypothesis testing entropy for $r \to \infty$
› data processing inequality
› unitarily invariant
by design!
› notions of "strong complexity" for a mixed state $\rho$
Brandão *et al* PRXQ 2021;
Caceres *et al* JHEP 2020; …
Kothakonda *et al.*
in preparation
Philippe Faist


Part I: Linear Growth of Quantum Circuit Complexity

(random circuits, pure states)

Part II: Complexity & Entropy

(operational tasks, resource theory)

Part III: Discussion & Outlook


- Operational meaning of complexity entropy = amount of work required to reset a state *in the eyes of an agent who can perform at most r gates*

- Monotone in the resource theory of complexity?

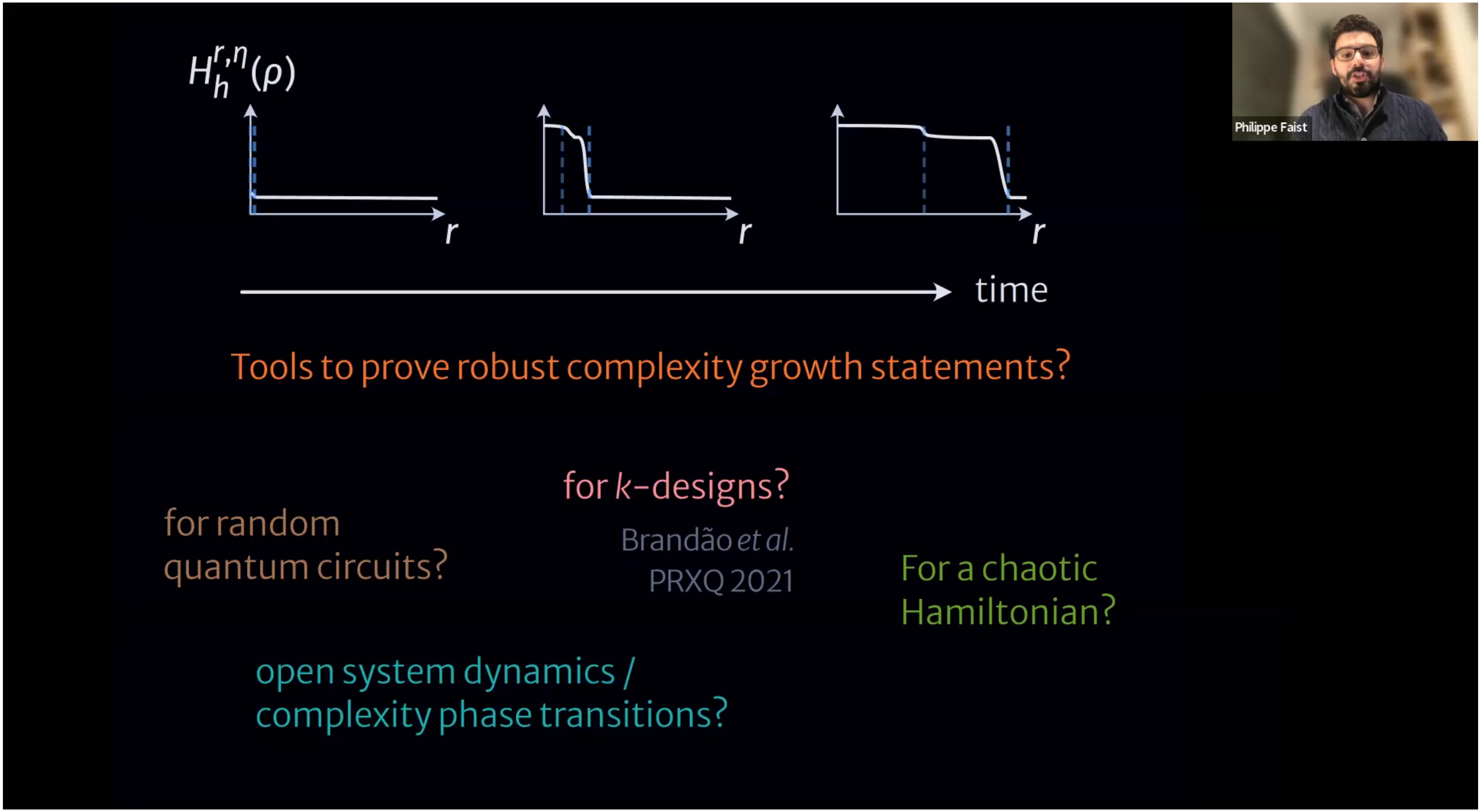
  Yunger Halpern *et al* 2110.11371

- Towards a unification of the second laws of thermodynamics and of complexity:

