Title: QFT III 2021/2022

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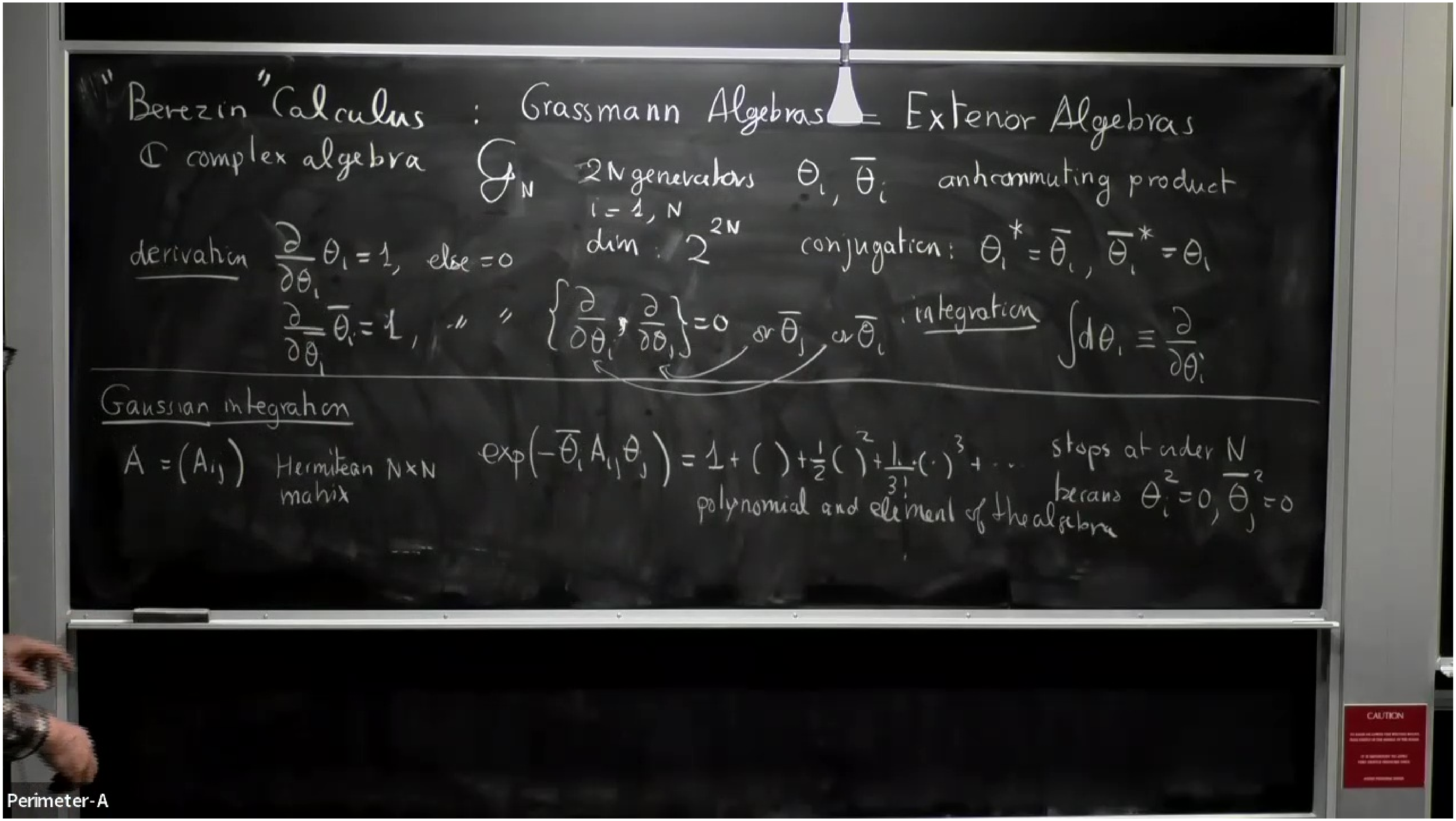
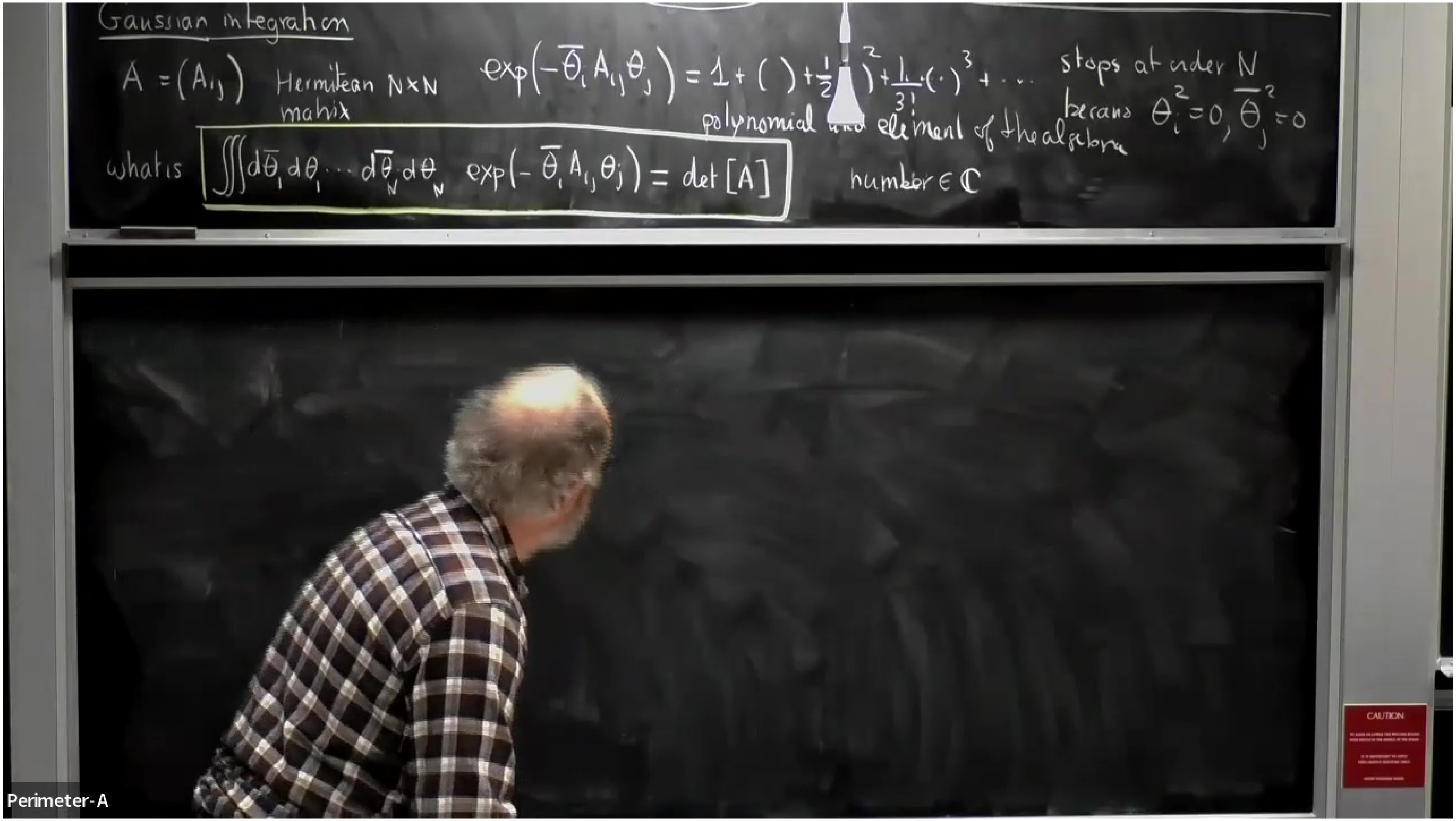

