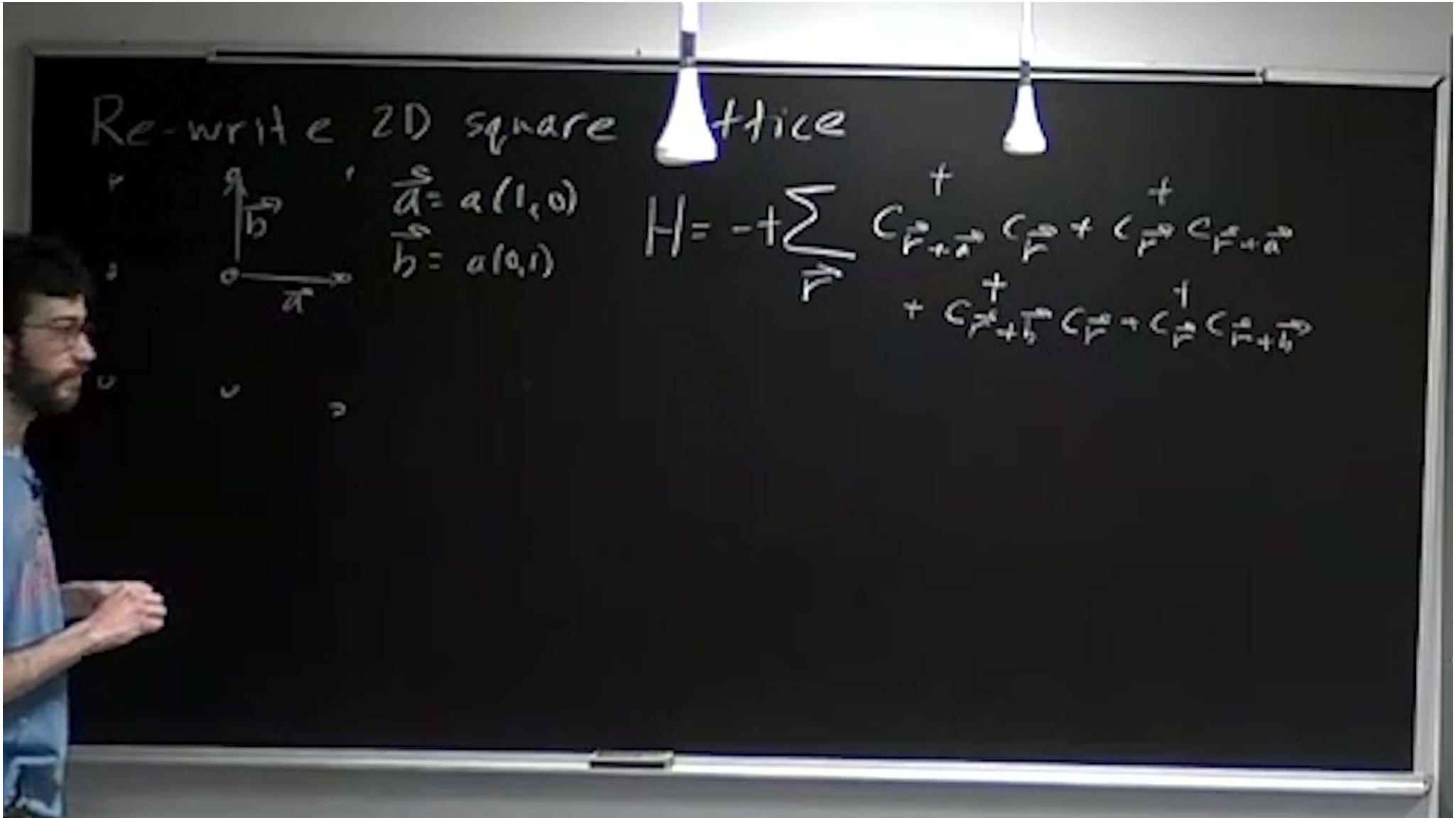
Title: PSI Lecture - Condensed Matter - Lecture 8

