

Title: QFT III 2021/2022

Speakers: Francois David

Collection: QFT III 2021/2022

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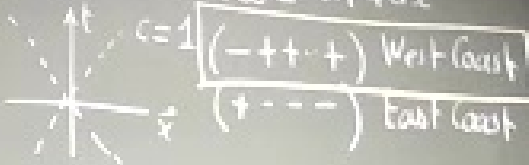
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- Functional Integral Quantization of QFT : spin 0 and $1/2$ fields
- Renormalization & R. Group theory + in QFT and Wilsonian renormalization
- Gauge theories spin 1 non-abelian
- BRST sym. \leftrightarrow CFT and String, anomalies, instantons

Today scalar free field

- Real time and Euclidean time QFT course

Minkowski $ds^2 = -dt^2 + d\vec{x}^2$



$c=1$

| | |
|----------|------------|
| $(-+++)$ | West Coast |
| $(+---)$ | East Coast |

Rotation

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Wick Rotation
time t



$$ds^2 = dt^2 + d\vec{x}^2$$

Eucledian metric

Eucledian
time $\tau = it$

Scalar Free Field (Klein-Gordon)

dim is d

time 1
space $d-1$

Classically $x = (t, \vec{x})$ $\phi = \phi(x)$ real

Classical action Lagrangian density

$$S[\phi] = \int dt \int d\vec{x} \left[\underbrace{\frac{1}{2}(\partial_t \phi)^2}_{\text{kinetic}} - \underbrace{\left(\frac{1}{2}(\vec{\nabla}_x \phi)^2 + \frac{m^2}{2} \phi^2 \right)}_{\text{potential}} \right]$$

action

kinetic

potential

Lagrangian

E-L Equ. K.G $\left[\frac{\partial}{\partial t^2} - \vec{\nabla}_x^2 + m^2 \right] \phi = 0$

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Wick Rotation $\text{red time } t$



$ds^2 = dt^2 + d\vec{x}^2$
Euclidean metric

Euclidean time $\tau = it$



Scalar Free Field (Klein-Gordon)

dim is d time 1
space $d-1$

Classically $x = (t, \vec{x})$ $x \rightarrow \phi(x)$ real

Classical action Lagrangian density

$S[\phi] = \int dt \int d\vec{x} \left[\frac{1}{2} (\partial_t \phi)^2 - \left(\frac{1}{2} (\vec{\nabla}_x \phi)^2 + \frac{m^2}{2} \phi^2 \right) \right]$

Quadratic in ϕ

action $\left\{ \begin{array}{l} \text{kinetic} \\ \text{potential} \end{array} \right.$

E.L. Equ. K.G $\left[\frac{\partial}{\partial t^2} - \vec{\nabla}_x^2 + m^2 \right] \phi = 0$

Linear in ϕ

$m = \text{"mass"}$
of the excitations
of $\phi = \text{boson}$

In Euclidean S.F. $x = (t, \vec{x}) = (x^0, \mu = 1, \dots, d)$

Action $S_E[\phi] = \int d^d x \left[\frac{1}{2} (\partial_\mu \phi)^2 + \frac{m^2}{2} \phi^2 \right]$

$t = -ix^0: S_E[\phi] \stackrel{''}{=} -i S[\phi]$

"Equ of Motion" $(-\Delta + m^2) \phi = 0$

Δ Laplace-Beltrami op $\Delta = \left(\frac{\partial}{\partial x^\mu} \right)^2$

Functional integral quantization (Sketchy)

$Z \rightarrow$ Functional Integral $Z = \frac{1}{Z_0} \int \mathcal{D}\phi e^{-S[\phi]}$



Fields

formally $D[\phi] = \prod_{x \in \mathbb{H}^{d-1}} d\phi(x)$
spacetime

formal measure
direct product of measures

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Wick Rotation
time t



$ds^2 = dt^2 + d\vec{x}^2$
Euclidean metric

Euclidean time $\tau = it$



Scalar Free Field (Klein-Gordon)

dim is d time \pm space $d-1$

Classically $x = (t, \vec{x})$ $\phi = \phi(x)$ real

Classical action Lagrangian density

$S[\phi] = \int dt \int d\vec{x} \left[\frac{1}{2} (\partial_t \phi)^2 - \left(\frac{1}{2} (\nabla_{\vec{x}} \phi)^2 + \frac{m^2}{2} \phi^2 \right) \right]$

Quadratic in ϕ

action $\underbrace{\hspace{10em}}_{\text{Lagrangian}}$
kinetic potential

E.L. Equ. K.G $\left[\frac{\partial}{\partial t^2} - \nabla_{\vec{x}}^2 + m^2 \right] \phi = 0$