  $H_h^{r,\eta}(\rho)$ tends to only ever increase ?

- Complexity-aware replacement of the standard entropy for physics in high complexity regimes?

  (e.g. spin-glass, phase transitions, exotic phases of matter?)



$H_h^{r,\eta}(\rho)$
$r$
$r$
$r$
time
Tools to prove robust complexity growth statements?
for $k$-designs?
for random quantum circuits?
Brandão *et al.* PRXQ 2021
For a chaotic Hamiltonian?
open system dynamics / complexity phase transitions?
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The complexity entropy gives the resource costs for **data compression** with computational limitations

Quantifier for quantum **pseudorandomness**?

Ji, Liu, Song CRYPTO '18

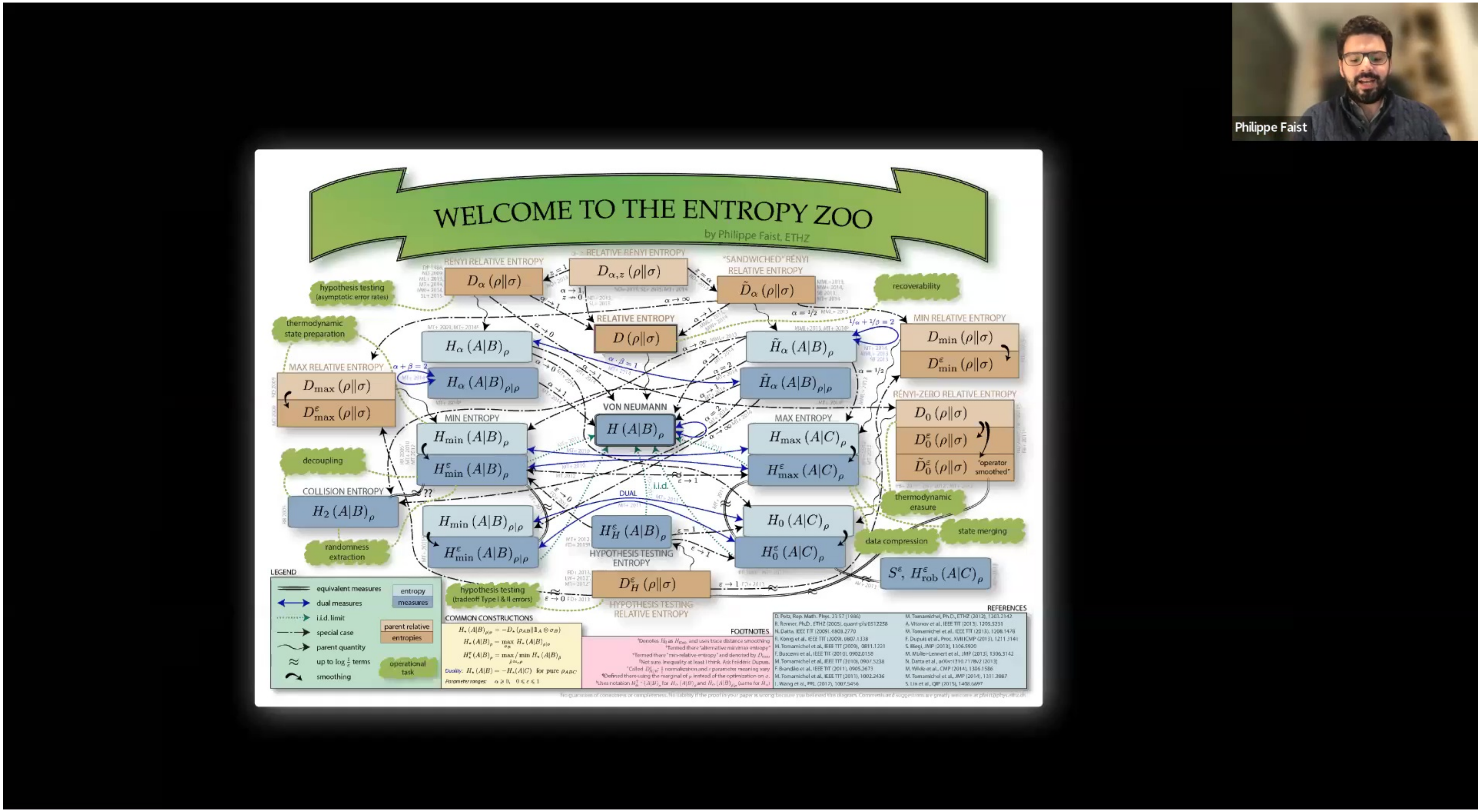
Thermodynamic erasure with a **quantum memory** and with computational limitations?

