

Title: QFT III 2021/2022

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Collection: QFT III 2021/2022

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Real Time $X = (t, \vec{x})$ $ds^2 = -dt^2 + d\vec{x}^2$
 $S = \frac{i}{2} \int dt d^3\vec{x} \left[(\partial_t \phi)^2 - (\vec{\nabla} \phi)^2 - m^2 \phi^2 \right] \quad (-+++)$ West Coast

$$Z = \int \mathcal{D}[\phi] \exp\left(\frac{i}{\hbar} S[\phi]\right)$$

ϕ spacetime $\rightarrow \mathbb{R}$ real scalar field

Operators? Local operators \vec{x}, t Heisenberg picture

Field operator $\phi(x)$

Euclidean Spacetime $X = (X^0, \vec{x})$ $ds^2 = dx_0^2 + d\vec{x}^2$

$$S_E[\phi] = \int d^4x \frac{1}{2} (\partial_\mu \phi)^2 + \frac{m^2}{2} \phi^2$$

$$Z_E = \int_E \mathcal{D}[\phi] \exp\left(-\frac{1}{\hbar} S_E[\phi]\right)$$

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$$\frac{\int \mathcal{D}[\phi] e^{\frac{i}{\hbar} S[\phi]} \phi(x_1) \phi(x_2)}{\int \mathcal{D}[\phi] e^{\frac{i}{\hbar} S[\phi]}} = \langle 0 | T \left[\hat{\phi}(x_1) \hat{\phi}(x_2) \right] | 0 \rangle$$

proper sp. time limit time ordered product

path/integrals \Rightarrow time ordered automatically

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Gaut

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$$\frac{\int \mathcal{D}[\phi] e^{-\frac{1}{\hbar} S_E[\phi]} \phi(x_1) \phi(x_2)}{\int \mathcal{D}[\phi] e^{-\frac{1}{\hbar} S_E[\phi]}} = \langle \phi(x_1) \cdot \phi(x_2) \rangle$$

correlation function

$$\frac{\int \mathcal{D}[\phi] e^{-\frac{1}{\hbar} S[\phi]} \phi(x_1) \phi(x_2)}{\int \mathcal{D}[\phi] e^{-\frac{1}{\hbar} S[\phi]}}$$

path/functionals integrals \Rightarrow time ordered locality

$$= \langle 0 | T [\hat{\phi}(x_1) \hat{\phi}(x_2)] | 0 \rangle$$

proper spacetime limit \uparrow time ordered product

$$\frac{\int \mathcal{D}[\phi] e^{iS[\phi]} \phi(x_1) \phi(x_2)}{\int \mathcal{D}[\phi] e^{iS[\phi]}}$$

$$= \langle 0 | T [\hat{\phi}(x_1) \hat{\phi}(x_2)] | 0 \rangle$$

proper to separate time limit

time ordered product

path/order interests \Rightarrow time ordered automatically

Green Function
Wightman Function (more Math.)
Quantum F-T

A simple example $A(\phi)$ etc.

$-\frac{x_0}{c} + t$ - unphysical direction

2 point function $\langle \phi(x_1) \phi(x_2) \rangle$

$$= \frac{\int \mathcal{D}[\phi] e^{-\frac{1}{\hbar} S_E[\phi]} \phi(x_1) \phi(x_2)}{\int \mathcal{D}[\phi] e^{-\frac{1}{\hbar} S_E[\phi]}}$$

$$Z = \det \left[\frac{-\Delta + m^2}{\hbar \cdot 2\pi} \right]^{-1/2}$$

Functional determinant

Some care to define it

$$\langle \phi(x_1) \phi(x_2) \rangle = \hbar \langle x_1 | \frac{1}{-\Delta + m^2} | x_2 \rangle$$

x_1, x_2 replacement of the inverse of $(-\Delta + m^2)$

Kernel of the operator

$$[\phi] = \int d^4x \left[\frac{1}{2} (\partial_\mu \phi)^2 + \frac{m^2}{2} \phi^2 \right] \text{quadrate.}$$

$$= \int d^4x \frac{1}{2} \phi (-\partial_\mu \partial^\mu + m^2) \phi$$

\hbar for $m > 0$
 $(-\Delta + m^2) \cdot \phi$
vector

operator $(-\Delta + m^2)$
Laplace

$$\frac{\int \mathcal{D}[\phi] e^{\frac{i}{\hbar} S[\phi]} \phi(x_1) \phi(x_2)}{\int \mathcal{D}[\phi] e^{\frac{i}{\hbar} S[\phi]}}$$

path (and integrals \Rightarrow time ordered automatically)
 $= \langle 0 | T [\hat{\phi}(x_1) \hat{\phi}(x_2)] | 0 \rangle$ Green Function
 Wightman Function (non Math.)
 Quantum F.T

