Title: Error-corrected quantum metrology

Speakers: Sisi Zhou

Series: Perimeter Institute Quantum Discussions

Date: January 10, 2022 - 3:00 PM

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Abstract: Quantum metrology, which studies parameter estimation in quantum systems, has many important applications in science and technology, ranging from frequency spectroscopy to gravitational wave detection. Quantum mechanics imposes a fundamental limit on the estimation precision, called the Heisenberg limit, which is achievable in noiseless quantum systems, but is in general not achievable in noisy systems. This talk is a summary of some recent works by the speaker and collaborators on error-corrected quantum metrology. Specifically, we present a necessary and sufficient condition for achieving the Heisenberg limit in noisy quantum systems. When the condition is satisfied, the Heisenberg limit is recovered by a quantum error correction protocol which corrects all noises while maintaining the signal; when it is violated, we show the estimation limit still in general has a constant factor improvement over classical strategies, and is achievable using approximate quantum error correction. Both error correction protocols can be optimized using semidefinite programs. Examples in some typical noisy systems will be provided.

Zoom Link: https://pitp.zoom.us/j/95698542740?pwd=OWJtcTViKzZqNDg1bWk4cDFtaTRxZz09

Error-corrected quantum metrology

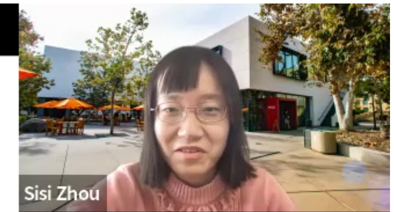
Sisi Zhou

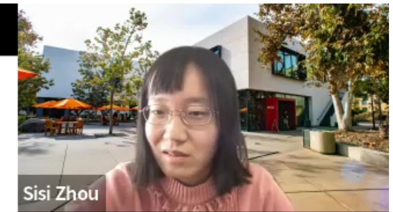
Perimeter Institute Quantum Discussions

2022/01/10

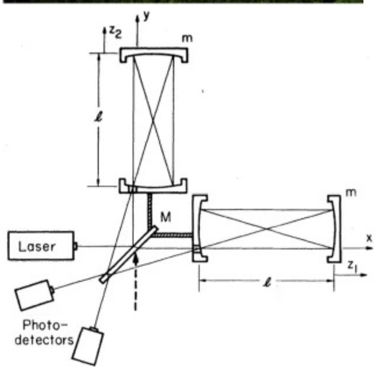
INSTITUTE FOR QUANTUM INFORMATION AND MATTER

Caltech

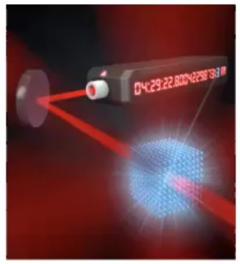


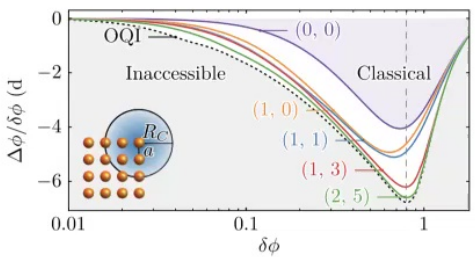


Quantum metrology is the science of estimation in quantum systems.

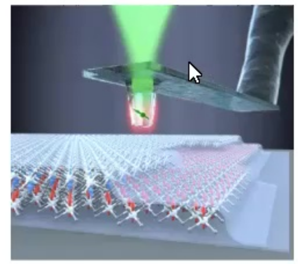


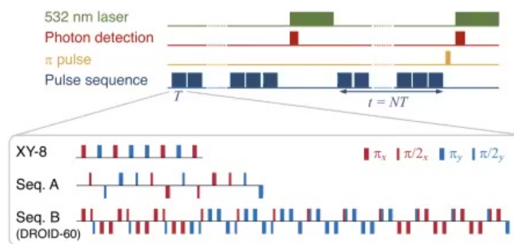
Optical interferometry





Atomic clock





Nitrogen-vacancy centers

Figures from: https://www.ligo.caltech.edu/news/ligo20191015, https://www.photonicsviews.com/atomic-clock-with-a-3d-optical-lattice/
[…]l.com/a/diamond-quantum-microscope-images-nanoscale-features-in-2d-magnets/, PhysRevD.23.1693, PhysRevX.11.041045, PhysRevX.10.031003

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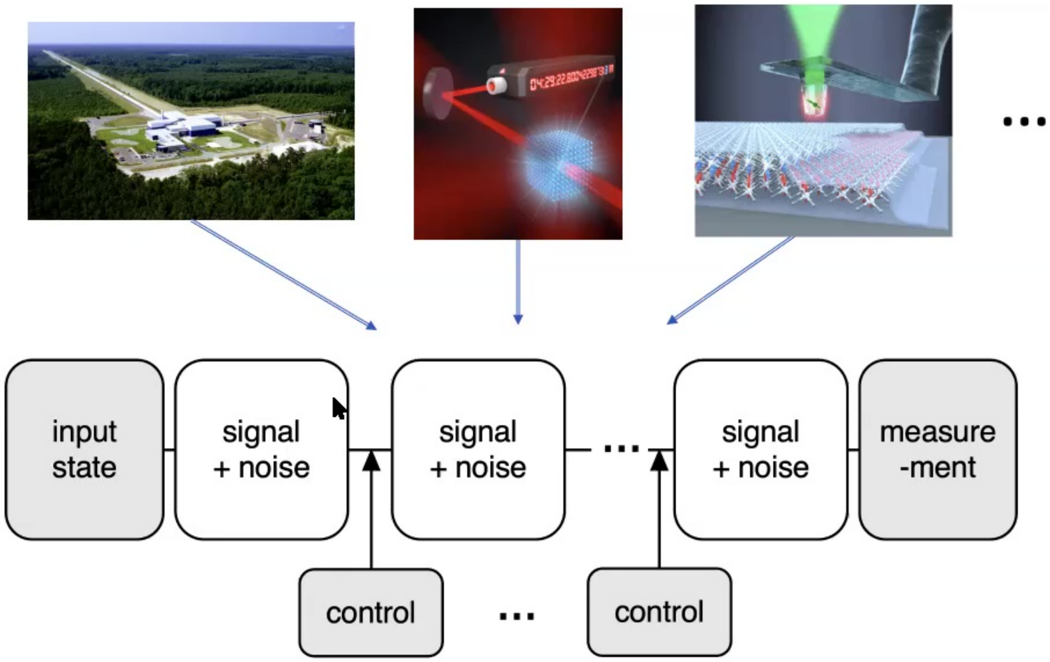
Quantum metrology is the science of estimation in quantum systems.



Figures from: https://www.ligo.caltech.edu/news/ligo20191015, https://www.photonicsviews.com/atomic-clock-with-a-3d-optical-lattice/,
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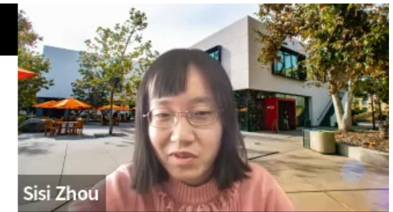

Outline

- Review of quantum metrology
  - Classical and quantum estimation theory
  - Effect of noise

- Quantum channel estimation (with quantum error correction)
  - Heisenberg limit vs. standard quantum limit
  - Optimizing estimation precision

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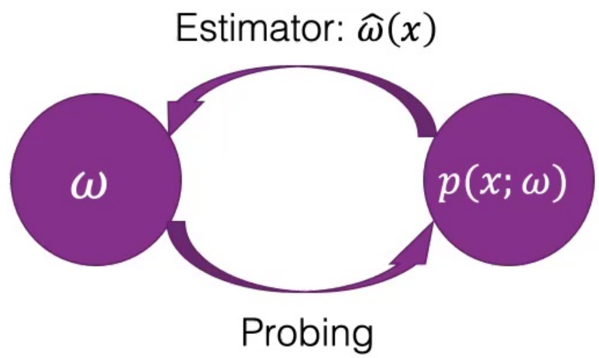
Classical estimation theory



Estimation precision:

$$\Delta\omega = (\mathbb{E}[(\widehat{\omega}(x) - \omega)^2])^{\frac{1}{2}},$$

where $\widehat{\omega}$ is an unbiased estimator: $\mathbb{E}[\widehat{\omega}(x)] = \omega$.

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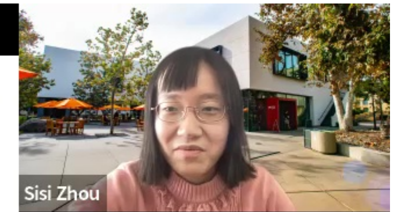
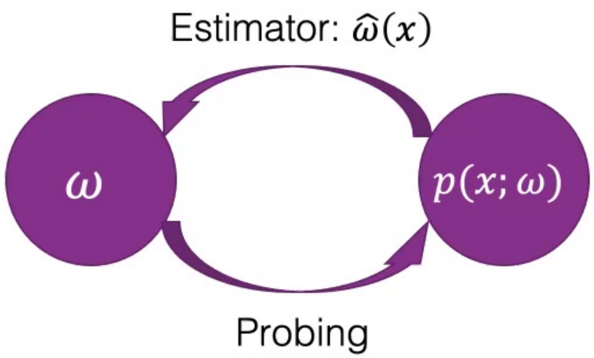
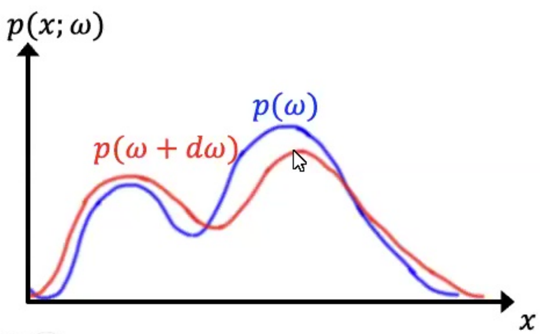


Classical estimation theory





Estimation precision:

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where $\hat{\omega}$ is an unbiased estimator: $\mathbb{E}[\hat{\omega}(x)] = \omega$.