$m =$ "mass" of the excitations of $\phi =$ boson

Linear in ϕ

In Euclidean

Action S

$S[\phi] = \int d\tau \int d\vec{x} \left[\frac{1}{2} (\partial_{\mu} \phi)^2 + \frac{m^2}{2} \phi^2 \right]$

$t = -i\tau$

$S[\phi]$

Eqn of

$\phi = 0$

Δ

$\Delta = \left(\frac{\partial}{\partial \mu} \right)^2$

Functional Integral quantization (Sketchy)

Path Integral \rightarrow Functional Integral $\hbar = \frac{\hbar}{2\pi i}$

$\int D[\phi] e^{\frac{i}{\hbar} S[\phi]}$

Fields

Measure over Field configuration

formally

$D[\phi] = \prod_{x \in \mathbb{R}^{d+1}} d\phi(x)$

formal measure direct product of measures

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In Euclidean SF $x = (t, \vec{x}) = (x^\mu, \mu = 0, d-1)$

Action $S_E[\phi] = \int d^d x \left[\frac{1}{2} (\partial_\mu \phi)^2 + \frac{m^2}{2} \phi^2 \right]$

$t = -ix^0$; $S_E[\phi] \stackrel{?}{=} -i S[\phi]$

"Eqn of Motion" $(-\Delta + m^2)\phi = 0$

Δ Laplacian-Beltrami op $\Delta = \left(\frac{\partial}{\partial x^\mu}\right)^2$

Functional Integral quantization (Sketchy)

Path Integral \rightarrow Functional Integral $\hbar = \frac{\hbar}{2\pi}$

$\int \mathcal{D}[\phi] e^{\frac{i}{\hbar} S[\phi]}$

measure over Field configuration phase

Fields

formally

$\mathcal{D}[\phi] = \prod_{x \in \mathbb{R}^{d-1}} d\phi(x)$
space-time

local measure direct product of measures

Euclidean Functional Integral

$Z = \int \mathcal{D}[\phi] e^{-\frac{1}{\hbar} S_E[\phi]}$ energy

positive "weight factor"

stat mech

$Z = \sum_{\text{microstates}} e^{-\beta E[\text{microstate}]}$ temperature
 $\beta = \frac{1}{k_B T}$ $\hbar \sim \frac{1}{k_B T}$

partition function

Scalar Free Field (Klein-Gordon)

dim is d time z space $d-1$

Classically $x = (t, \vec{x})$ $x \rightarrow \phi(x)$ real

Classical action Lagrangian density

Real time

$S[\phi] = \int dt \int d^{d-1} \vec{x} \left[\frac{1}{2} (\partial_t \phi)^2 - \left(\frac{1}{2} (\vec{\nabla}_x \phi)^2 + \frac{m^2}{2} \phi^2 \right) \right]$ Quadratic in ϕ

kinetic potential

E.L. Equ. K.6 $\left[\frac{\partial}{\partial t^2} - \vec{\nabla}_x^2 + m^2 \right] \phi = 0$

Linear in ϕ

Lagrangian $\int d^d x$

$m =$ "mass" of the excitations of $\phi =$ boson

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In Euclidean ST $x = (t, \vec{x}) = (x^0, \mu = 0, 1)$

Action $S_E[\phi] = \int d^4x \left[\frac{1}{2} (\partial_\mu \phi)^2 + \frac{m^2}{2} \phi^2 \right]$

$t = -ix^4$; $S_E[\phi] = -i S[\phi]$

Eqn of Motion $(-\Delta + m^2)\phi = 0$

Δ Laplacian-Beltrami op $\Delta = \left(\frac{\partial}{\partial x^\mu}\right)^2$

Functional Integral quantization (Schematics)

Path Integral \rightarrow Functional Integral $\hbar = \frac{\hbar}{2\pi}$

$\int \mathcal{D}[\phi] e^{i S[\phi]}$

measure over Field configuration phase

formally $\mathcal{D}[\phi] = \prod_x d\phi(x)$
 local measure direct product of measures

Euclidean Functional Integral

$Z = \int \mathcal{D}_E[\phi] e^{-\frac{1}{\hbar} S_E[\phi]}$ energy

positive "weight factor"

stat. mech

$Z = \sum_{\text{microstates}} e^{-\beta E[\text{microstate}]}$ temporal $\beta = \frac{1}{k_B T}$ $\hbar \sim k_B T$

partition function

Classically $x = (t, \vec{x})$ $x \rightarrow \phi(x)$ real
 Classical action Lagrangian density