A simple operator $A(\phi)$ at time

$\frac{-x_0}{c \hbar} H = \text{unnormalized density}$

2 point function $\langle \phi(x_1) \phi(x_2) \rangle$

$$= \frac{\int \mathcal{D}[\phi] e^{\frac{i}{\hbar} S[\phi]} \phi(x_1) \phi(x_2)}{\int \mathcal{D}[\phi]}$$

$$Z = \det \left[\frac{-\Delta + m^2}{\hbar \cdot 2\pi} \right]$$

Functional determinant

$$\langle \phi(x_1) \phi(x_2) \rangle =$$

Gaussian

Some

$$\frac{1}{(-\Delta + m^2) |x \rangle}$$

prop of the operator

Euclidean Spacetime $X = (X^0, \vec{x})$ $dx^2 = dx_0^2 + d\vec{x}^2$

$$S_E[\phi] = \int d^4x \left[\frac{1}{2} (\partial_\mu \phi)^2 + \frac{m^2}{2} \phi^2 \right] \text{quadratic in } \phi$$

$$= \int d^4x \frac{1}{2} \phi (-\partial_\mu \partial^\mu + m^2) \phi \text{ integration by parts}$$

$$\frac{1}{2} \phi (-\Delta + m^2) \phi$$

quadratic form $m > 0$
 vector vector

operator $(-\Delta + m^2)$
 Laplace identity

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$$\frac{\int \mathcal{D}[\phi] e^{iS[\phi]} \phi(x) \phi(x_0)}{\int \mathcal{D}[\phi] e^{iS[\phi]}}$$

paths (and integrals) \Rightarrow time ordered automatically

$$= \langle 0 | T [\hat{\phi}(x) \hat{\phi}(x_0)] | 0 \rangle$$

Green Function
Wightman Function (math)
Quantum F.T

proper to space-time limit
time ordered product

A simple example: $A(a)$ of H

$$e^{-\frac{x_0}{\hbar} H} = \text{unnormalized density matrix}$$

2 point function $\langle \phi(x_1) \phi(x_2) \rangle$

$$= \frac{\int \mathcal{D}[\phi] e^{-\frac{1}{\hbar} S_E[\phi]} \phi(x_1) \phi(x_2)}{\int \mathcal{D}[\phi] e^{-\frac{1}{\hbar} S_E[\phi]}}$$

$$Z = \det \left[\frac{-\Delta + m^2}{\hbar \cdot 2\pi} \right]^{-1/2}$$

Functional determinant

Some care to define it

Gaussian integrals!

$a = (a_1, \dots, a_n)$ sym. matrix $[A_{ij}]$ $A \in \mathbb{R}^{n \times n}$

$$\frac{\int_{-\infty}^{\infty} da_1 \dots da_n \exp\left[-\frac{1}{2} \sum_{i,j} a_i A_{ij} a_j\right]}{\int da_1 \dots da_n} = \frac{1}{\sqrt{\det A}}$$

G_0 is the "propagator" associated to the K-G equation

$$\langle \phi(x_1) \phi(x_2) \rangle = \frac{1}{\hbar} \langle 0 | \frac{1}{-\Delta + m^2} | 0 \rangle = \frac{1}{\hbar} G_0(x_1, x_2) \quad \text{satisfy } (-\Delta + m^2) \cdot G_0(x_1, x_2) = \delta(x_1 - x_2)$$

x_1, x_2 neighborhood of the inverse of $(-\Delta + m^2)$

Kernel of the operator

"Dirac function"

2 point function $\langle \phi(x) \phi(x_2) \rangle$

$$= \frac{\int \mathcal{D}[\phi] e^{-\frac{1}{\hbar} S[\phi]} \phi(x) \phi(x_2)}{\int \mathcal{D}[\phi] e^{-\frac{1}{\hbar} S[\phi]}}$$

$\int_{\mathbb{R}^n} dx_1 \dots dx_n \exp\left[-\frac{1}{2} \sum_{ij} x_i A_{ij} x_j\right] = \frac{(2\pi)^{n/2}}{\sqrt{\det A}}$

Functional determinant $Z = \det \left[\frac{-\Delta + m^2}{\hbar 2\pi} \right]$

Gaussian integrals!