Cramér-Rao bound:

$$\Delta\omega \geq \frac{1}{\sqrt{N_{\mathrm{expr}} \cdot F(p(x;\omega))}}$$

N_{expr}: number of experiments

$F(p(x;\omega))$: Fisher information

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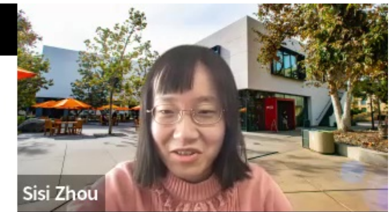
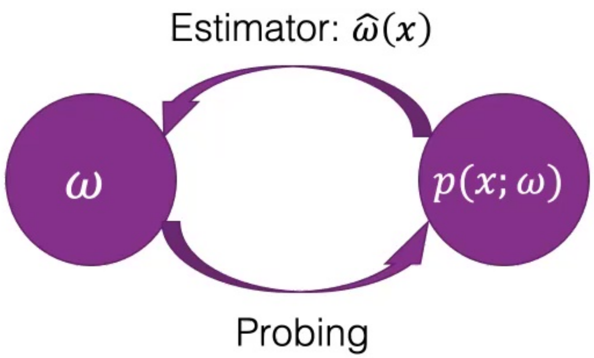
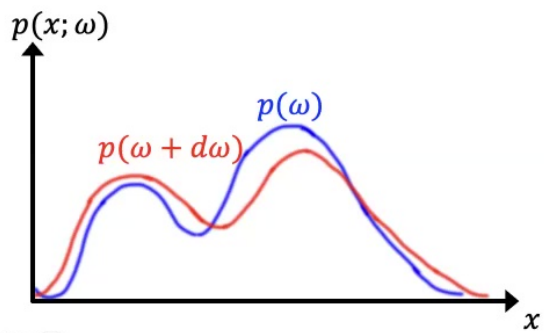


Classical estimation theory





Estimation precision:

$$\Delta\omega = (\mathbb{E}[(\hat{\omega}(x) - \omega)^2])^{\frac{1}{2}},$$

where $\hat{\omega}$ is an unbiased estimator: $\mathbb{E}[\hat{\omega}(x)] = \omega$.

Cramér-Rao bound:

$$\Delta\omega \geq \frac{1}{\sqrt{N_{\text{expr}} \cdot F(p(x;\omega))}}$$

N_{expr}: number of experiments

$F(p(x;\omega))$: Fisher information

The bound is saturable asymptotically using the maximum likelihood estimator.

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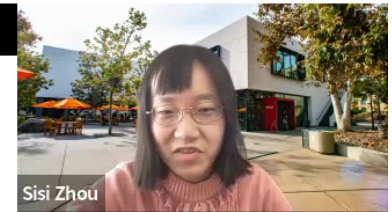


Classical Fisher information

Classical Fisher information:

$$F\big(p(x;\omega)\big) = \sum_x \frac{1}{p(x;\omega)}\left(\frac{\partial p(x;\omega)}{\partial \omega}\right)^2$$

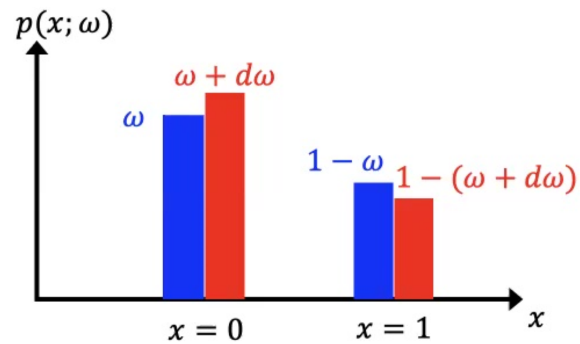
Example (Bernoulli distribution):

$0 < \omega < 1$ and $x = 0$ or 1.

$p(x = 0; \omega) = \omega$

$p(x = 1; \omega) = 1 - \omega$

$$F\big(p(x;\omega)\big) = \frac{1}{\omega} + \frac{1}{1-\omega} = \frac{1}{\omega(1-\omega)}$$



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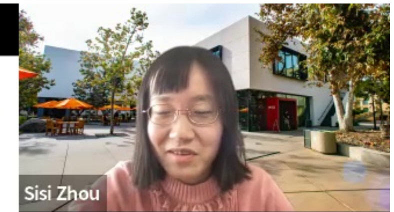
Classical and quantum Fisher information



Classical Fisher information:

$$F\big(p(x;\omega)\big) = \sum_x \frac{1}{p(x;\omega)}\left(\frac{\partial p(x;\omega)}{\partial \omega}\right)^2 = \mathbb{E}[L(x;\omega)^2]$$

$p(x;\omega) \Leftrightarrow \rho_\omega$

$L(x;\omega) \Leftrightarrow L_\omega$

Logarithmic derivative $L(x;\omega) = \partial_\omega \log p(x;\omega)$:

$$p(x;\omega) \cdot L(x;\omega) = \partial_\omega p(x;\omega)$$

$p(x;\omega) \Leftrightarrow \rho_\omega$

$L(x;\omega) \Leftrightarrow L_\omega$

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Classical and quantum Fisher information

Classical Fisher information:

$$F\big(p(x;\omega)\big) = \sum_x \frac{1}{p(x;\omega)} \left(\frac{\partial p(x;\omega)}{\partial \omega}\right)^2 = \mathbb{E}[L(x;\omega)^2]$$

Quantum analogue:

$$F(\rho_\omega) = \mathbb{E}[L_\omega^2] = \mathrm{Tr}(\rho_\omega L_\omega^2)$$

$p(x;\omega) \Leftrightarrow \rho_\omega$

$L(x;\omega) \Leftrightarrow L_\omega$

Logarithmic derivative $L(x;\omega) = \partial_\omega \log p(x;\omega)$:

$$p(x;\omega) \cdot L(x;\omega) = \partial_\omega p(x;\omega)$$

Quantum analogue L_ω :

$$\frac{L_\omega \rho_\omega + \rho_\omega L_\omega}{2} = \partial_\omega \rho_\omega$$

$p(x;\omega) \Leftrightarrow \rho_\omega$

$L(x;\omega) \Leftrightarrow L_\omega$

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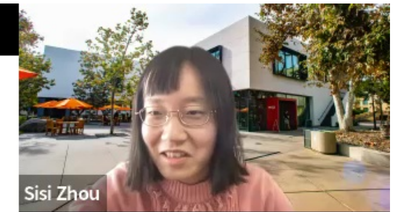
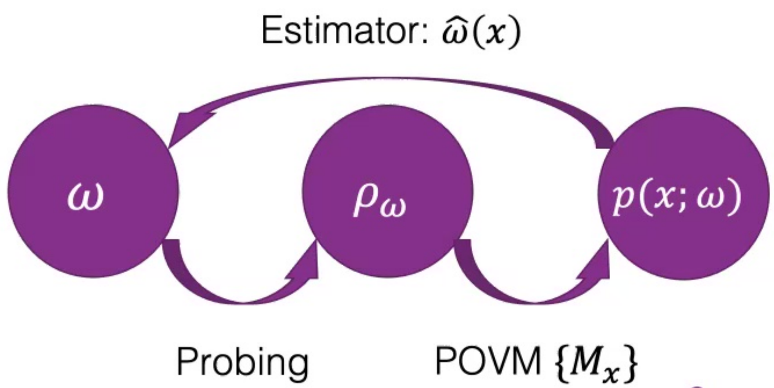
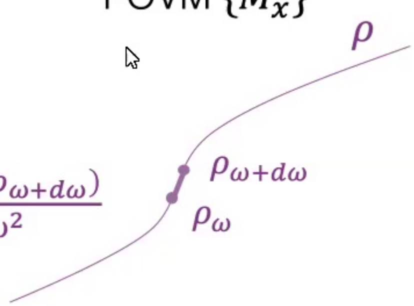


Quantum estimation theory



$$F(\rho_\omega) \propto \frac{d_B^2(\rho_\omega, \rho_{\omega+d\omega})}{d\omega^2}$$



Helstrom 1976, Holevo 1982
Braunstein & Caves 1994

$$F(\rho_\omega) = \max_{\{M_x\}} F(p(x;\omega))$$

Quantum Cramér-Rao bound:

$$\Delta\omega \geq \frac{1}{\sqrt{N_{\text{expr}} \cdot F(\rho_\omega)}}$$

N_{expr}: number of experiments

$F(\rho_\omega)$: quantum Fisher information

The quantum Cramér-Rao bound is also saturable asymptotically.