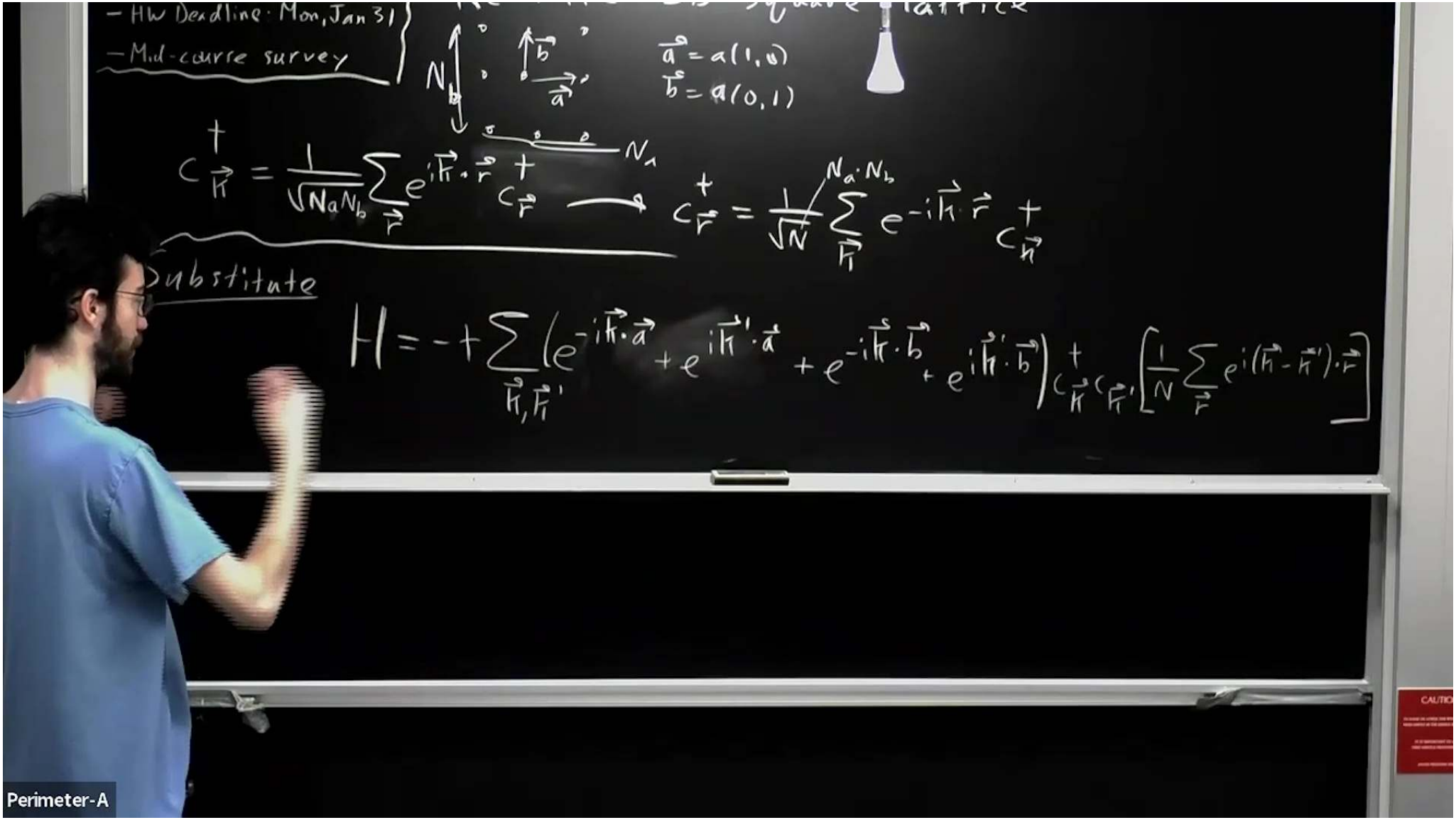
Speakers: Aaron Szasz

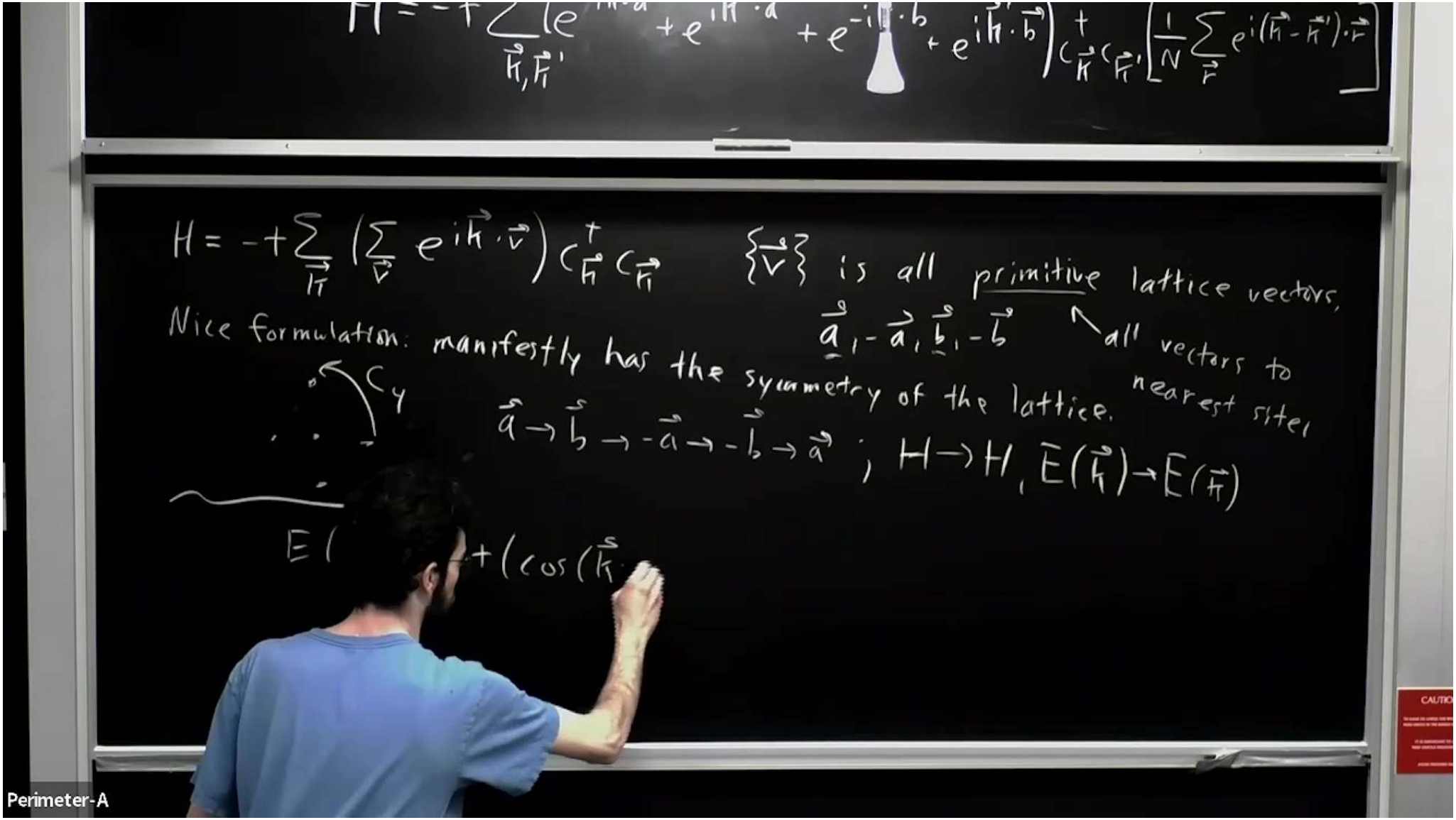
Collection: PSI Lecture - Condensed Matter

Date: January 25, 2022 - 11:30 AM

URL: https://pirsa.org/22010019


Re-write 2D square lattice
$\vec{a} = a(1,0)$
$\vec{b} = a(0,1)$
$H = -t \sum_{\vec{r}} c^{\dagger}_{\vec{r}+\vec{a}} c_{\vec{r}} + c^{\dagger}_{\vec{r}} c_{\vec{r}+\vec{a}} + c^{\dagger}_{\vec{r}+\vec{b}} c_{\vec{r}} + c^{\dagger}_{\vec{r}} c_{\vec{r}+\vec{b}}$


- HW Deadline: Mon, Jan 31
- Mid-course survey
$\vec{a} = a(1,0)$
$\vec{b} = a(0,1)$
N_a
N_b
$c^\dagger_{\vec{k}} = \frac{1}{\sqrt{N_a N_b}} \sum_{\vec{r}} e^{i\vec{k}\cdot\vec{r}} c^\dagger_{\vec{r}}$
$c^\dagger_{\vec{r}} = \frac{1}{\sqrt{N}} \sum_{\vec{k}} e^{-i\vec{k}\cdot\vec{r}} c^\dagger_{\vec{k}}$
Substitute
$H = -t \sum_{\vec{k},\vec{k}'} \left(e^{-i\vec{k}\cdot\vec{a}} + e^{i\vec{k}'\cdot\vec{a}} + e^{-i\vec{k}\cdot\vec{b}} + e^{i\vec{k}'\cdot\vec{b}} \right) c^\dagger_{\vec{k}} c_{\vec{k}'} \left[\frac{1}{N} \sum_{\vec{r}} e^{i(\vec{k}-\vec{k}')\cdot\vec{r}} \right]$
Perimeter-A


$H = -t \sum_{\vec{r}} \left(\sum_{\vec{v}} e^{i\vec{k}\cdot\vec{v}} \right) c^{\dagger}_{\vec{k}} c_{\vec{r}}$
$\{\vec{v}\}$ is all primitive lattice vectors,
$\vec{a}, -\vec{a}, \vec{b}, -\vec{b}$ all vectors to nearest sites
Nice formulation: manifestly has the symmetry of the lattice.
C_4
$\vec{a} \to \vec{b} \to -\vec{a} \to -\vec{b} \to \vec{a}$; $H \to H$, $E(\vec{k}) \to E(\vec{k})$
$E($
$+(\cos(\vec{k}$
Perimeter-A

$$H = -t \sum_{\vec{k},\vec{k}'} \left(e^{i\vec{k}\cdot\vec{a}} + e^{i\vec{k}\cdot\vec{a}} + e^{-i\vec{k}\cdot\vec{b}} + e^{i\vec{k}\cdot\vec{b}} \right) c^{\dagger}_{\vec{k}} c_{\vec{k}'} \left[\frac{1}{N} \sum_{\vec{r}} e^{i(\vec{k}-\vec{k}')\cdot\vec{r}} \right]$$