Work cost of **quantum processes** with computational limitations?

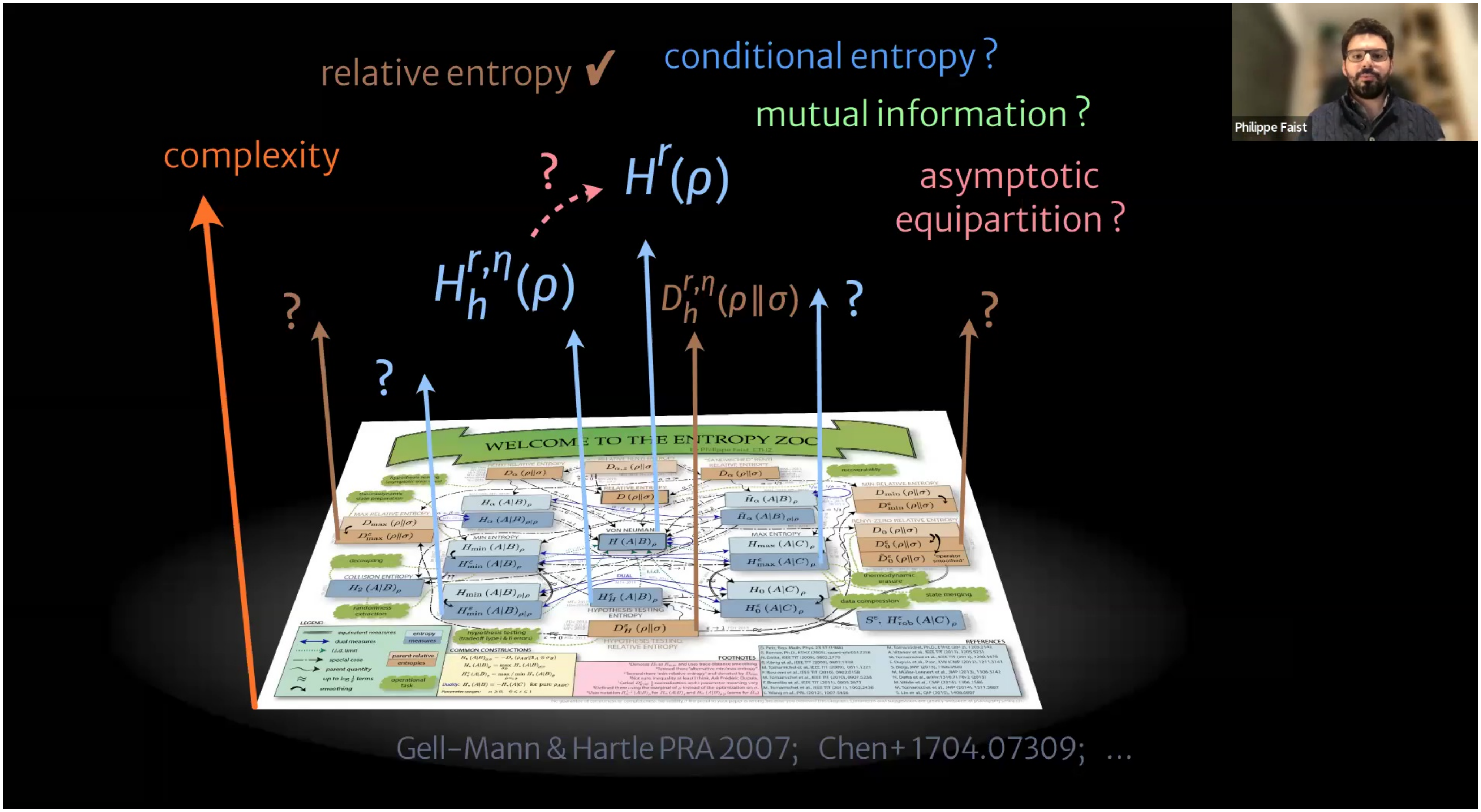
del Rio *et al* Nat 2011
PhF *et al* Nat Comm 2015