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Quantum Fisher information

Example 1 (pure state):

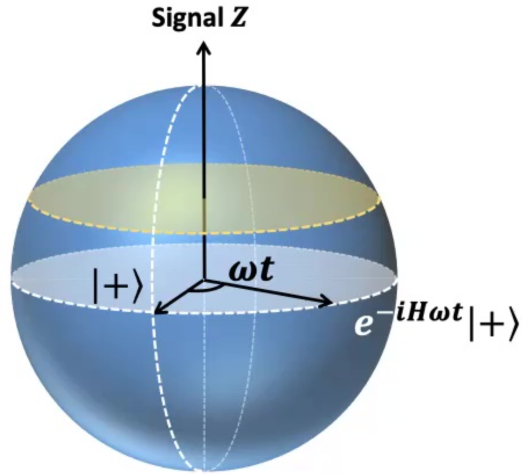
$\rho_\omega = |\psi_\omega\rangle\langle\psi_\omega|$, $|\psi_\omega\rangle = e^{-i\omega H t}|\psi_0\rangle$,

$$F(\rho_\omega) = 4t^2\langle\Delta^2 H\rangle = \Theta(t^2),$$

where $\langle\Delta^2 H\rangle = (\langle\psi_0|H^2|\psi_0\rangle - \langle\psi_0|H|\psi_0\rangle^2)$.

- For a single qubit state, when $H = Z/2$, an optimal initial state is

$$|\psi_0\rangle = |+\rangle = \frac{(|0\rangle + |1\rangle)}{\sqrt{2}}.$$



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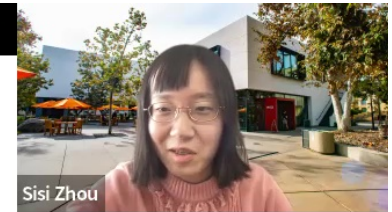


Quantum Fisher information

Example 2 (single-qubit dephasing):

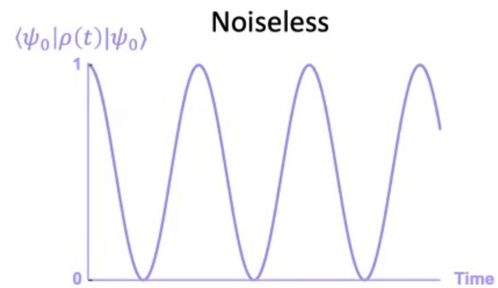
$$\frac{d\rho}{dt} = -i\left[\frac{\omega Z}{2}, \rho\right] + \frac{\gamma}{2}(Z\rho Z - \rho)$$

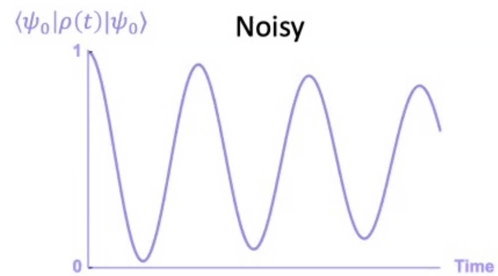
Input state: $|\psi_0\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$.

Noiseless case ($\gamma = 0$): $F(\rho_\omega(t)) = t^2 = \Theta(t^2)$.

Noisy case ($\gamma > 0$): $F(\rho_\omega(t)) = t^2 e^{-2\gamma t}$.

If we measure and renew the qubit every constant time, we get only QFI $= \Theta(t)$.





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Quantum Fisher information

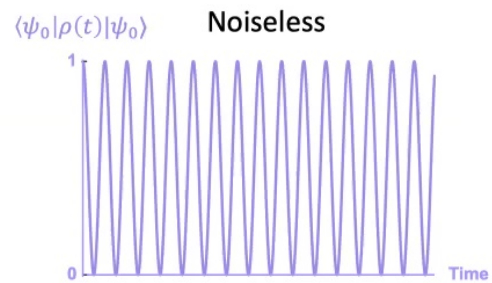
Example 3 (N-qubit dephasing):

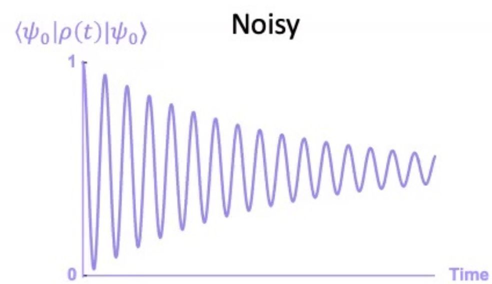
Input state (GHZ state):

$$|\psi_0\rangle = \frac{|0\rangle^{\otimes N} + |1\rangle^{\otimes N}}{\sqrt{2}}$$

Noiseless case ($\gamma = 0$): $F(\rho_\omega(t)) = N^2 t^2 = \Theta(N^2)$.

Noisy case ($\gamma > 0$): $F(\rho_\omega(t)) = N^2 t^2 e^{-2N\gamma t}$.





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Quantum Fisher information

Example 3 (N-qubit dephasing):

Input state (GHZ state):

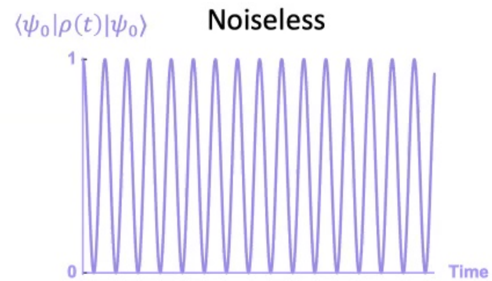
$$|\psi_0\rangle = \frac{|0\rangle^{\otimes N} + |1\rangle^{\otimes N}}{\sqrt{2}}$$

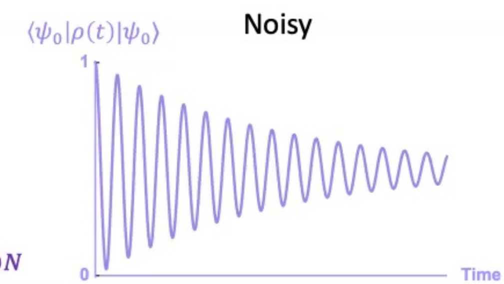
Noiseless case ($\gamma = 0$): $F(\rho_\omega(t)) = N^2 t^2 = \Theta(N^2)$.

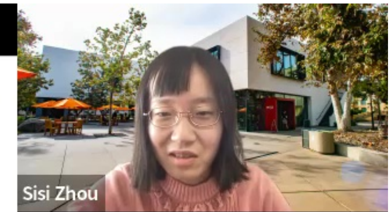
Noisy case ($\gamma > 0$): $F(\rho_\omega(t)) = N^2 t^2 e^{-2N\gamma t}$.

Average QFI over time: $\max_{t>0} F(\rho_\omega(t))/t = N/2e\gamma = \Theta(N)$.

same as $|\psi_0\rangle = \left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)^{\otimes N}$







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Quantum Fisher information

In quantum metrology, the resource we care about is the number of channels used N or the probing time t. There are two types of estimation precision limits:

- The Heisenberg limit (HL): QFI $= \Theta(N^2)$ or $\Theta(t^2)$

— the ultimate estimation precision limit allowed by quantum mechanics.

- The standard quantum limit (SQL): QFI $= \Theta(N)$ or $\Theta(t)$

— achievable using “classical” strategies (no need to maintain the coherence in & between probes for a long time).

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Outline

- Review of quantum metrology
  - Classical and quantum estimation theory
  - Effect of noise

- Quantum channel estimation (with quantum error correction)
  - Heisenberg limit vs. standard quantum limit
  - Optimizing estimation precision

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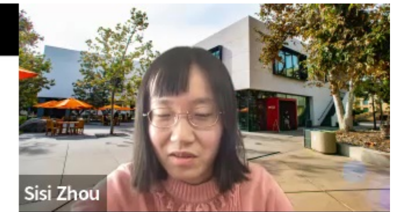
Quantum channel estimation



$$\rho_{\text{in}} \;—\; \boxed{\mathcal{E}_\omega} \;—\; \rho_{\text{out}} = \mathcal{E}_\omega(\rho_{\text{in}})$$

- Unitary channel: $\mathcal{E}_\omega(\rho) = \mathcal{U}_\omega(\rho) = e^{-i\omega H}\rho e^{i\omega H}$.
- Hamiltonian estimation under noise: $\mathcal{E}_\omega(\rho) = \mathcal{N} \circ \mathcal{U}_\omega(\rho)$, $\mathcal{N}(\rho) = \sum_{i=1}^{r} K_i \rho K_i^\dagger$.
- General quantum channel: $\mathcal{E}_\omega(\rho) = \sum_{i=1}^{r} K_{\omega,i} \rho K_{\omega,i}^\dagger$.