$S[\phi] = \int dt \int d^3x \left[\frac{1}{2} (\partial_t \phi)^2 - \left(\frac{1}{2} (\vec{\nabla}_x \phi)^2 + \frac{m^2}{2} \phi^2 \right) \right]$

kinetic potential

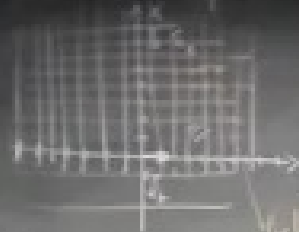
E.L. Eqn. K.6 $\left[\frac{\partial}{\partial t^2} - \vec{\nabla}_x^2 + m^2 \right] \phi = 0$
 Linear in ϕ Lagrangian $\frac{\partial}{\partial x^\mu} \frac{\delta \mathcal{L}}{\delta \phi} = \frac{\delta \mathcal{L}}{\delta \phi}$

Real \hbar

$m =$ "mass" of the excitation
 ϕ boson

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Discretize spacetime (Lattice Field Theory)



$d = 1 + 1$ dim
 $ds^2 = -dt^2 + dx^2$
 $\epsilon_t = \epsilon_x = \epsilon$ simplification

spacetime is a lattice \sum^d
 $x = \epsilon(t, \vec{j}) \quad t, j \in \mathbb{Z}$
 $\phi(x)$ is defined on the sites
 \rightarrow discretization of $\phi_{i,j}$

$t = \epsilon \cdot i \quad i \in \mathbb{Z}$
 $x = \epsilon \cdot \vec{j} \quad \vec{j} \in \mathbb{Z}^{d-1}$

Functional Integral quantization (Sketchy)

Path Integral \rightarrow Functional Integral $\hbar = \frac{\hbar}{2\pi}$
 partial Fields

$$\int \mathcal{D}[\Phi] e^{i/\hbar S[\Phi]}$$

measure over Field configuration phase

formally $\mathcal{D}[\Phi] = \prod_{x \in \mathbb{H}^{d+1}} d\phi(x)$
 spacetime
 local measure

Follow the recipe to construct a path integral

Action $\int dt \int dx \frac{1}{2} (\partial_t \phi)^2 - (\partial_x \phi)^2 - m^2 \phi^2$

discrete action $\sum_{\text{sites}} \epsilon^2 \left[\frac{1}{2} \left(\frac{\phi_{i,j} - \phi_{i-1,j}}{\epsilon} \right)^2 - \frac{1}{2} \left(\frac{\phi_{i,j} - \phi_{i,j-1}}{\epsilon} \right)^2 - \frac{m^2}{2} \phi_{i,j}^2 \right]$

$\partial \rightarrow$ discrete derivatives

$\int \rightarrow \sum_{\text{sites}}$

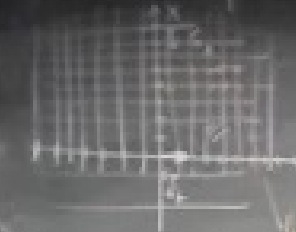
Measure

$$\mathcal{D}[\phi] = \prod_{\text{sites}} d\phi_{i,j} \cdot \left(\frac{2i\pi\hbar}{\epsilon} \right)^{1/2}$$

\uparrow
 real num

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Discretize spacetime (Lattice Field Theory)



$d = 3$ dim
 $ds^2 = -dt^2 + dx^2$
 $\epsilon_T = \epsilon_x = \epsilon$ simplification

spacetime is a lattice \sum^d
 $x = \epsilon(i, j) \quad i, j \in \mathbb{Z}$
 $\phi(x)$ defined on the sites
 \rightarrow discrete collection of $\phi_{(i,j)}$

$t = \epsilon_T i \quad i \in \mathbb{Z}$
 $x = \epsilon_x j \quad j \in \mathbb{Z}^{d-1}$

Functional Integral quantization (Sketchy)

Path Integral \rightarrow Functional Integral $\hbar = \frac{\hbar}{2\pi}$

$\int \mathcal{D}[\phi] e^{\frac{i}{\hbar} S[\phi]}$

measure over Field configuration phase

Fields formally $\mathcal{D}[\phi] = \prod_{x \in \mathbb{H}^{d+1}} d\phi(x)$
 local measure

to construct a path integral

$\int dx \frac{1}{2} (\partial_t \phi)^2 - (\partial_x \phi)^2 - m^2 \phi^2$
 $\int \frac{1}{2} \left[\frac{1}{\epsilon} (\phi_{i,j+1} - \phi_{i,j})^2 - \frac{1}{\epsilon} (\phi_{i+1,j} - \phi_{i,j})^2 - \frac{m^2}{2} \phi_{i,j}^2 \right]$

derivatives
 Lattice in Euclidean space

Measure $\mathcal{D}[\phi] = \prod_{(i,j)} \left[d\phi_{i,j} \cdot \left(\frac{2\pi\hbar}{\epsilon^{d-2}} \right)^{\frac{1}{2}} \right]$
 sites real measure factor depends on ϵ and \hbar , but not on m

Euclidean $\left(\frac{2\pi\hbar}{\epsilon^{d-2}} \right)^{\frac{1}{2}} \Rightarrow$ lattice of coupled oscillators

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