"Berezin" Calculus : Grassmann Algebras = Exterior Algebras
$\mathbb{C}$ complex algebra $\mathcal{G}_N$ 2N generators $\theta_i, \bar{\theta}_i$ $i=1,N$ anticommuting product
derivation $\frac{\partial}{\partial\theta_i}\theta_i = 1$, else $=0$
dim : 2^{2N}
conjugation: $\theta_i^* = \bar{\theta}_i$, $\bar{\theta}_i^* = \theta_i$
$\frac{\partial}{\partial\bar{\theta}_i}\bar{\theta}_i = 1$, " "
$\left\{\frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\theta_j}\right\} = 0$ or $\bar{\theta}_j$ or $\bar{\theta}_i$
integration $\int d\theta_i \equiv \frac{\partial}{\partial\theta_i}$
Gaussian integration
$A = (A_{ij})$ Hermitean $N\times N$ matrix
$\exp(-\bar{\theta}_i A_{ij}\theta_j) = 1 + (\ ) + \frac{1}{2}(\ )^2 + \frac{1}{3!}(\cdot)^3 + \dots$
stops at order N because $\theta_i^2 = 0$, $\bar{\theta}_j^2 = 0$
polynomial and element of the algebra
Perimeter-A
CAUTION


Gaussian integration
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$\exp(-\bar{\theta}_i A_{ij} \theta_j) = 1 + (\ ) + \frac{1}{2}(\ )^2 + \frac{1}{3!}(\cdot)^3 + \dots$ stops at order N
because $\theta_i^2 = 0$, $\bar{\theta}_j^2 = 0$
polynomial and element of the algebra
what is $\boxed{\iiint d\bar{\theta}_1 d\theta_1 \cdots d\bar{\theta}_N d\theta_N \exp(-\bar{\theta}_i A_{ij} \theta_j) = \det[A]}$
number $\in \mathbb{C}$
Perimeter-A

Gaussian integration

$A = (A_{ij})$ (Hermitean) $N\times N$ matrix complex

$\exp(-\bar\theta_i A_{ij}\theta_j) = 1 + (\ ) + \frac{1}{2}(\ )^2 + \frac{1}{3!}(\cdot)^3 + \dots$ stops at order N because $\theta_i^2 = 0,\ \bar\theta_j^2 = 0$

polynomial and element of the algebra

what is $\boxed{\iint d\bar\theta_1 d\theta_1 \cdots d\bar\theta_N d\theta_N \ \exp(-\bar\theta_i A_{ij}\theta_j) = \det[A]}$ number $\in \mathbb{C}$

$N=1 \quad A = (A) \quad \exp(-\bar\theta A\theta) = 1 - \bar\theta A \theta = A$

$N=2 \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{pmatrix}$ " $= 1 + \cdots + (A_{12}A_{21} - A_{11}A_{22})\,\theta_1\theta_2\bar\theta_1\bar\theta_2 = A_{11}A_{22} - A_{12}A_{21}$