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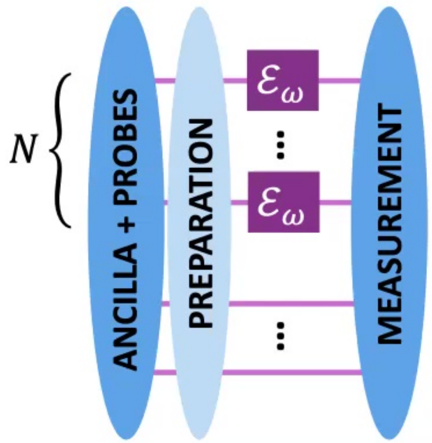
Quantum channel estimation

$$\mathcal{E}_\omega(\rho) = \sum_{i=1}^{r} K_{\omega,i}\rho K_{\omega,i}^\dagger$$



Parallel strategy

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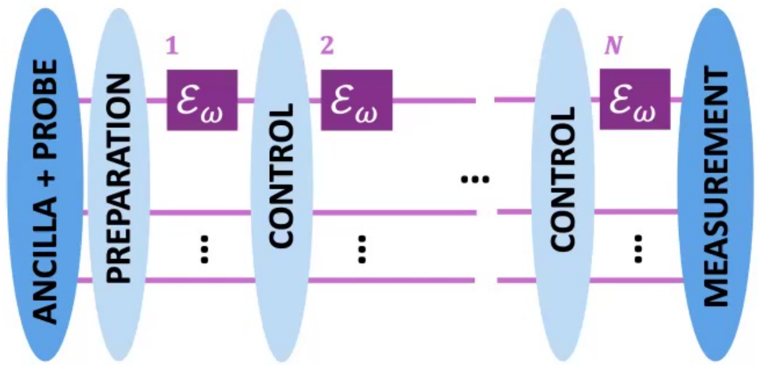

Sequential strategy

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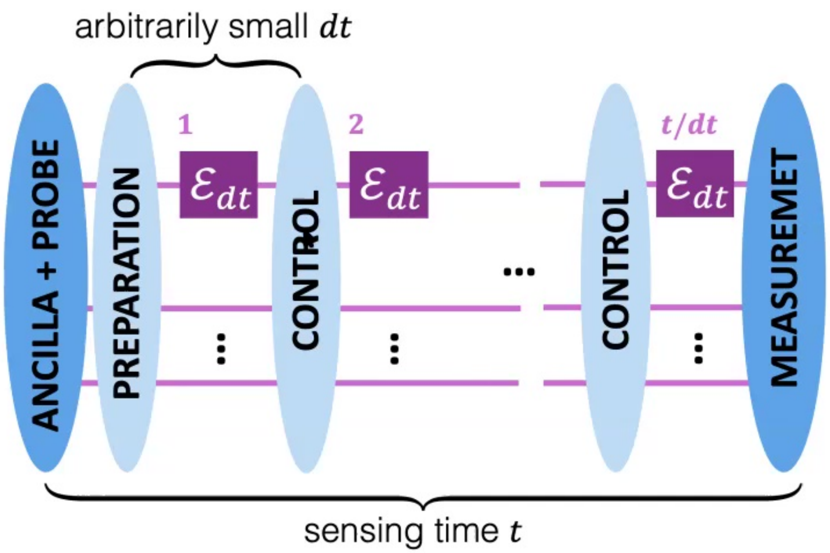
Sisi Zhou

Hamiltonian estimation under Markovian noise

$$\mathcal{E}_{\omega,dt}(\rho) = \rho + \left(-i[\omega H, \rho] + \sum_{i=1}^{r} L_i \rho L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \rho\}\right) dt + O(dt^2)$$



Sekatski et al. 2017

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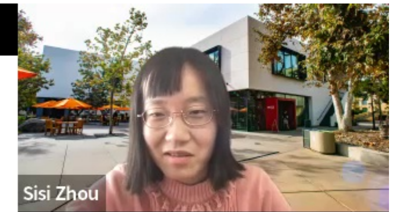

Asymptotic quantum channel estimation

- HL vs. SQL

— Given an arbitrary quantum channel $\mathcal{E}_\omega$, is it possible to achieve the HL using parallel or sequential strategies?

$$\mathrm{QFI} \propto N^2 \quad / \quad \mathrm{QFI} \propto N$$

- Optimal QFI

— What is the optimal QFI coefficient, both when the HL is achievable and when it is not?

$$\lim_{N\to\infty} \frac{\mathrm{QFI}}{N^2} \quad / \quad \lim_{N\to\infty} \frac{\mathrm{QFI}}{N}$$

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<table>
<tr><th>QFI</th><th>Upper bound</th><th>Lower bound</th></tr>
<tr><td>HL vs. SQL</td><td rowspan="3">Channel extension method</td><td rowspan="3">Quantum error correction</td></tr>
<tr><td>HL coefficient</td></tr>
<tr><td>SQL coefficient</td></tr>
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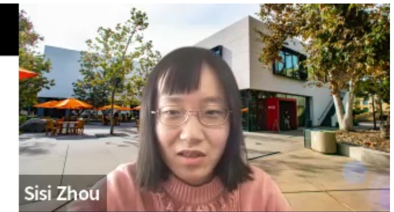

QFI	Upper bound	Lower bound
HL vs. SQL	Channel extension method	Quantum error correction
HL coefficient		
SQL coefficient		

Fujiwara & Imai 2008
Escher et al. 2011, Demkowicz-Dobrzanski et al. 2012
Kołodyński & Demkowicz-Dobrzanski 2013
Demkowicz-Dobrzanski & Maccone 2014

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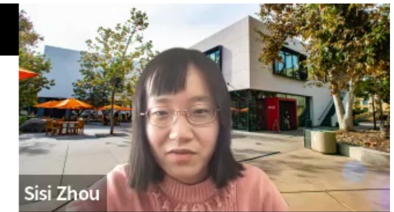


Channel extension method

(Ancilla-assisted) Channel QFI:

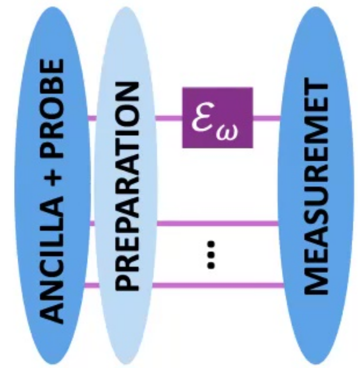
$$F(\mathcal{E}_\omega) = \max_\rho F((\mathcal{E}_\omega \otimes \mathbb{I})(\rho))$$

Let $\mathcal{E}_\omega(\rho) = \sum_{i=1}^{r} K_{\omega,i}\rho K_{\omega,i}^\dagger$,

$\|\cdot\|$ is the operator norm

$$F(\mathcal{E}_\omega) = 4\min_h \|\alpha\|, \quad \alpha = (\dot{\mathbf{K}} - ih\mathbf{K})^\dagger(\dot{\mathbf{K}} - ih\mathbf{K}).$$

Efficiently computable using semidefinite programming



$$\mathbf{K} = \begin{pmatrix} K_{\omega,1} \\ \vdots \\ K_{\omega,r} \end{pmatrix}, \quad \dot{\mathbf{K}} = \begin{pmatrix} \partial_\omega K_{\omega,1} \\ \vdots \\ \partial_\omega K_{\omega,r} \end{pmatrix}$$

Hermitian $h \in \mathbb{C}^{r\times r}$

Fujiwara & Imai 2008

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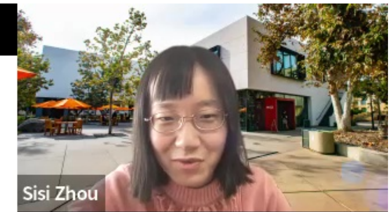
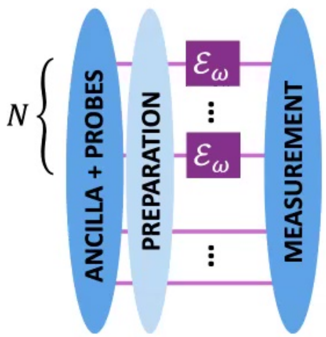
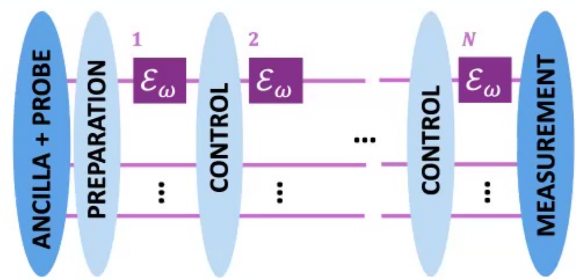


Channel extension method





$$F^{(\mathrm{par})} \leq 4N\|\alpha\| + 4N(N-1)\|\beta\|^2$$

$$F^{(\mathrm{seq})} \leq 4N\|\alpha\| + 4N(N-1)\|\beta\|\left(\|\beta\| + 2\sqrt{\|\alpha\|}\right)$$

where $\alpha = \left(\dot{\mathbf{K}} - ih\mathbf{K}\right)^\dagger\left(\dot{\mathbf{K}} - ih\mathbf{K}\right)$ and $\beta = i\mathbf{K}^\dagger(\dot{\mathbf{K}} - ih\mathbf{K})$.

The HL is not achievable if there exists an h such that $\beta = 0$.

Example: depolarizing noise, Pauli-Z signal with dephasing noise.

Demkowicz-Dobrzanski et al. 2012, Demkowicz-Dobrzanski & Maccone 2014

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Channel extension method

(Ancilla-assisted) Channel QFI:

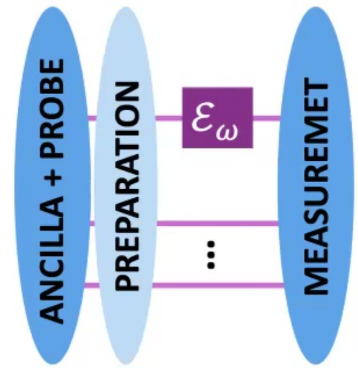
$$F(\mathcal{E}_\omega) = \max_\rho F((\mathcal{E}_\omega \otimes \mathbb{I})(\rho))$$

Let $\mathcal{E}_\omega(\rho) = \sum_{i=1}^{r} K_{\omega,i}\rho K_{\omega,i}^\dagger$,

$\|\cdot\|$ is the operator norm

$$F(\mathcal{E}_\omega) = 4\min_h \|\alpha\|, \quad \alpha = (\dot{\mathbf{K}} - ih\mathbf{K})^\dagger(\dot{\mathbf{K}} - ih\mathbf{K}).$$

Efficiently computable using semidefinite programming



$$\mathbf{K} = \begin{pmatrix} K_{\omega,1} \\ \vdots \\ K_{\omega,r} \end{pmatrix}, \quad \dot{\mathbf{K}} = \begin{pmatrix} \partial_\omega K_{\omega,1} \\ \vdots \\ \partial_\omega K_{\omega,r} \end{pmatrix}$$