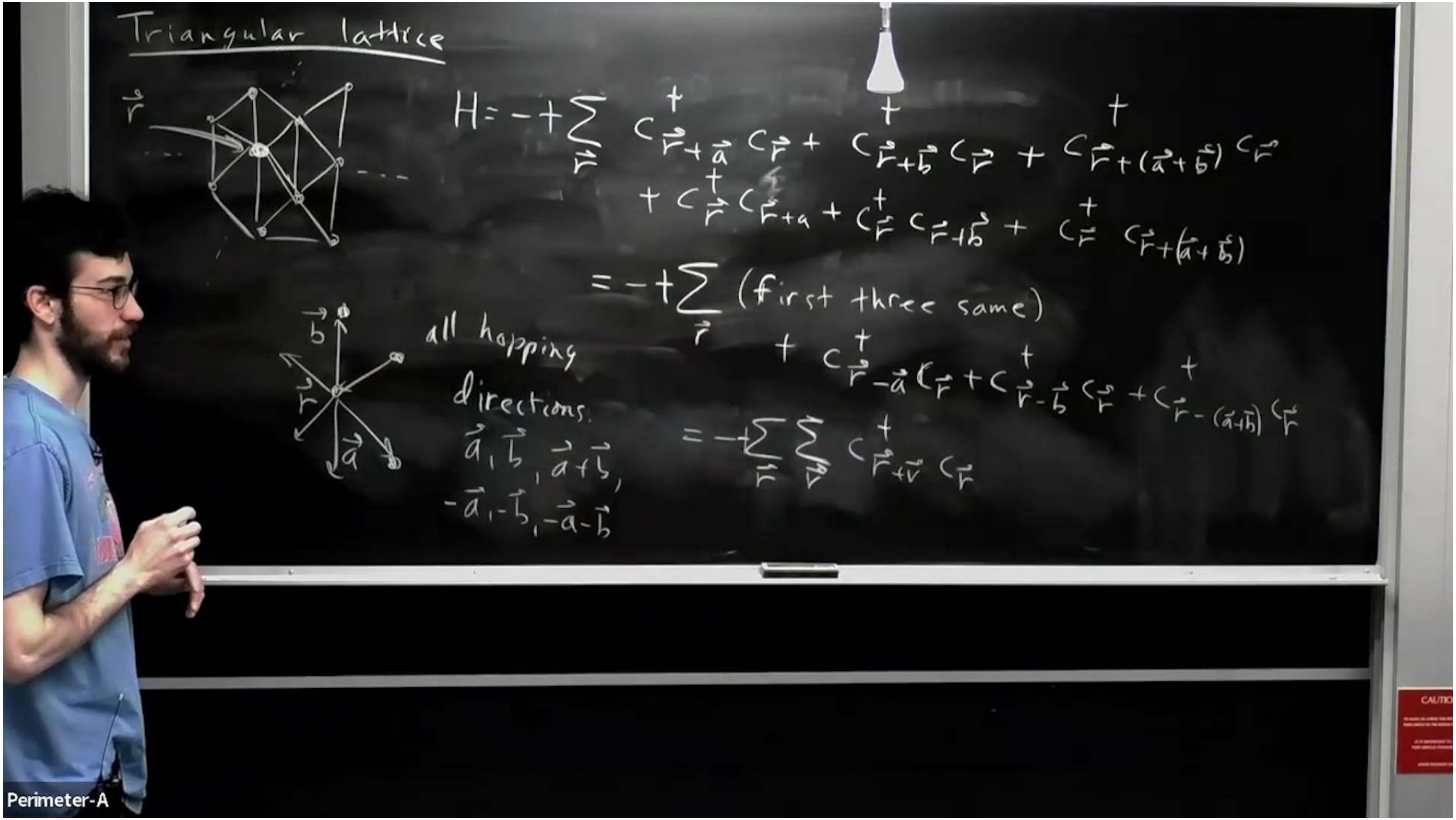
$$H = -t \sum_{\vec{r}} \left(\sum_{\vec{v}} e^{i\vec{k}\cdot\vec{v}} \right) c^{\dagger}_{\vec{k}} c_{\vec{r}} \qquad \{\vec{v}\} \text{ is all primitive lattice vectors, } \vec{a}, -\vec{a}, \vec{b}, -\vec{b}$$

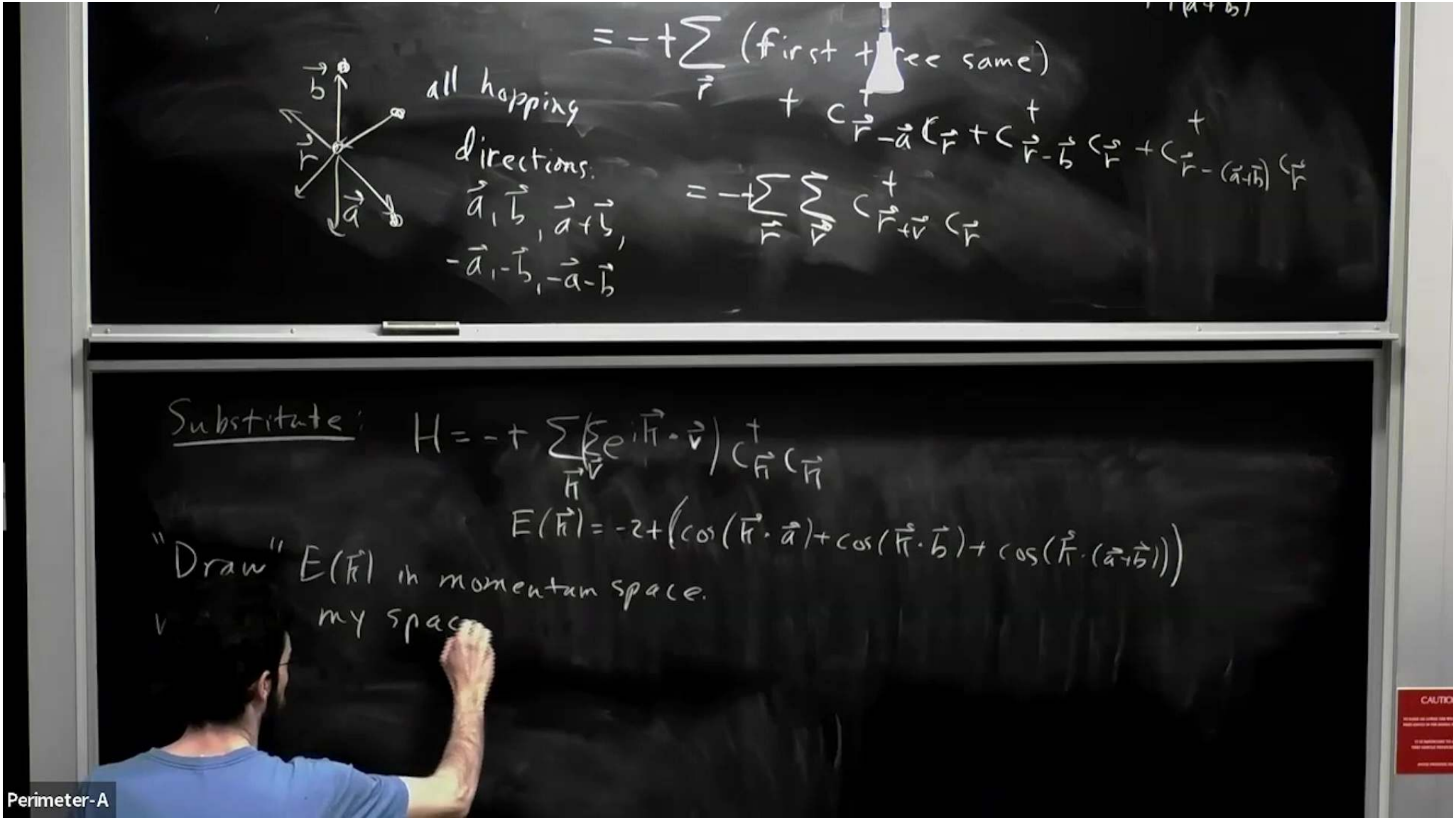
Nice formulation: manifestly has the symmetry of the lattice. ↖ all vectors to nearest sites