$A_{21} = \bar A_{12}$, $\underbrace{A_{11}, A_{22}}_{\text{real}}$

This is valid for any N

Exercise: prove it by recursion on N

Wick Theorem

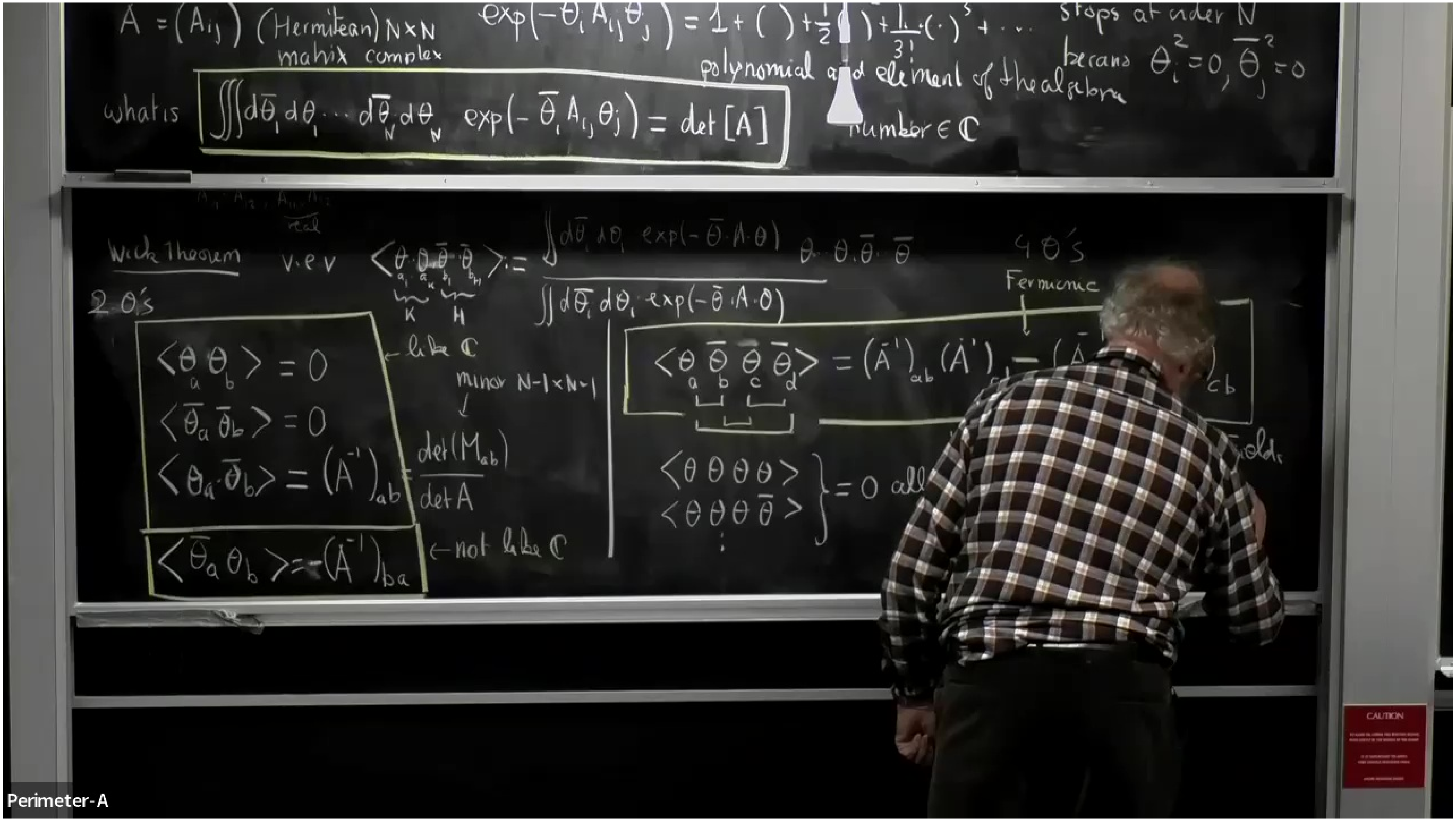
v.e.v. $\langle \underbrace{\theta_{a_1}\cdots\theta_{a_K}}_{K}\underbrace{\bar\theta_{b_1}\cdots\bar\theta_{b_H}}_{H}\rangle := \dfrac{\iint d\bar\theta_i\, d\theta_i\ \exp(-\bar\theta\cdot A\cdot\theta)\ \theta\cdots\theta\,\bar\theta\cdots\bar\theta}{\iint d\bar\theta_i\, d\theta_i\ \exp(-\bar\theta\cdot A\cdot\theta)}$