Hermitian $h \in \mathbb{C}^{r\times r}$

Fujiwara & Imai 2008

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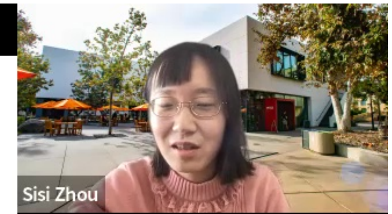
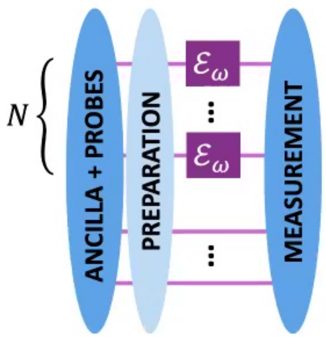
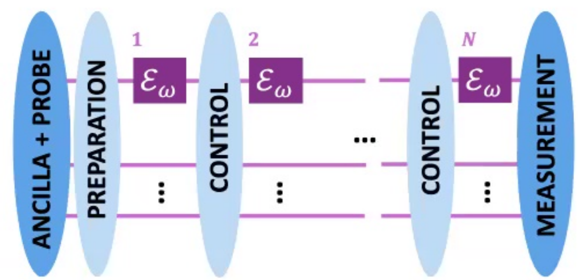


Channel extension method





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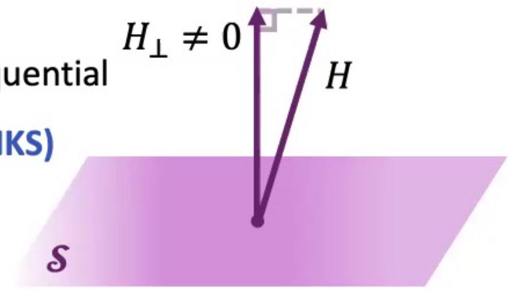
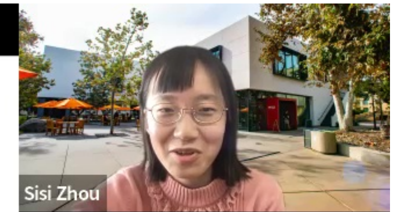
Necessary and sufficient condition





Theorem 1: The HL is achievable using parallel strategies (or sequential strategies) if and only if **the Hamiltonian-not-in-Kraus-Span (HNKS) condition** is satisfied:

$$H(\mathcal{E}_\omega) \notin \mathcal{S}(\mathcal{E}_\omega),$$

where

Hamiltonian: $H(\mathcal{E}_\omega) = i\mathbf{K}^\dagger\dot{\mathbf{K}}$,

Kraus span: $\mathcal{S}(\mathcal{E}_\omega) = \mathrm{span}_{\mathbb{H}}\{K_{\omega,i}^\dagger K_{\omega,j}, \forall i,j\}$.

Fujiwara 2008, Demkowicz-Dobrzanski et al. 2012,
Demkowicz-Dobrzanski & Maccone 2014, SZ & Jiang 2021

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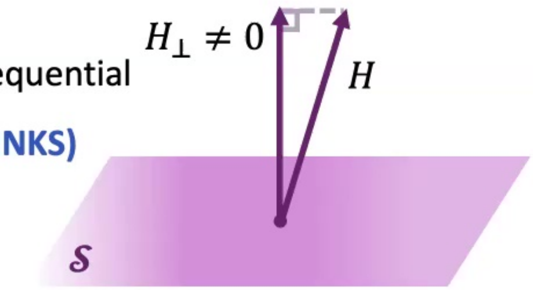
Necessary and sufficient condition



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$\mathbb{H}$ means the set of Hermitian matrices

Remark: For unitary channel $U_\omega = e^{-iH\omega}$, or $\mathcal{E}_\omega = \mathcal{N} \circ \mathcal{U}_\omega$, $H(\mathcal{E}_\omega) = H$.

Fujiwara 2008, Demkowicz-Dobrzanski et al. 2012,
Demkowicz-Dobrzanski & Maccone 2014, SZ & Jiang 2021

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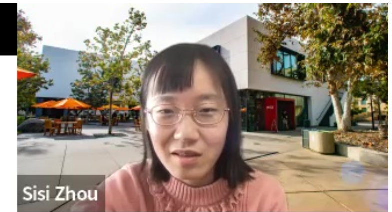
Example: Pauli Z signal with bit-flip noise



$$\mathcal{E}_\omega(\rho) = \mathcal{N} \circ \mathcal{U}_\omega(\rho),$$

$$\mathcal{N}(\rho) = (1-p)\rho + pX\rho X, \quad U_\omega = e^{-\frac{i\omega Z}{2}},$$

$$H = Z/2 \notin \mathcal{S} = \mathrm{span}\{I, X\}$$

Optimal code: $|0_L\rangle = |0\rangle_S \otimes |0\rangle_A$, $|1_L\rangle = |1\rangle_S \otimes |1\rangle_A$.

After error:

$(X \otimes I)|0_L\rangle = |1\rangle_S \otimes |0\rangle_A$, $(X \otimes I)|1_L\rangle = |0\rangle_S \otimes |1\rangle_A$.

Recovery:

Measure $S = Z \otimes Z$. If $S = -1$, flip the probe with X; otherwise do nothing.

Kessler et al. 2014, Arrad et al. 2014, Dur et al. 2014, Unden et al. 2016

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Quantum error correction

Given quantum channel $\mathcal{N}(\rho) = \sum_k K_k \rho K_k^\dagger$ and $\rho = P\,\rho\,P$,

$\exists$ A recovery channel $\mathcal{R}$, s.t.

$$\rho = \mathcal{R} \circ \mathcal{N}(\rho)$$

$$\Leftrightarrow P\,K_j^\dagger K_k\,P \propto P, \forall j, k$$

(the Knill—Laflamme condition)

where P is the projection onto the code space.

Bennett et al. 1996
Knill & Laflamme 1997

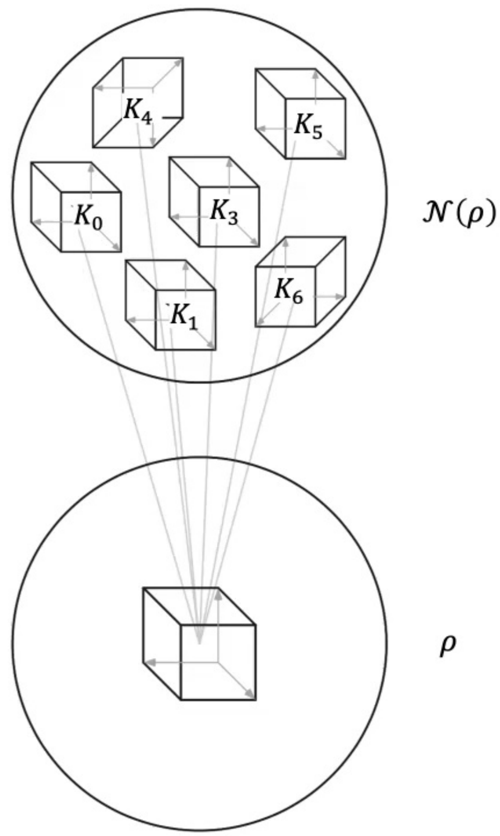
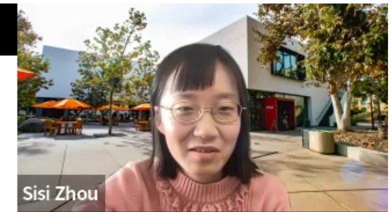


Figure from: Nielsen & Chuang



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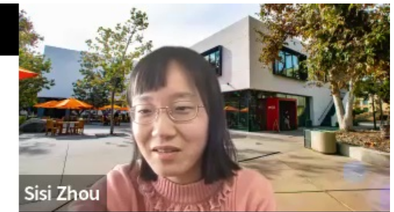
Reduction to unitary channel



To correct noise:

$$P\left(K_j^\dagger K_k \otimes I_A\right)P \propto P, \forall j, k$$

To detect signal:

$$P(H \otimes I_A)P \neq c \cdot P$$

SZ et al. 2018, SZ & Jiang 2021

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Reduction to unitary channel

Assumptions:

$$\mathcal{E}_\omega(\rho) = \mathcal{N} \circ \mathcal{U}_\omega(\rho) = \mathcal{N}\left(e^{-i\omega H} \rho e^{i\omega H}\right)$$

$$\omega \approx 0, \qquad \rho = P\rho P$$

$$\forall \rho = P\rho P,\ \mathcal{R} \circ \mathcal{N}(\rho) = \rho$$

To correct noise:

$$P\left(K_j^\dagger K_k \otimes I_A\right)P \propto P, \forall j, k$$

Effective unitary channel after QEC:

$$\mathcal{E}_\omega(\rho) \approx \mathcal{N}(\rho - i\omega[H, \rho])$$

$$\mathcal{R} \circ \mathcal{E}_\omega(\rho) \approx \mathcal{R} \circ \mathcal{N}(\rho) - \mathcal{R} \circ \mathcal{N}(-i\omega[H, \rho])$$

$$\mathcal{R} \circ \mathcal{E}_\omega(\rho) \approx e^{-i\omega H_{\text{eff}}} \rho e^{i\omega H_{\text{eff}}}, \quad H_{\text{eff}} := PHP$$

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SZ et al. 2018, SZ & Jiang 2021