Some care to define it

G_0 is the "propagator" associated to the K.G. equation

$\langle \phi(x) \phi(x_2) \rangle = \hbar \langle \phi(x) | \frac{1}{-\Delta + m^2} | \phi(x_2) \rangle = \hbar G_0(x, x_2)$ satisfy $(-\Delta + m^2) \cdot G_0(x, x_2) = \delta(x - x_2)$

$\delta(x - x_2)$ Dirac function

$\hbar \langle \phi(x) | \frac{1}{-\Delta + m^2} | \phi(x_2) \rangle$ Kernel of the operator

$G_0(x_1, x_2) = G_0(x_2, x_1)$

$G_0(x) \rightarrow$ Fourier T. $\hat{G}_0(k) = \int dx e^{ikx} G_0(x)$

$x = (x^\mu)$ vector
 $k = (k_\mu)$ co-vector
 $x \cdot k = x^\mu k_\mu$
 $k^2 = k_\mu k^\mu$ Euclidean

K.G. equation $\hat{G}_0(k) \cdot (k^2 + m^2) = 1$ $\hat{G}_0(k) = \frac{1}{k^2 + m^2}$

Inverse Fourier Transform
 Euclidean propagator $G_0(x) = \int \frac{d^4 k}{(2\pi)^4} \frac{e^{ikx}}{k^2 + m^2}$

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$$G_0(x, x') = G_0(x - x') \quad G_0(x) \rightarrow \text{Fourier T.} \quad \hat{G}_0(k) = \int d^d x e^{-i k x} G_0(x)$$

$$\text{KG equation} \quad \hat{G}_0(k) \cdot (k^2 + m^2) = 1 \quad \hat{G}_0(k) = \frac{1}{k^2 + m^2}$$

$$\text{Inverse Fourier Transform} \quad G_0(x) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{i k x}}{k^2 + m^2}$$

Euclidean propagator

$x = (x^\mu)$ vector
 $k = (k_\mu)$ co-vector
 $x \cdot k = x^\mu k_\mu$
 $k^2 = k_\mu k_\mu$ Euclidean

The prescription



The prescription for getting the Feynman Prop

Euclidean \rightarrow Minkowski

$$x_E = (x^0, \vec{x}) \quad x^0 = i t \quad x = (t, \vec{x})$$

Euclidean time \rightarrow time

analytic continuation

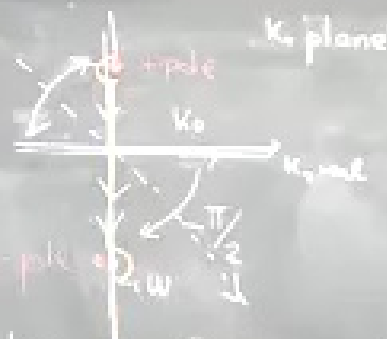
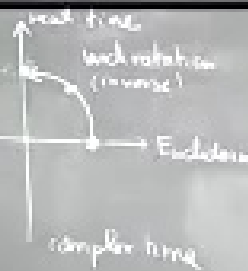
$$G_0(x, \vec{x}) = \int \frac{dK_0}{2\pi} \int \frac{d\vec{k}}{(2\pi)^d} \frac{e^{i(K_0 x^0 + \vec{k} \cdot \vec{x})}}{K_0^2 + \vec{k}^2 + m^2}$$

$$K_E = (K_0, \vec{k})$$

Euclidean

$$K_0 \rightarrow -i \omega$$

Realtime
 real frequency



pole at
 $\omega = \pm \sqrt{\vec{k}^2 + m^2}$
 $K_0 = \pm i \sqrt{\vec{k}^2 + m^2}$

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \int \frac{d\vec{k}}{(2\pi)^d} \frac{\exp[i(\omega t + \vec{k} \cdot \vec{x})]}{-\omega^2 + \vec{k}^2 + m^2 - i\epsilon}$$

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Cont

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correlation function

Euclidean F.T

$\langle A(0) \rangle$

$$\frac{\int D[\phi] e^{-\frac{1}{\hbar} S[\phi]} \phi(x_1) \phi(x_2)}{\int D[\phi] e^{-\frac{1}{\hbar} S[\phi]}}$$

path/integral \Rightarrow time ordered automatically
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proper to spacetime limit time ordered product

Green Function
 Feynman Function (more math.)
 Quantum F.T

A single operator $A(0)$ at the origin: e.g. $\phi(0)$
 evolution operator $e^{-\frac{t}{\hbar} H}$ real time $t = -i X^0$
 $e^{-\frac{X^0}{\hbar} H}$ euclidean time

$e^{-\frac{X^0}{\hbar} H}$ = unnormalized density matrix Gibbs State

at temperature T
 $e^{-\frac{X^0}{k_B T} H}$
 $X^0 = \frac{\hbar}{k_B T}$
 Euclidean time