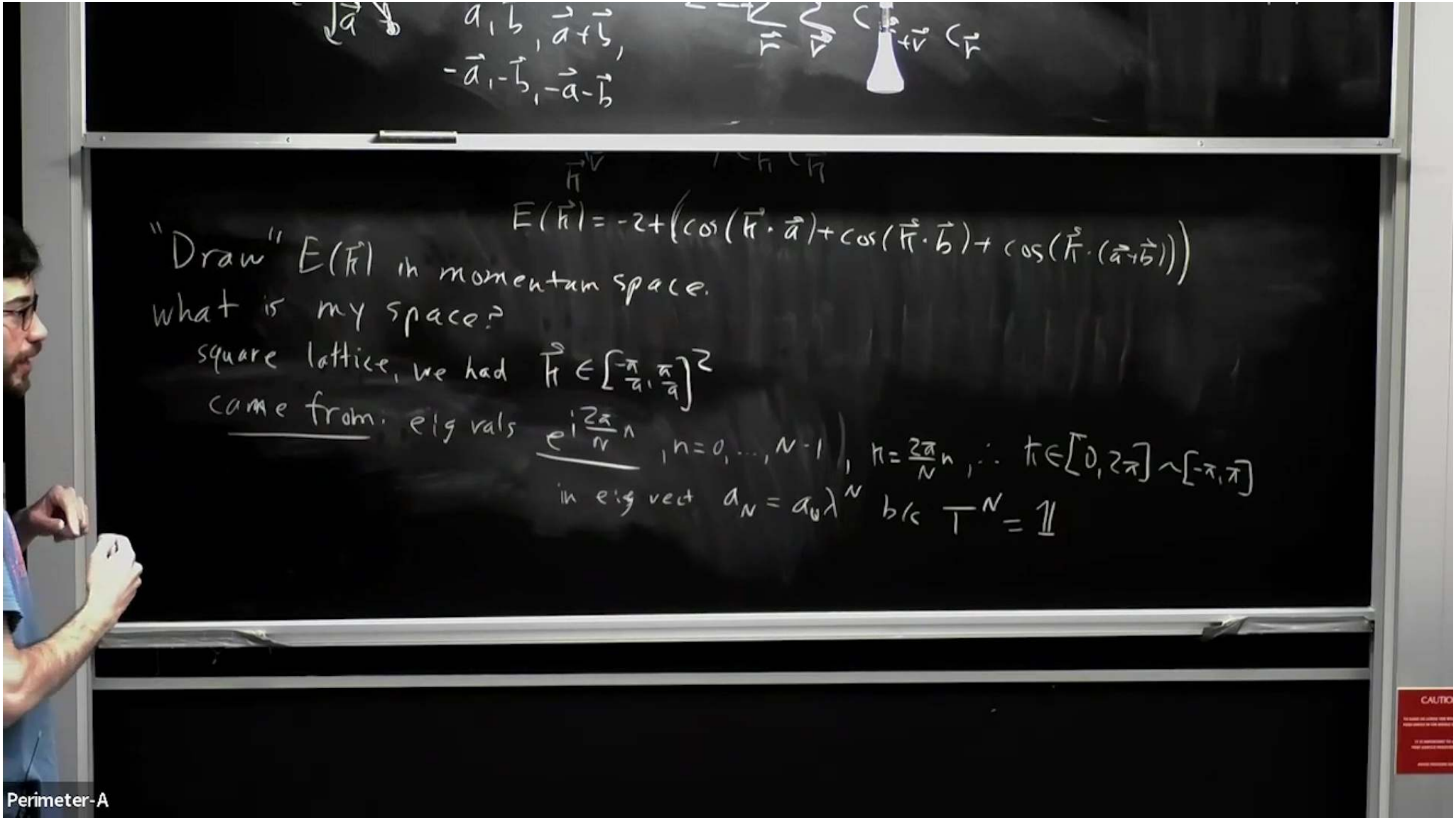
$\vec{a} \to \vec{b} \to -\vec{a} \to -\vec{b} \to \vec{a}$; $H \to H$, $E(\vec{k}) \to E(\vec{k})$

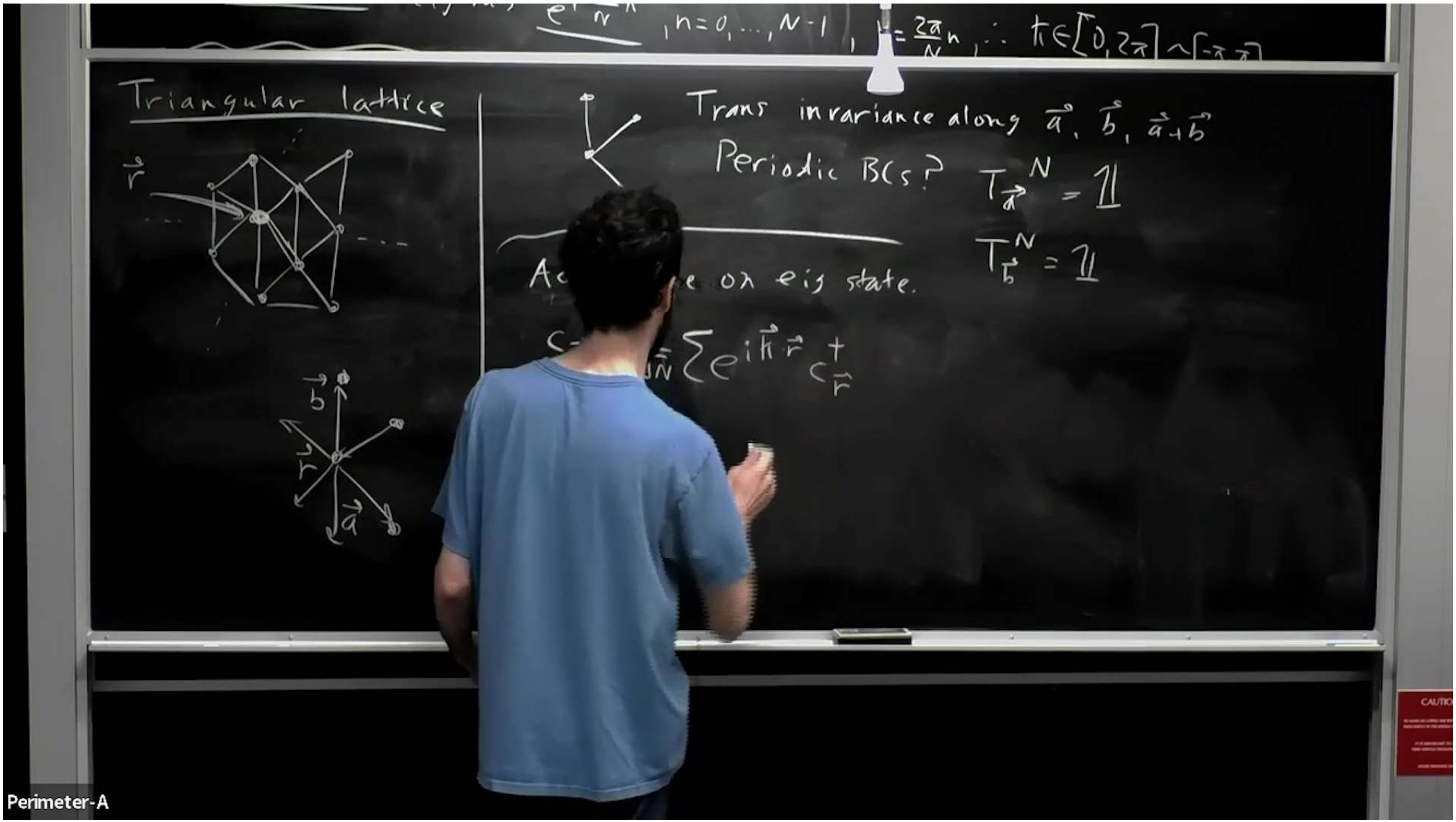
$$E(\vec{k}) = -2t\left(\cos(\vec{k}\cdot\vec{a}) + \cos(\vec{k}\cdot\vec{b})\right)$$
$$= -2t\left(\cos(k_x a) + \cos(k_y a)\right)$$