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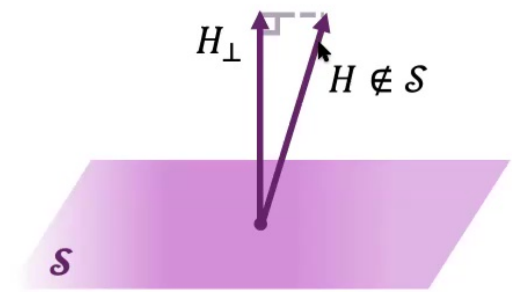

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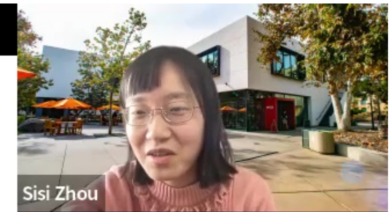
SZ et al. 2018, SZ & Jiang 2021



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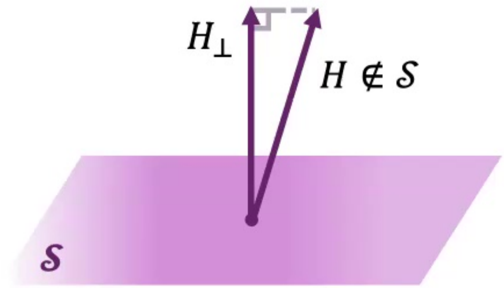
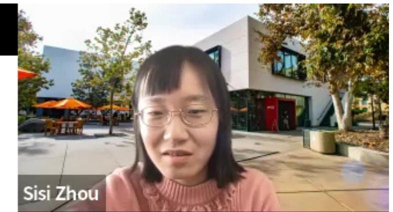
Attaining the HL





$$H = H_\parallel + H_\perp,$$

where $H_\parallel \in \mathcal{S}$ and $H_\perp \perp \mathcal{S}$ is a non-zero traceless Hermitian matrix.

$$H_\perp \propto \sigma_0 - \sigma_1,$$

where $\sigma_{0,1}$ are density matrices and $\sigma_0 \perp \sigma_1$.

The code space is

$$|c_0\rangle = \sum_{ij}\left(\sqrt{\sigma_0}\right)_{ij}|i\rangle_S|j\rangle_A, \quad |c_1\rangle = \sum_{ij}\left(\sqrt{\sigma_1}\right)_{ij}|i\rangle_S|j\rangle_A.$$

$$P\left(K_j^\dagger K_k \otimes I_A\right)P \propto P, \forall j,k$$
$$P(H \otimes I_A)P \neq c \cdot P$$

Then the logical GHZ state recovers the HL.

SZ et al. 2018, SZ & Jiang 2021

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<table>
<tr><th>QFI</th><th>Upper bound</th><th>Lower bound</th></tr>
<tr><td>HL vs. SQL</td><td rowspan="3">Channel extension method</td><td rowspan="3">Quantum error correction</td></tr>
<tr><td>HL coefficient</td></tr>
<tr><td>SQL coefficient</td></tr>
</table>

SZ et al. 2018
SZ & Jiang 2021

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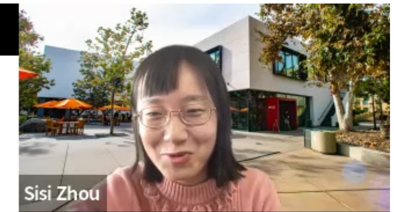

Asymptotic HL coefficient

$$F_{\mathrm{HL}}^{(\mathrm{par})}(\mathcal{E}_\omega) = \lim_{N\to\infty} \frac{F\left(\mathcal{E}_\omega^{\otimes N}\right)}{N^2}$$

Upper bound:

$$F^{(\mathrm{par})} \leq 4(N\|\alpha\| + N(N-1)\|\beta\|^2)$$

Theorem 2: When $H \notin \mathcal{S}$,

$$F_{\mathrm{HL}}^{(\mathrm{par})}(\mathcal{E}_\omega) = 4\min_h \|\beta\|^2, \quad \beta = H - \mathbf{K}^\dagger h \mathbf{K}.$$

$\min_h \|\beta\|$ is a distance between H and $\mathcal{S}$

SZ & Jiang 2021

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Attaining the optimal HL coefficient

$|\mathfrak{c}_0\rangle = \sum_{ij} (\sqrt{\sigma_0})_{ij} |i\rangle_S |j\rangle_A, \quad |\mathfrak{c}_1\rangle = \sum_{ij} (\sqrt{\sigma_1})_{ij} |i\rangle_S |j\rangle_A.$

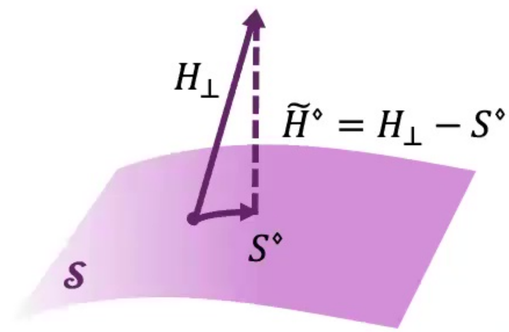
Semidefinite programming: $\tilde{C} = \sigma_0 - \sigma_1$

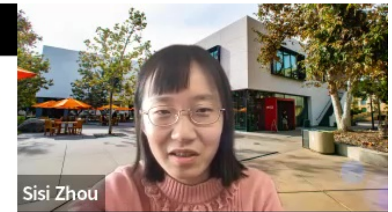
- maximize $F^{(\mathrm{par})} \propto \mathrm{Tr}(\tilde{C}H)^2$
- subject to $\mathrm{Tr}(|\tilde{C}|) \le 2$ and $\mathrm{Tr}(\tilde{C}S) = 0, \forall S \in \mathcal{S}$

The optimal QFI:

$$F^{(\mathrm{par})} = 4N^2 \|H - \mathcal{S}\|^2 = 4N^2 \|\widetilde{H}^\diamond\|^2.$$

The solution $\tilde{C}^\diamond$ is directly related to $\widetilde{H}^\diamond$.





SZ et al. 2018, SZ & Jiang 2021

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QFI	Upper bound	Lower bound
HL vs. SQL		
HL coefficient	Channel extension method	Quantum error correction
SQL coefficient		

SZ & Jiang 2020
SZ & Jiang 2021

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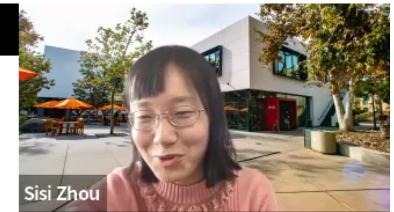

Asymptotic SQL coefficient

$$F_{\mathrm{SQL}}^{(\mathrm{par})}(\mathcal{E}_\omega) = \lim_{N\to\infty} \frac{F\left(\mathcal{E}_\omega^{\otimes N}\right)}{N}$$

Upper bounds:

$$F^{(\mathrm{par})} \leq 4(N\|\alpha\| + N(N-1)\|\beta\|^2)$$
$$F^{(\mathrm{seq})} \leq 4N\|\alpha\| + 4N(N-1)\|\beta\|(\|\beta\| + 2\sqrt{\|\alpha\|})$$

Theorem 3: When $H \in \mathcal{S}$, $F_{\mathrm{SQL}}^{(\mathrm{par})}(\mathcal{E}_\omega) = F_{\mathrm{SQL}}^{(\mathrm{seq})}(\mathcal{E}_\omega)$ and

$$F_{\mathrm{SQL}}^{(\mathrm{par})}(\mathcal{E}_\omega) = 4 \min_{h:\beta=0} \|\alpha\|, \quad \alpha = \left(\dot{\mathbf{K}} - ih\mathbf{K}\right)^\dagger (\dot{\mathbf{K}} - ih\mathbf{K}).$$

Attainability:

approximate QEC protocol

Fujiwara 2008, Demkowicz-Dobrzanski et al. 2012,
Demkowicz-Dobrzanski & Maccone 2014, SZ & Jiang 2021

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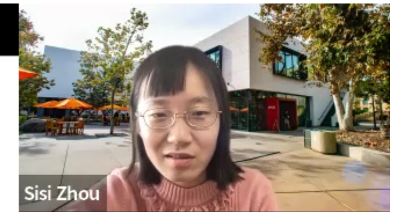
Asymptotic SQL coefficient



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Attaining the optimal SQL coefficient

- **Step 1: QFI coefficients for dephasing channels**
- Step 2: Reduction to dephasing channels
- Step 3: Code optimization

Dephasing channels $\mathcal{D}_\omega$:

$$\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \xrightarrow{\mathcal{D}_\omega} \begin{pmatrix} \rho_{00} & \xi\rho_{01} \\ \xi^*\rho_{10} & \rho_{11} \end{pmatrix}$$

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Attaining the optimal SQL coefficient

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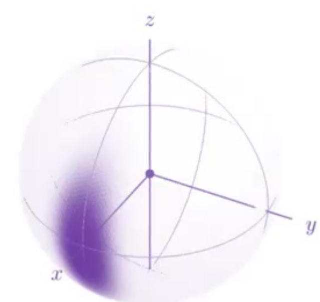
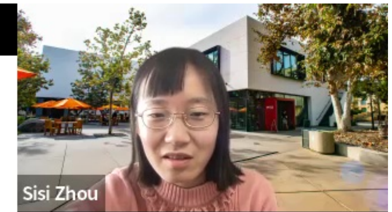
When $|\xi| = 1$, $F_{\mathrm{HL}}(\mathcal{D}_\omega) = |\dot{\xi}|^2$,