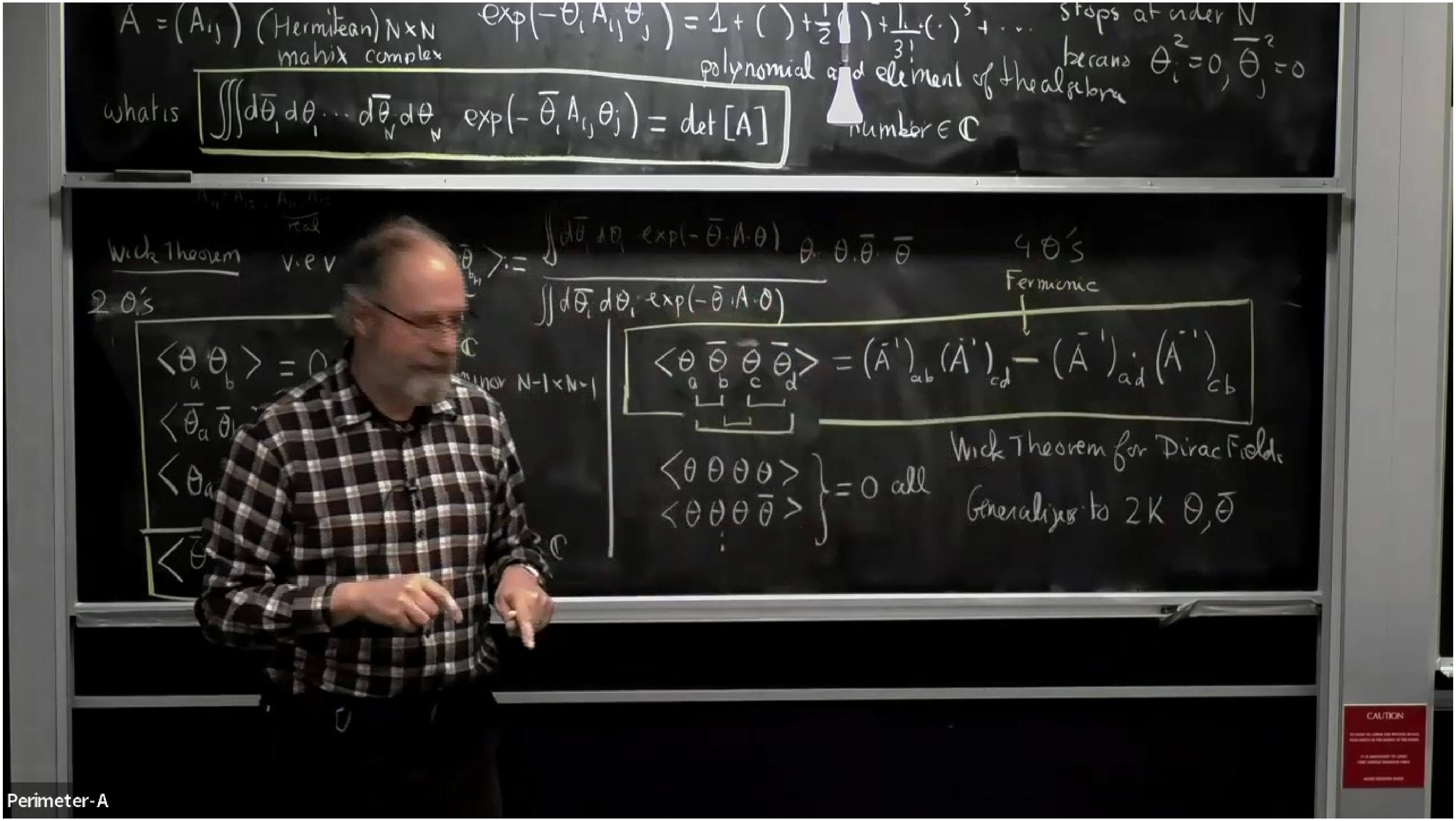
$\langle \theta_a \theta_b \rangle$

$\langle \bar\theta_a \bar\theta_b \rangle$

$\langle \theta_a \bar\theta_b \rangle$

Perimeter-A


A = (Aij) (Hermitean) N×N matrix complex
exp(−θi Aij θj) = 1 + ( ) + 1/2 ( ) + 1/3! (·)^3 + . . .
stops at order N because θi^2 = 0, θ̄j^2 = 0
polynomial and element of the algebra
number ∈ ℂ
what is ∭dθ̄1 dθ1 · · · dθ̄N dθN exp(− θ̄i Aij θj) = det[A]
real
Wick Theorem
v.e.v
<θa1 · θak θ̄b1 · θ̄bH> := ∫dθ̄i dθi exp(−θ̄·A·θ) θ·θ·θ̄·θ̄ / ∫dθ̄i dθi exp(−θ̄·A·θ)
K H
4 θ's
Fermionic
2 θ's
<θa θb> = 0
<θ̄a θ̄b> = 0
<θa θ̄b> = (A^−1)ab = det(Mab) / det A
<θ̄a θb> = −(A^−1)ba
← like ℂ
minor N−1×N−1
← not like ℂ
<θa θ̄b θc θ̄d> = (A^−1)ab (A^−1)
cb
<θθθθ>
<θθθθ̄>
= 0 all
Perimeter-A


(Hermitean) N×N matrix complex
stops at order N
polynomial and element of the algebra
what is
number ∈ ℂ
Wick Theorem
Fermionic
minor N-1×N-1
=0 all
Wick Theorem for Dirac Fields
Generalizes to 2K θ, θ̄
Perimeter-A
CAUTION

$$\boxed{\langle \theta_a \bar\theta_b \rangle = (A^{-1})_{ab}}\ \frac{\det(M_{ab})}{\det A}$$

$$\boxed{\langle \bar\theta_a \theta_b \rangle = -(A^{-1})_{ba}}$$ ← not like $\mathbb{C}$

$$\left.\begin{matrix}\langle \theta\theta\theta\theta \rangle \\ \langle \theta\theta\theta\bar\theta \rangle \\ \vdots\end{matrix}\right\} = 0 \text{ all}$$