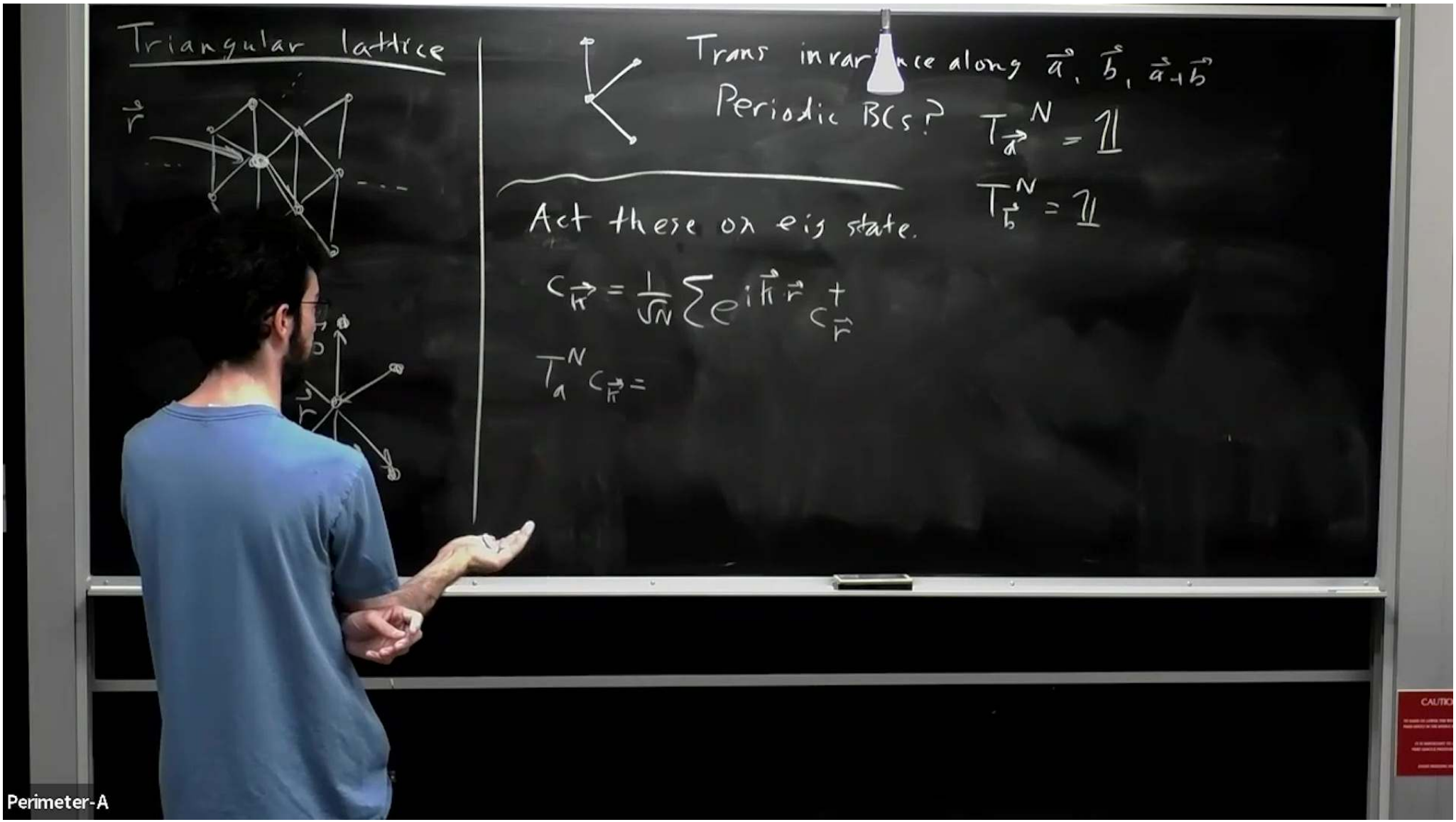
Perimeter-A

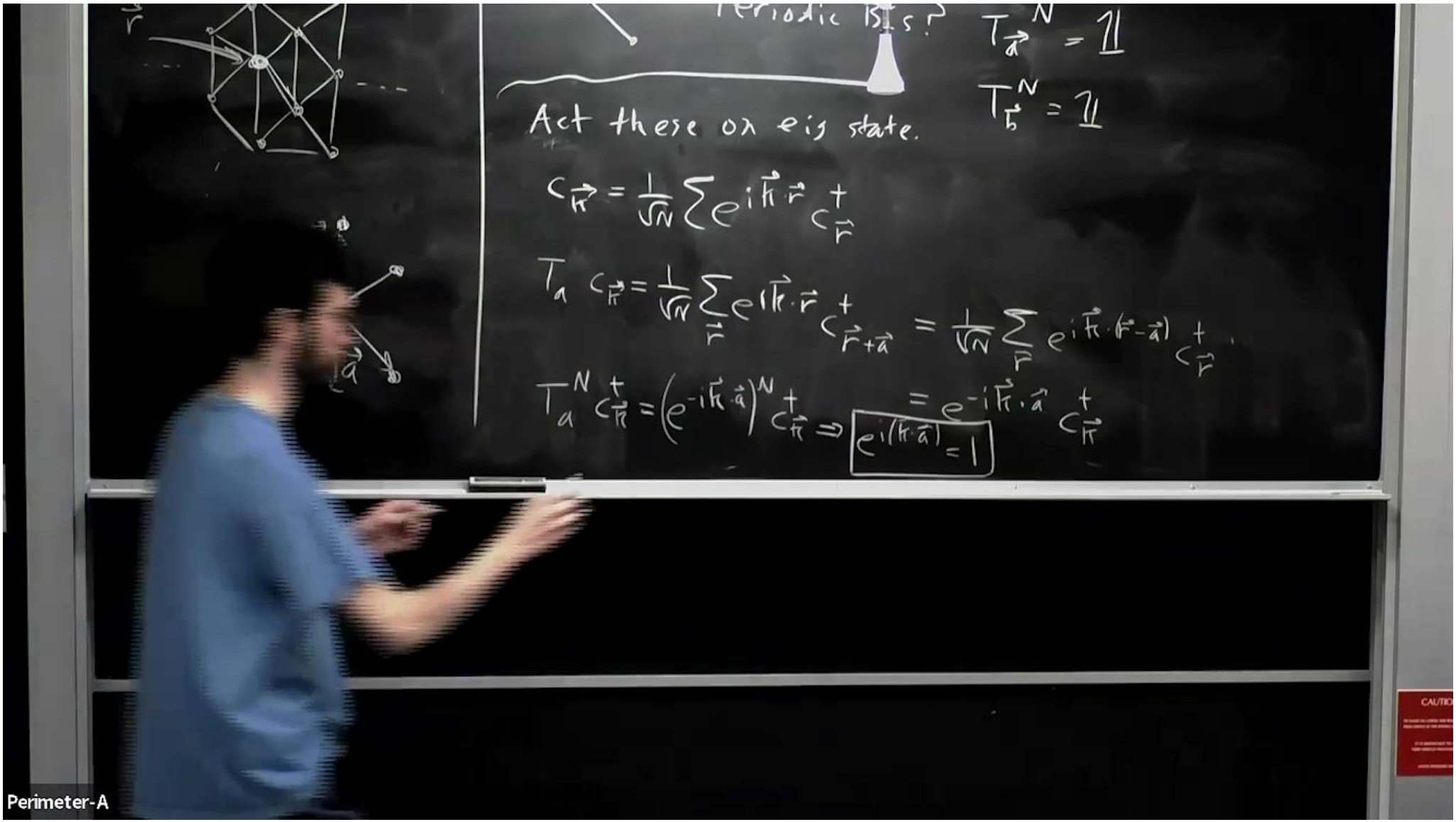

Triangular lattice
$H = -t \sum_{\vec{r}} c^\dagger_{\vec{r}+\vec{a}} c_{\vec{r}} + c^\dagger_{\vec{r}+\vec{b}} c_{\vec{r}} + c^\dagger_{\vec{r}+(\vec{a}+\vec{b})} c_{\vec{r}} + c^\dagger_{\vec{r}} c_{\vec{r}+a} + c^\dagger_{\vec{r}} c_{\vec{r}+b} + c^\dagger_{\vec{r}} c_{\vec{r}+(\vec{a}+\vec{b})}$
$= -t \sum_{\vec{r}}$ (first three same) $+ c^\dagger_{\vec{r}-\vec{a}} c_{\vec{r}} + c^\dagger_{\vec{r}-\vec{b}} c_{\vec{r}} + c^\dagger_{\vec{r}-(\vec{a}+\vec{b})} c_{\vec{r}}$
all hopping directions:
$\vec{a}, \vec{b}, \vec{a}+\vec{b}, -\vec{a}, -\vec{b}, -\vec{a}-\vec{b}$
$= -t \sum_{\vec{r}} \sum_{\vec{v}} c^\dagger_{\vec{r}+\vec{v}} c_{\vec{r}}$
Perimeter-A

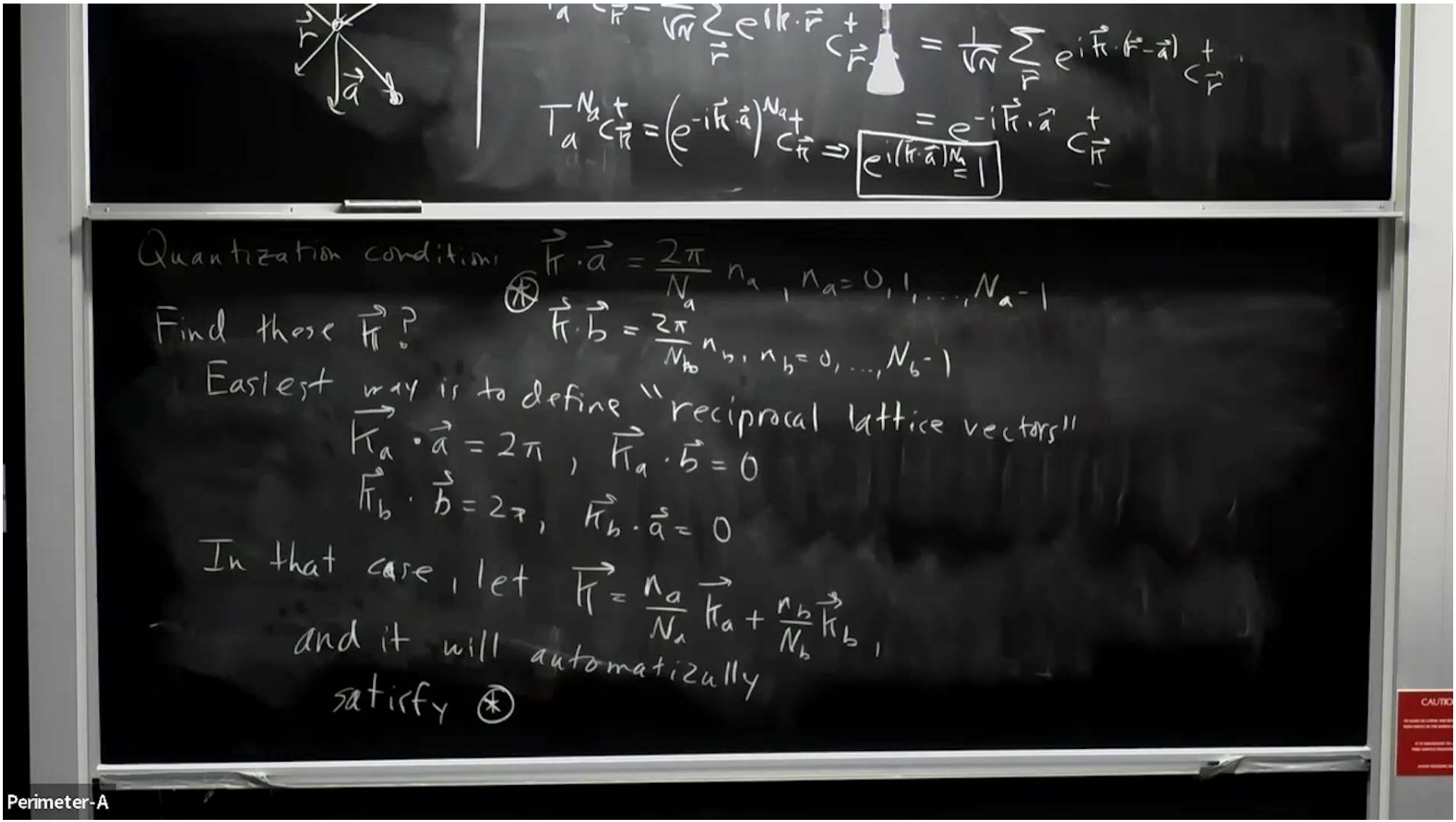

$= -t \sum_{\vec{r}}$ (first + three same)
$+ c^\dagger_{\vec{r}-\vec{a}} c_{\vec{r}} + c^\dagger_{\vec{r}-\vec{b}} c_{\vec{r}} + c^\dagger_{\vec{r}-(\vec{a}+\vec{b})} c_{\vec{r}}$