When $|\xi| < 1$, $F_{\mathrm{SQL}}(\mathcal{D}_\omega) = \frac{|\dot{\xi}|^2}{1-|\xi|^2}$,



logical spin-squeezed state



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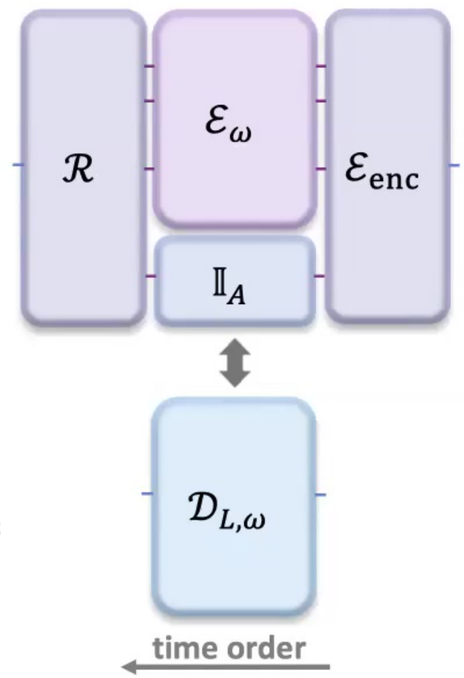
Attaining the optimal SQL coefficient

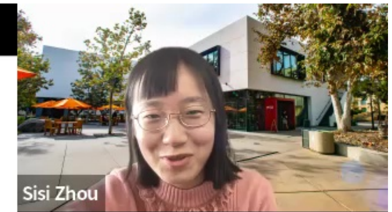
- Step 1: QFI coefficients for dephasing channels
- **Step 2: Reduction to dephasing channels**
- Step 3: Code optimization

Two-dimensional code:

Encoding:

$|0_L\rangle = \sum_{ij}(C_0)_{ij}\ |i\rangle_S|j\rangle_{A_1}|0\rangle_{A_2},\quad |1_L\rangle = \sum_{ij}(C_1)_{ij}\ |i\rangle_S|j\rangle_{A_1}|1\rangle_{A_2}$





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Attaining the optimal SQL coefficient

- Step 1: QFI coefficients for dephasing channels
- Step 2: Reduction to dephasing channels
- **Step 3: Code optimization**

Two-dimensional code:

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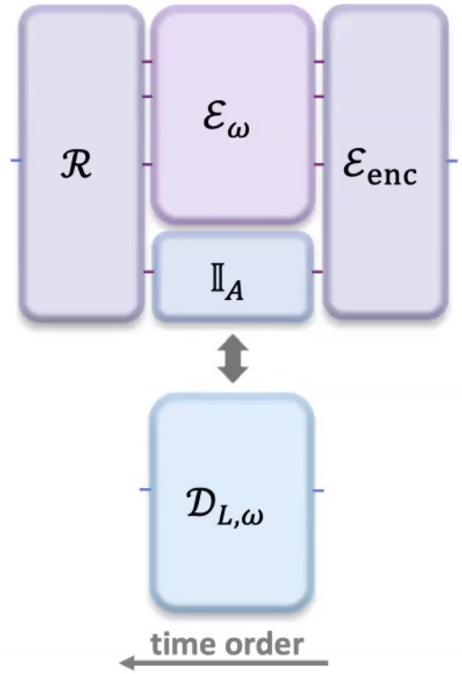
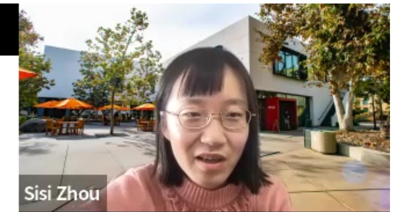
$$|0_L\rangle = \sum_{ij} (C_0)_{ij} \, |i\rangle_S |j\rangle_{A_1} |0\rangle_{A_2}, \quad |1_L\rangle = \sum_{ij} (C_1)_{ij} \, |i\rangle_S |j\rangle_{A_1} |1\rangle_{A_2}$$

Decoding:

$$\mathcal{R}(\cdot) = \sum_m R_m (\cdot) R_m^\dagger, \quad R_m = P_{|0\rangle_L} R_m P_{|0\rangle_{A_2}} + P_{|1\rangle_L} R_m P_{|1\rangle_{A_2}}$$

Optimization of $F_{\mathrm{SQL}}(\mathcal{D}_{L,\omega})$ over C_0, C_1 and $\mathcal{R}$ gives $F_{\mathrm{SQL}}(\mathcal{E}_\omega)$.





SZ & Jiang 2021

Example: Pauli Z signal with amplitude damping

$$\mathcal{E}_\omega(\rho) = \mathcal{N} \circ \mathcal{U}_\omega(\rho),$$

$$\mathcal{N}(\rho) = K_1 \rho K_1^\dagger + K_2 \rho K_2^\dagger, \quad U_\omega = e^{-\frac{i\omega Z}{2}},$$

$$K_1 = |0\rangle\langle 0| + \sqrt{1-p}|1\rangle\langle 1|, \quad K_2 = \sqrt{p}|0\rangle\langle 1|.$$

$$H = Z \in \mathcal{S} = \mathbb{H}_2$$

The QFI coefficient:

$$F_{\mathrm{SQL}}(\mathcal{E}_\omega) = \frac{4(1-p)}{p} = O(1/p).$$

Demkowicz-Dobrzanski et al. 2012, SZ & Jiang 2021

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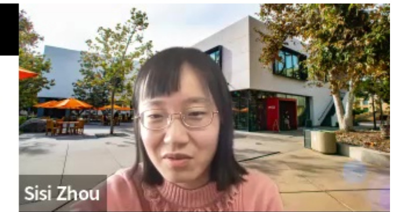
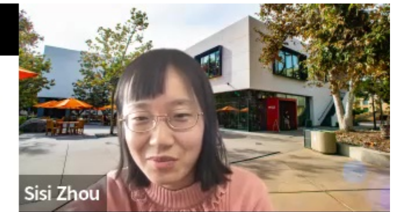


Example: Pauli Z signal with amplitude damping



$$\mathcal{E}_\omega(\rho) = \mathcal{N} \circ \mathcal{U}_\omega(\rho),$$

$$\mathcal{N}(\rho) = K_1 \rho K_1^\dagger + K_2 \rho K_2^\dagger, \quad U_\omega = e^{-\frac{i\omega Z}{2}},$$

$$K_1 = |0\rangle\langle 0| + \sqrt{1-p}|1\rangle\langle 1|, \quad K_2 = \sqrt{p}|0\rangle\langle 1|.$$

$$H = Z \in \mathcal{S} = \mathbb{H}_2$$

Near-optimal code ($\delta > \epsilon > 0$):

$$|0_L\rangle = \sin(\delta+\epsilon)\,|0\rangle_S|00\rangle_A + \cos(\delta+\epsilon)\,|1\rangle_S|10\rangle_A,$$

$$|1_L\rangle = \sin(\delta-\epsilon)\,|0\rangle_S|01\rangle_A + \cos(\delta-\epsilon)\,|1\rangle_S|11\rangle_A.$$

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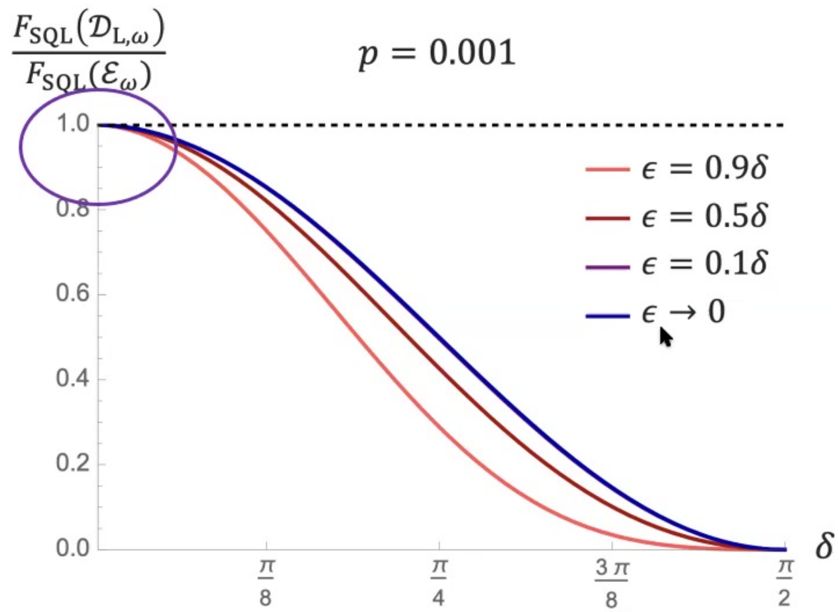
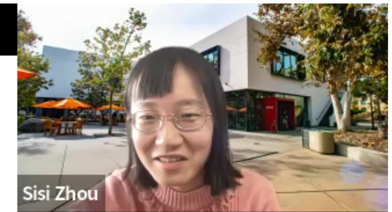
Example: Pauli Z signal with amplitude damping



Near-optimal coc



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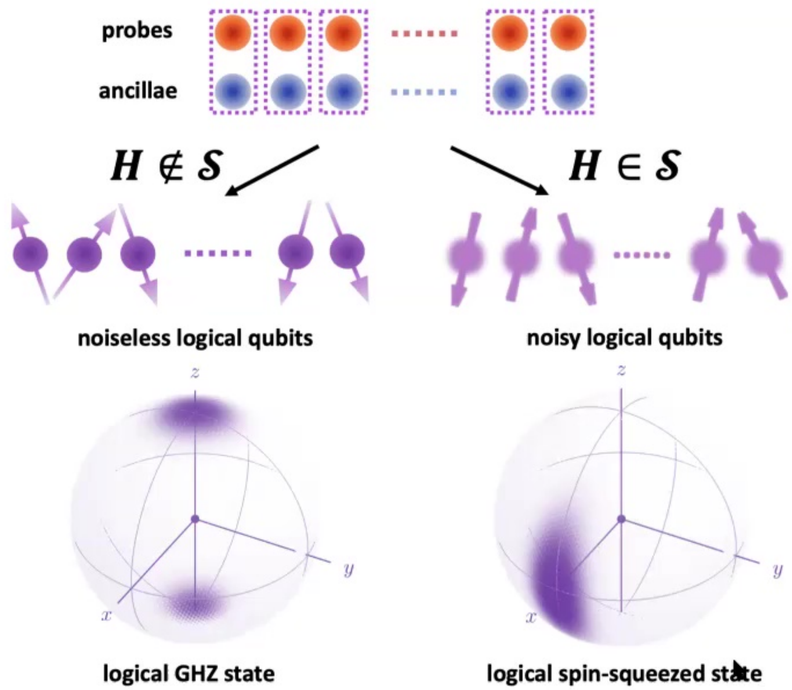
Summary: the QEC protocol

The HL is recovered.