Wick Theorem for Dirac Fields

Generalizes to 2K $\theta, \bar\theta$

Free Dirac Field

Functional Integral ⟹ Apply the rules for ϕ, but make the Dirac Field Grassmann

4 dim Minkowski. $\Psi = (\Psi_a)$ $a = 1 \ldots 4 = 2^{[\frac{4}{2}]}$ a dirac Indices

$\bar\Psi = (\bar\Psi_a)$ complex field: w.f. of a single electron $\{\gamma^\mu, \gamma^\nu\} = -2h^{\mu\nu}$ $h^{\mu\nu} = (-1, 1, 1, 1)$

Action Dirac $S[\bar\Psi, \Psi] = \int d^4x\, \bar\Psi(x)(i\partial\!\!\!/ - m)\Psi(x)$ $\partial\!\!\!/ = \gamma^\mu \frac{\partial}{\partial x_\mu}$ Dirac operator $\prod_i d\bar\theta_i\, d\theta_i$

2nd quantization

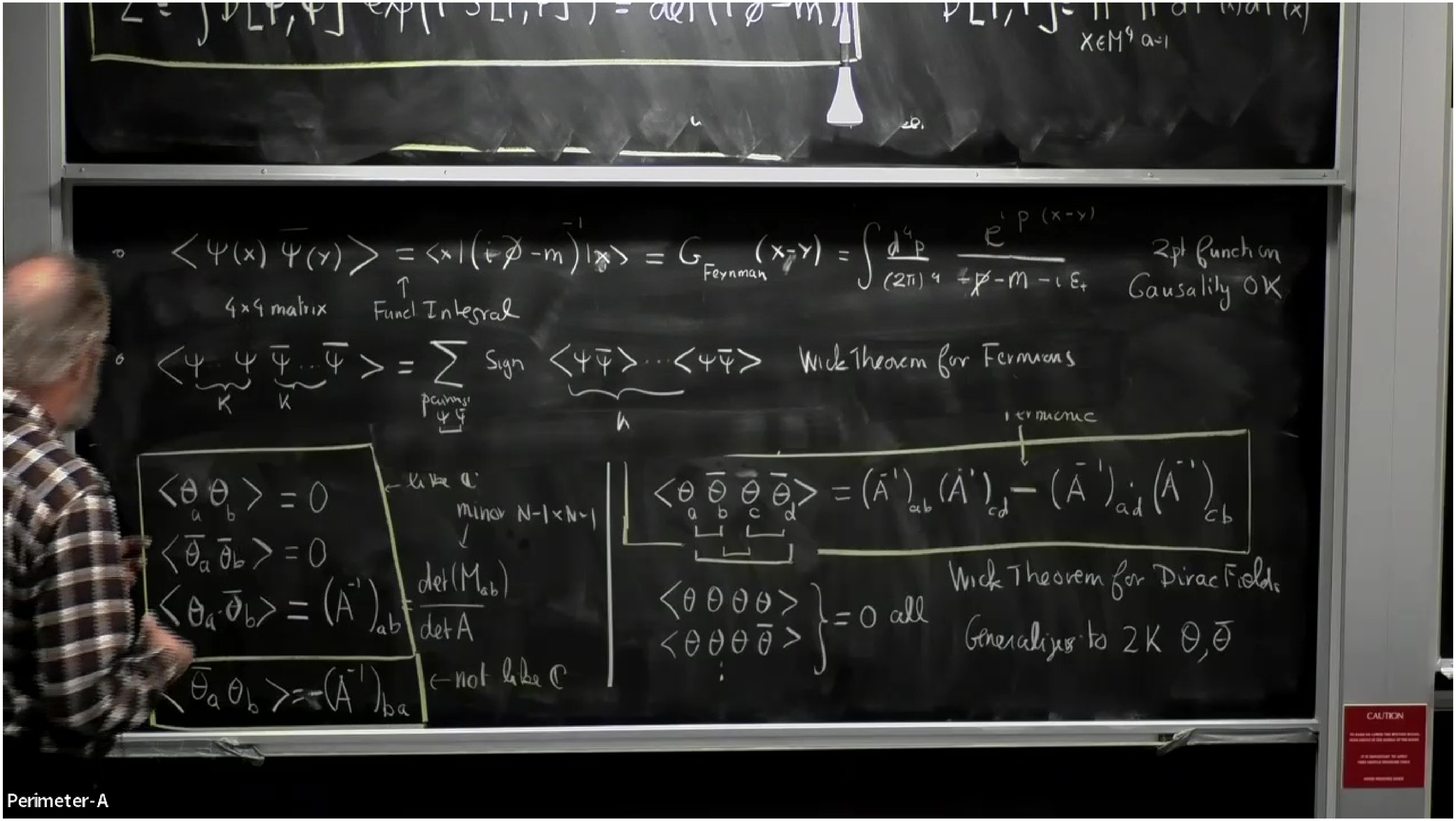
Just make $\Psi^a(x), \bar\Psi^a(x)$ Grassmann numbers ⟹ big Grassmann Algebra