all hopping directions:
$\vec{a}, \vec{b}, \vec{a}+\vec{b}, -\vec{a}, -\vec{b}, -\vec{a}-\vec{b}$
$= -t \sum_{\vec{r}} \sum_{\vec{v}} c^\dagger_{\vec{r}+\vec{v}} c_{\vec{r}}$
Substitute: $H = -t \sum_{\vec{k}} \left(\sum_{\vec{v}} e^{i\vec{k}\cdot\vec{v}} \right) c^\dagger_{\vec{k}} c_{\vec{k}}$
$E(\vec{k}) = -2t \left(\cos(\vec{k}\cdot\vec{a}) + \cos(\vec{k}\cdot\vec{b}) + \cos(\vec{k}\cdot(\vec{a}+\vec{b})) \right)$
"Draw" $E(\vec{k})$ in momentum space.
my spac


$\vec{a}$, $\vec{b}$, $\vec{a}+\vec{b}$,
$-\vec{a}$, $-\vec{b}$, $-\vec{a}-\vec{b}$
$E(\vec{k}) = -2+\left(\cos(\vec{k}\cdot\vec{a})+\cos(\vec{k}\cdot\vec{b})+\cos(\vec{k}\cdot(\vec{a}+\vec{b}))\right)$
"Draw" $E(\vec{k})$ in momentum space.
what is my space?
square lattice, we had $\vec{k}\in\left[\frac{-\pi}{a},\frac{\pi}{a}\right]^2$
came from: eig vals $e^{i\frac{2\pi}{N}n}$, $n=0,\dots,N-1$, $k=\frac{2\pi}{N}n$, $\therefore k\in[0,2\pi]\sim[-\pi,\pi]$
in eig vect $a_N = a_0\lambda^N$ b/c $T^N = \mathbb{1}$
Perimeter-A