The optimal HL coefficient is achievable for parallel strategies.



The optimal SQL coefficient is achievable for parallel and sequential strategies.

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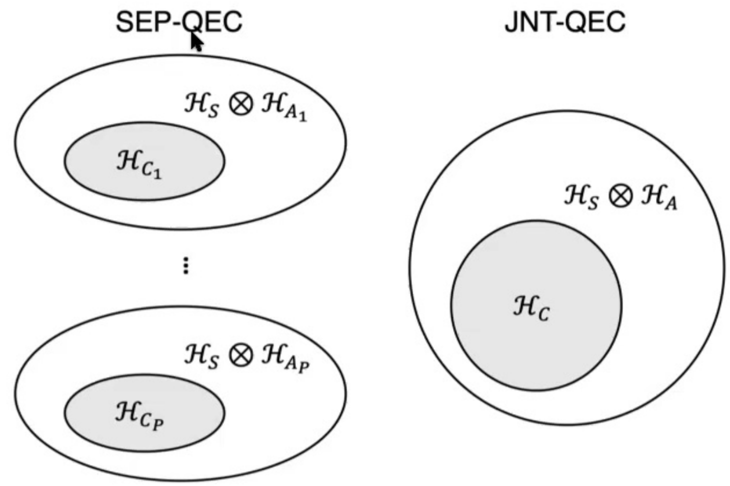
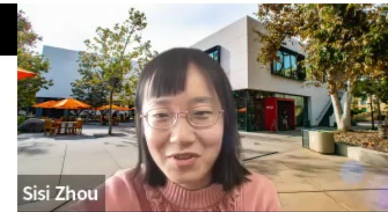
Summary and outlook

- The structure of the noise and the Hamiltonian determines the estimation precision limit in quantum metrology.
- Quantum error correction is powerful – recovering the HL, achieving the optimal HL and SQL coefficients.
- Ancilla-free QEC protocols.
- **Multi-parameter Hamiltonian estimation under Markovian noise.**

Gorecki & SZ et al. 2020





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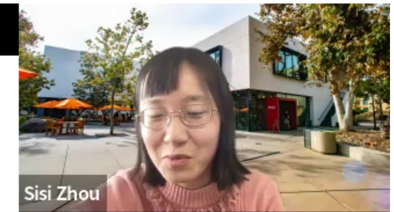
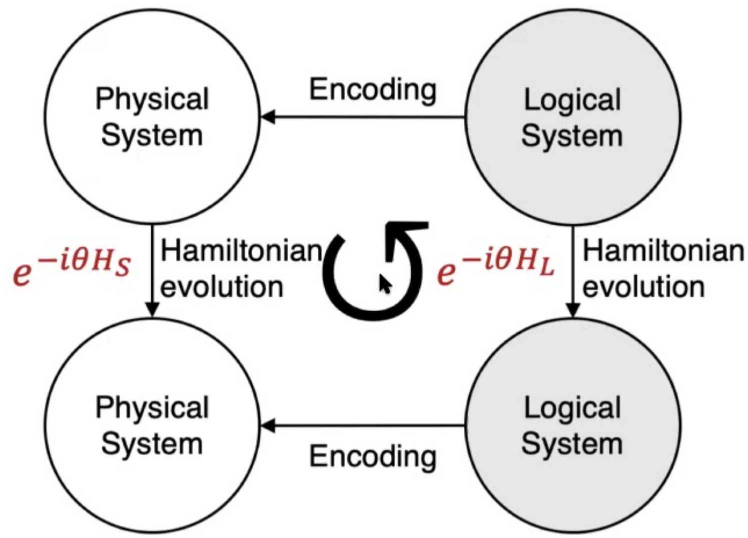


Summary and outlook

- The structure of the noise and the Hamiltonian determines the estimation precision limit in quantum metrology.
- Quantum error correction is powerfu SQL coefficients.
- Ancilla-free QEC protocols.
- Multi-parameter Hamiltonian estimation under Markovian noise.
- **Connection to covariant QEC.**

SZ et al. 2021, Liu & SZ 2021



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Summary and outlook

- The structure of the noise and the Hamiltonian determines the estimation precision limit in quantum metrology.
- Quantum error correction is powerful – recovering the HL, achieving the optimal HL and SQL coefficients.
- **Future directions:**
  - Parallel strategies vs. sequential strategies when the HL is achievable
  - Multi-parameter estimation for general quantum channels
  - Fault-tolerant QEC for quantum metrology

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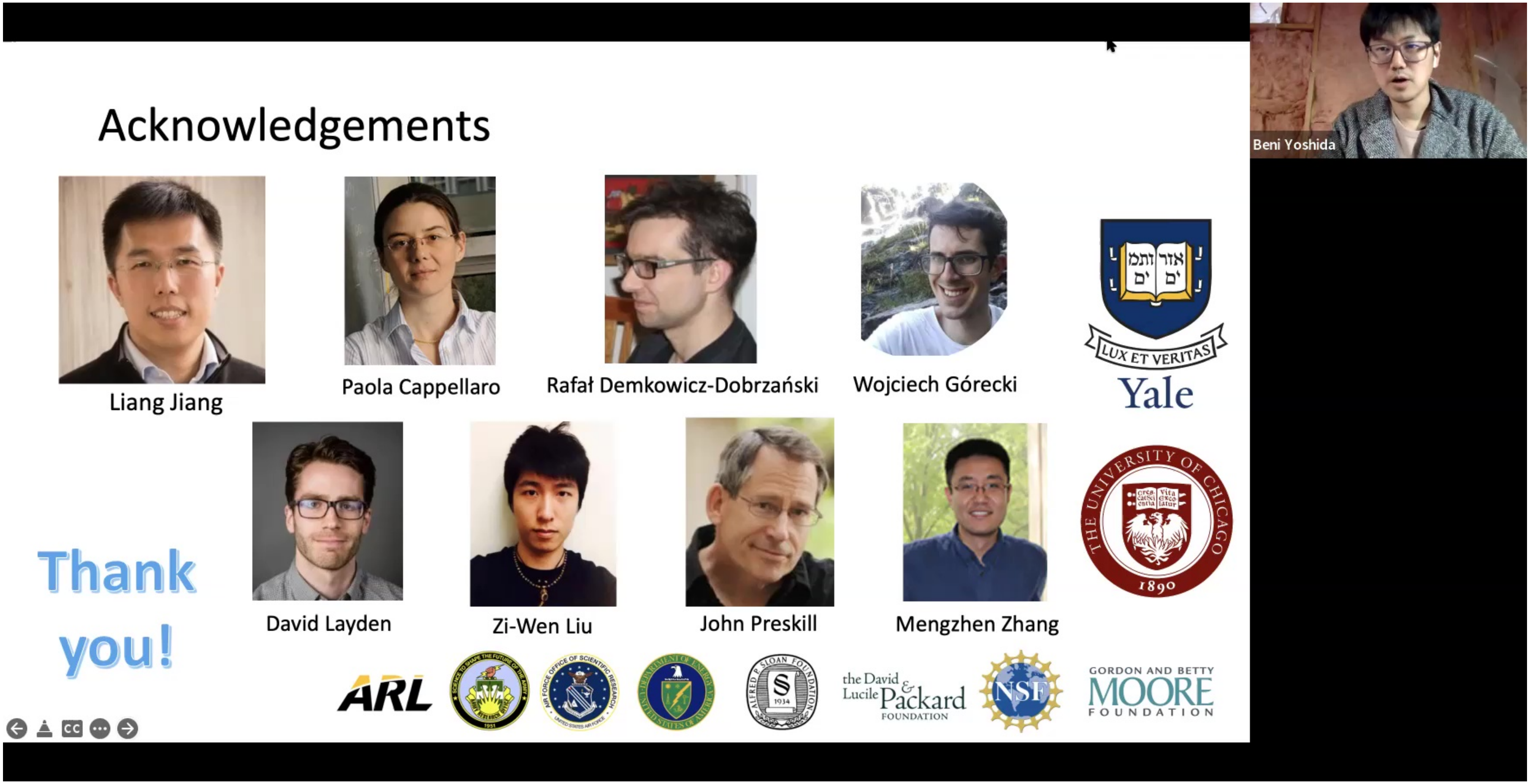
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