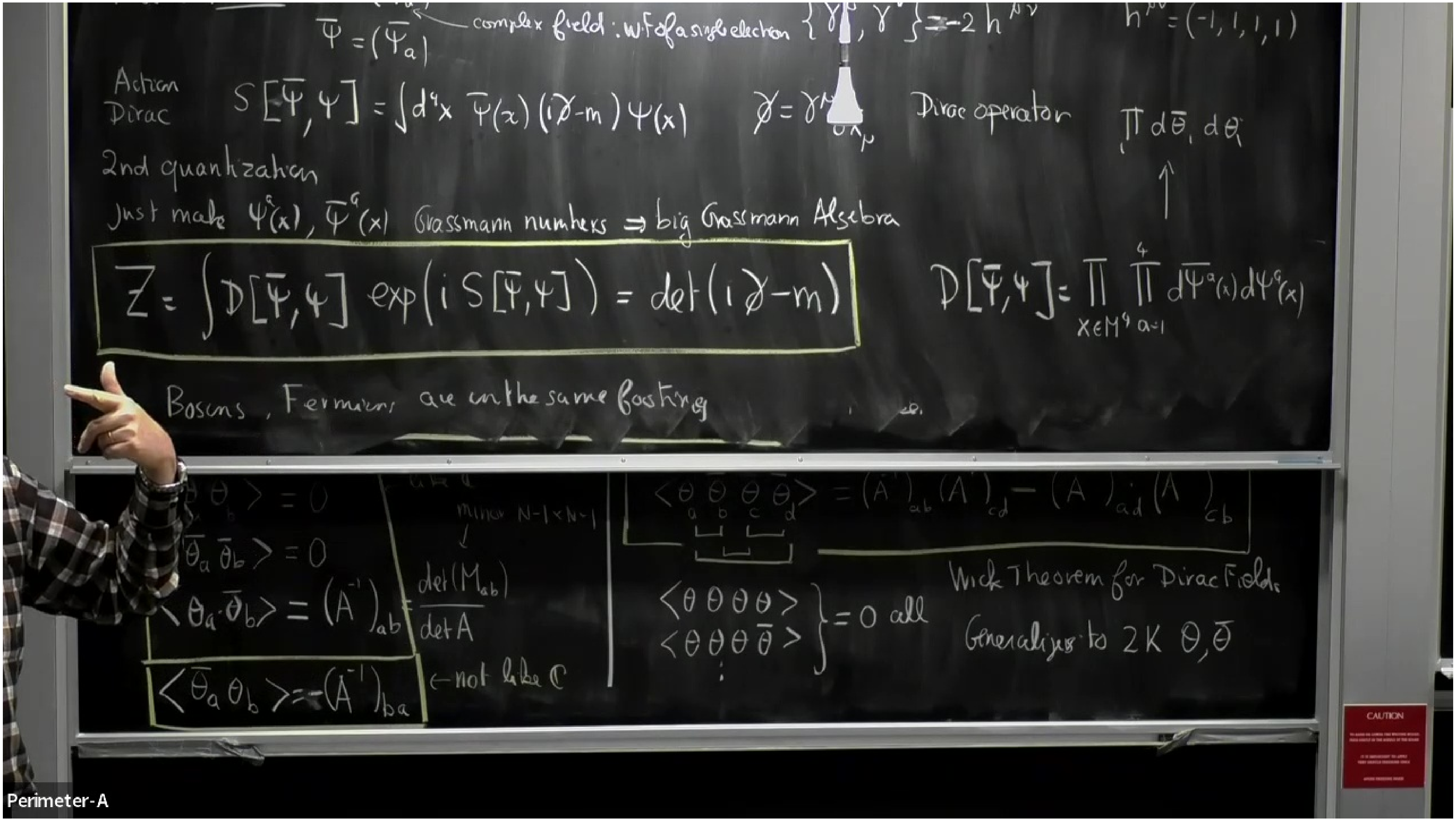
$$\boxed{Z = \int D[\bar\Psi, \Psi] \exp(iS[\bar\Psi, \Psi]) = \det(i\partial\!\!\!/ - m)}$$