Triangular lattice
Trans invariance along $\vec{a}$, $\vec{b}$, $\vec{a}+\vec{b}$
Periodic BCs?
$T_{\vec{a}}^{N} = \mathbb{1}$
$T_{\vec{b}}^{N} = \mathbb{1}$
$\vec{r}$
$\vec{b}$
$\vec{a}$
Perimeter-A


Triangular lattice
Trans invariance along $\vec{a}$, $\vec{b}$, $\vec{a}+\vec{b}$
Periodic BCs?
$T_{\vec{a}}^N = \mathbb{1}$
$T_{\vec{b}}^N = \mathbb{1}$
Act these on eig state.
$c_{\vec{k}} = \frac{1}{\sqrt{N}} \sum e^{i\vec{k}\vec{r}} c_{\vec{r}}^\dagger$
$T_a^N c_{\vec{k}} =$
Perimeter-A


periodic BCs?
$T_{\vec{a}}^N = \mathbb{1}$
$T_{\vec{b}}^N = \mathbb{1}$
Act these on eig state.
$c_{\vec{k}} = \frac{1}{\sqrt{N}} \sum e^{i\vec{k}\cdot\vec{r}} c_{\vec{r}}^\dagger$
$T_a c_{\vec{k}} = \frac{1}{\sqrt{N}} \sum_{\vec{r}} e^{i\vec{k}\cdot\vec{r}} c_{\vec{r}+\vec{a}}^\dagger = \frac{1}{\sqrt{N}} \sum_{\vec{r}} e^{i\vec{k}\cdot(\vec{r}-\vec{a})} c_{\vec{r}}^\dagger$
$T_a^N c_{\vec{k}}^\dagger = \left(e^{-i\vec{k}\cdot\vec{a}}\right)^N c_{\vec{k}}^\dagger \Rightarrow e^{i(\vec{k}\cdot\vec{a})} = 1$
$= e^{-i\vec{k}\cdot\vec{a}} c_{\vec{k}}^\dagger$
Perimeter-A


$\vec{r}$
$\vec{a}$
$c^\dagger_{\vec{k}} = \frac{1}{\sqrt{N}} \sum_{\vec{r}} e^{i\vec{k}\cdot\vec{r}}\, c^\dagger_{\vec{r}} = \frac{1}{\sqrt{N}} \sum_{\vec{r}} e^{i\vec{k}\cdot(\vec{r}-\vec{a})}\, c^\dagger_{\vec{r}}$
$T_a^{N_a} c^\dagger_{\vec{k}} = \left(e^{-i\vec{k}\cdot\vec{a}}\right)^{N_a} c^\dagger_{\vec{k}} = e^{-i\vec{k}\cdot\vec{a}}\, c^\dagger_{\vec{k}} \Rightarrow \boxed{e^{i(\vec{k}\cdot\vec{a})N_a} = 1}$
Quantization condition: $\vec{k}\cdot\vec{a} = \frac{2\pi}{N_a} n_a$, $n_a = 0, 1, \ldots, N_a - 1$ ⊛
$\vec{k}\cdot\vec{b} = \frac{2\pi}{N_b} n_b$, $n_b = 0, \ldots, N_b - 1$
Find these $\vec{k}$?
Easiest way is to define "reciprocal lattice vectors"
$\vec{k}_a \cdot \vec{a} = 2\pi$, $\vec{k}_a \cdot \vec{b} = 0$
$\vec{k}_b \cdot \vec{b} = 2\pi$, $\vec{k}_b \cdot \vec{a} = 0$
In that case, let $\vec{k} = \frac{n_a}{N_a}\vec{k}_a + \frac{n_b}{N_b}\vec{k}_b$,
and it will automatically satisfy ⊛

$$T_a^{N_a} c_{\vec{r}}^\dagger = \left(e^{-i\vec{k}\cdot\vec{a}}\right)^{N_a} c_{\vec{r}}^\dagger \Rightarrow \boxed{e^{i(\vec{k}\cdot\vec{a})N_a = 1}}$$

Quantization condition: $\vec{k}\cdot\vec{a} = \frac{2\pi}{N_a} n_a$, $n_a = 0, 1, \ldots, N_a - 1$ ⊛

$\vec{k}\cdot\vec{b} = \frac{2\pi}{N_b} n_b$, $n_b = 0, \ldots, N_b - 1$

Find these $\vec{k}$?

Easiest way is to define "reciprocal lattice vectors"

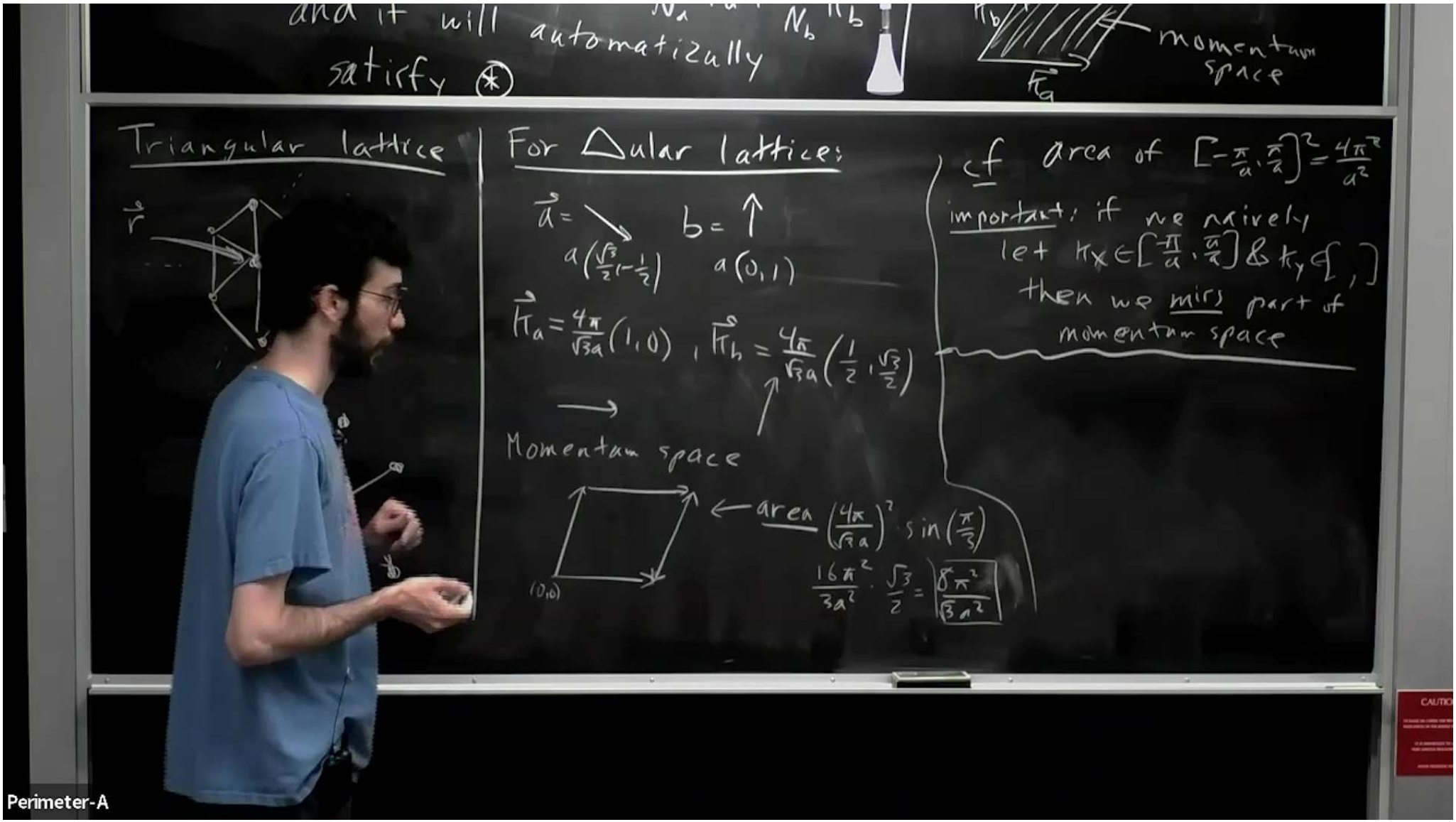
$\vec{k}_a \cdot \vec{a} = 2\pi$, $\vec{k}_a \cdot \vec{b} = 0$