Complexity in the resource theory of thermodynamics?

*cf.* Taranto *et al* 2106.05151



Philippe Faist
WELCOME TO THE ENTROPY ZOO
by Philippe Faist, ETHZ
RÉNYI RELATIVE ENTROPY
α-z RELATIVE RÉNYI ENTROPY
"SANDWICHED" RÉNYI RELATIVE ENTROPY
RELATIVE ENTROPY
MIN RELATIVE ENTROPY
MAX RELATIVE ENTROPY
RÉNYI-ZERO RELATIVE ENTROPY
VON NEUMANN
MIN ENTROPY
MAX ENTROPY
COLLISION ENTROPY
HYPOTHESIS TESTING ENTROPY
HYPOTHESIS TESTING RELATIVE ENTROPY
hypothesis testing (asymptotic error rates)
thermodynamic state preparation
recoverability
decoupling
randomness extraction
thermodynamic erasure
state merging
data compression
hypothesis testing (tradeoff Type I & II errors)
"operator smoothed"
DUAL
i.i.d.
LEGEND
equivalent measures
dual measures
i.i.d. limit
special case
parent quantity
smoothing
entropy measures
parent relative entropies
operational task
COMMON CONSTRUCTIONS
FOOTNOTES
REFERENCES


relative entropy ✔
conditional entropy ?
mutual information ?
complexity
? $H^r(\rho)$
asymptotic equipartition ?
$H_h^{r,\eta}(\rho)$
$D_h^{r,\eta}(\rho\|\sigma)$
?
?
?
?
WELCOME TO THE ENTROPY ZOO
Philippe Faist
Gell-Mann & Hartle PRA 2007; Chen+ 1704.07309; ...

Computing / bounding the complexity entropy in certain regimes?

entanglement bounds

$k$-designs

Eisert PRL 2021

relations to magic?

version for continuous variable systems?

classical case?

Properties of the complexity entropy defined with the **Nielsen complexity**?

Complexity entropy in AdS/CFT?

*e.g.* Bernamonti *et al* JHEP 2018; Caceres *et al* JHEP 2020; ...

quantum complexity and quantum error correction?

quantum machine learning?

Thank you for your attention!