$$D[\bar\Psi, \Psi] = \prod_{x \in M^4} \prod_{a=1}^{4} d\bar\Psi^a(x)\, d\Psi^a(x)$$

Perimeter-A

CAUTION


$\langle \Psi(x)\,\bar\Psi(y)\rangle = \langle x|(i\not\partial - m)^{-1}|y\rangle = G_{\text{Feynman}}(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip(x-y)}}{\not p - m - i\epsilon_+}$
2pt function
Causality OK
4×4 matrix
Funcl Integral
$\langle \underbrace{\Psi\cdots\Psi}_{K}\ \underbrace{\bar\Psi\cdots\bar\Psi}_{K}\rangle = \sum_{\text{pairings } \Psi\bar\Psi} \text{Sign}\ \underbrace{\langle\Psi\bar\Psi\rangle\cdots\langle\Psi\bar\Psi\rangle}_{h}$
Wick Theorem for Fermions
$\langle\theta_a\theta_b\rangle = 0$
$\langle\bar\theta_a\bar\theta_b\rangle = 0$
$\langle\theta_a\bar\theta_b\rangle = (A^{-1})_{ab} = \frac{\det(M_{ab})}{\det A}$
← like ℂ
minor N−1×N−1
$\langle\bar\theta_a\theta_b\rangle = -(A^{-1})_{ba}$
← not like ℂ
fermionic
$\langle\theta_a\bar\theta_b\theta_c\bar\theta_d\rangle = (A^{-1})_{ab}(A^{-1})_{cd} - (A^{-1})_{ad}(A^{-1})_{cb}$
Wick Theorem for Dirac Fields
$\langle\theta\theta\theta\theta\rangle$, $\langle\theta\theta\theta\bar\theta\rangle$, ⋮ $= 0$ all
Generalizes to 2K $\theta,\bar\theta$
Perimeter-A


$\bar\Psi = (\bar\Psi_a)$ ← complex field : w.f of a single electron
$\{\gamma^\mu, \gamma^\nu\} = -2\,\eta^{\mu\nu}$
$\eta^{\mu\nu} = (-1,1,1,1)$
Action Dirac $S[\bar\Psi,\Psi] = \int d^4x\, \bar\Psi(x)(i\not\partial - m)\Psi(x)$
$\not\partial = \gamma^\mu \partial_{x_\mu}$
Dirac operator
$\prod_i d\bar\theta_i\, d\theta_i$
2nd quantization
Just make $\Psi^a(x)$, $\bar\Psi^a(x)$ Grassmann numbers ⇒ big Grassmann Algebra
$Z = \int D[\bar\Psi,\Psi]\, \exp(iS[\bar\Psi,\Psi]) = \det(i\not\partial - m)$
$D[\bar\Psi,\Psi] = \prod_{x\in M^4} \prod_{a=1}^{4} d\bar\Psi^a(x)\, d\Psi^a(x)$
Bosons, Fermions are on the same footing
$\langle \theta_a \theta_b \rangle = 0$
$\langle \bar\theta_a \bar\theta_b \rangle = 0$
$\langle \theta_a \bar\theta_b \rangle = (A^{-1})_{ab} = \frac{\det(M_{ab})}{\det A}$
minor $N-1\times N-1$
$\langle \bar\theta_a \theta_b \rangle = -(A^{-1})_{ba}$ ← not like $\mathbb{C}$
$\langle \theta_a \bar\theta_b \theta_c \bar\theta_d \rangle = (A^{-1})_{ab}(A^{-1})_{cd} - (A^{-1})_{ad}(A^{-1})_{cb}$
$\langle \theta\theta\theta\theta \rangle$, $\langle \theta\theta\theta\bar\theta \rangle$ $\Big\} = 0$ all
Wick Theorem for Dirac Fields
Generalizes to $2K$ $\theta, \bar\theta$
Perimeter-A