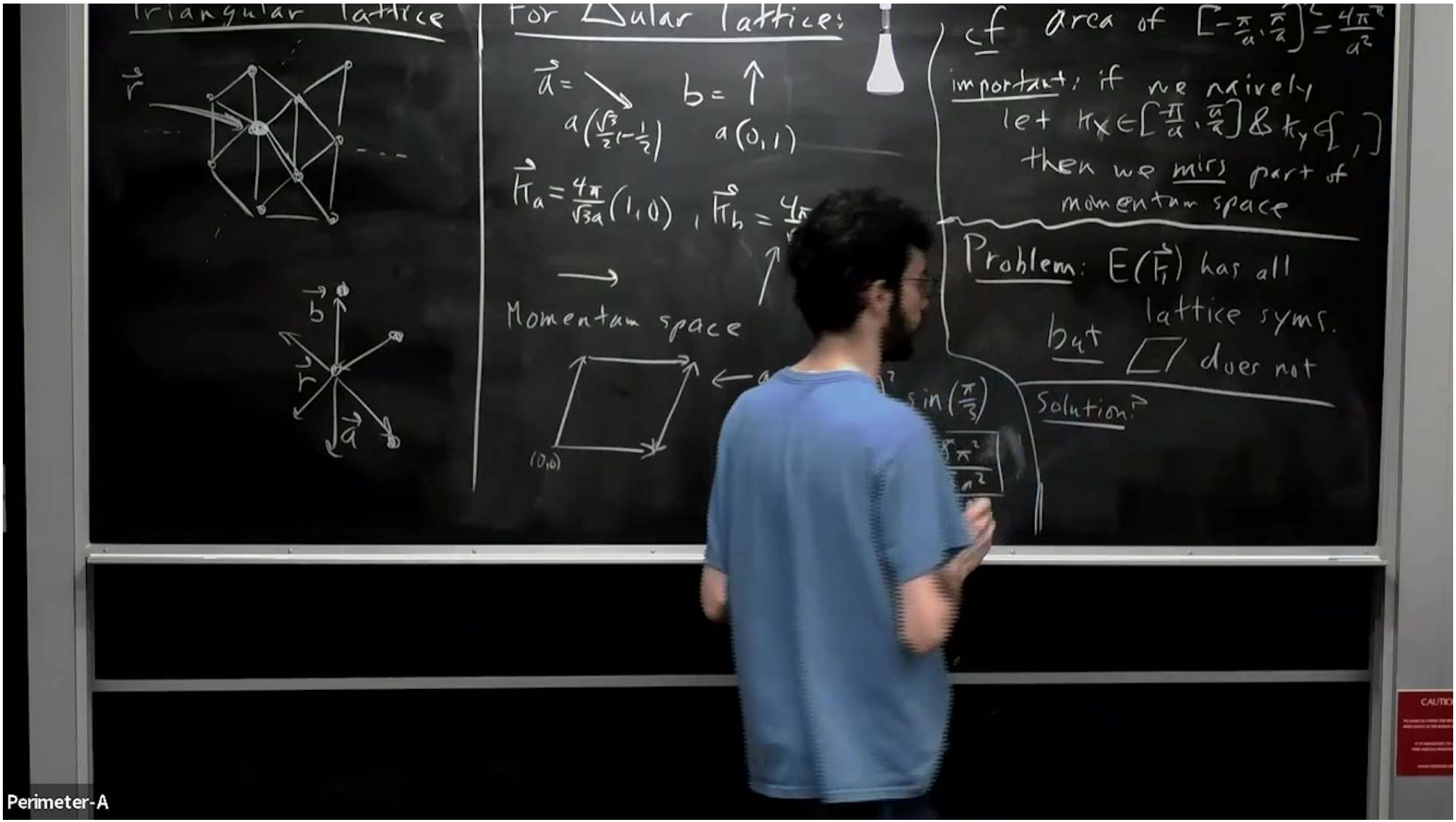
$\vec{k}_b \cdot \vec{b} = 2\pi$, $\vec{k}_b \cdot \vec{a} = 0$

In that case, let $\vec{k} = \frac{n_a}{N_a}\vec{k}_a + \frac{n_b}{N_b}\vec{k}_b$,

and it will automatically satisfy ⊛

The space of allowed $\vec{k}$ is the //ogram bounded by $\vec{k}_a$ and $\vec{k}_b$


and it will automatizully satisfy (*)
momentum space
Triangular lattice
For △ular lattice:
cf area of $[-\frac{\pi}{a},\frac{\pi}{a}]^2 = \frac{4\pi^2}{a^2}$
important: if we naively let $k_x \in [-\frac{\pi}{a},\frac{\pi}{a}]$ & $k_y \in [\ ,\ ]$ then we miss part of momentum space
$\vec{a} = a(\frac{\sqrt{3}}{2}, -\frac{1}{2})$
$b = a(0,1)$
$\vec{k}_a = \frac{4\pi}{\sqrt{3}a}(1,0)$, $\vec{k}_b = \frac{4\pi}{\sqrt{3}a}(\frac{1}{2},\frac{\sqrt{3}}{2})$
Momentum space
(0,0)
area $(\frac{4\pi}{\sqrt{3}a})^2 \cdot \sin(\frac{\pi}{3})$
$\frac{16\pi^2}{3a^2} \cdot \frac{\sqrt{3}}{2} = \frac{8\pi^2}{\sqrt{3}a^2}$
Perimeter-A


Triangular lattice
For △ular lattice:
cf area of
important: if we naively
Problem: E(
has all
lattice syms.
but
does not
Solution?
Momentum space
Perimeter-A

Solution? Use square lattice as an analogy

$\frac{16\pi^2}{3a^2}\frac{\sqrt{3}}{2} = \frac{8\pi^2}{\sqrt{3}a^2}$

Symmetrical

In our new language,

"reciprocal lattice"

$\vec{K}_a = \frac{2\pi}{a}(1,0)$

$\vec{K}_b = \frac{2\pi}{a}(0,1)$

"Brillouin zone" is the Wigner-Seitz cell of the recip latt.

Triangular

BZ

Claim

given $2a$ periodicity in $\vec{K}\cdot\vec{a}$ and $\vec{K}\cdot\vec{b}$, equivalent to $\parallel$ogram

Quick check: area $6\cdot\frac{4\pi^2}{3\sqrt{3}a^2} = \frac{8\pi^2}{\sqrt{3}a}$

Perimeter-A