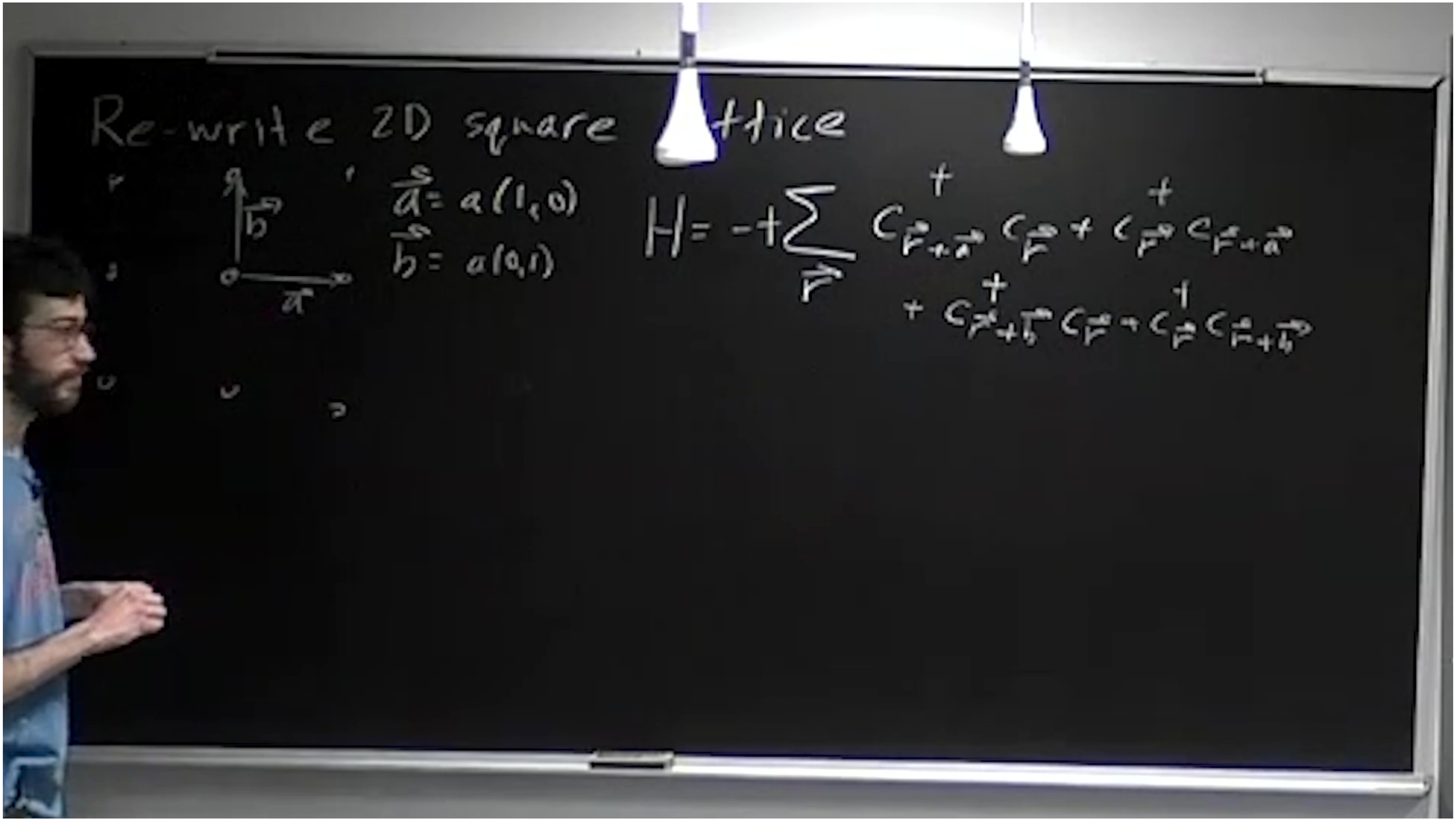
Title: PSI Lecture - Condensed Matter - Lecture 8

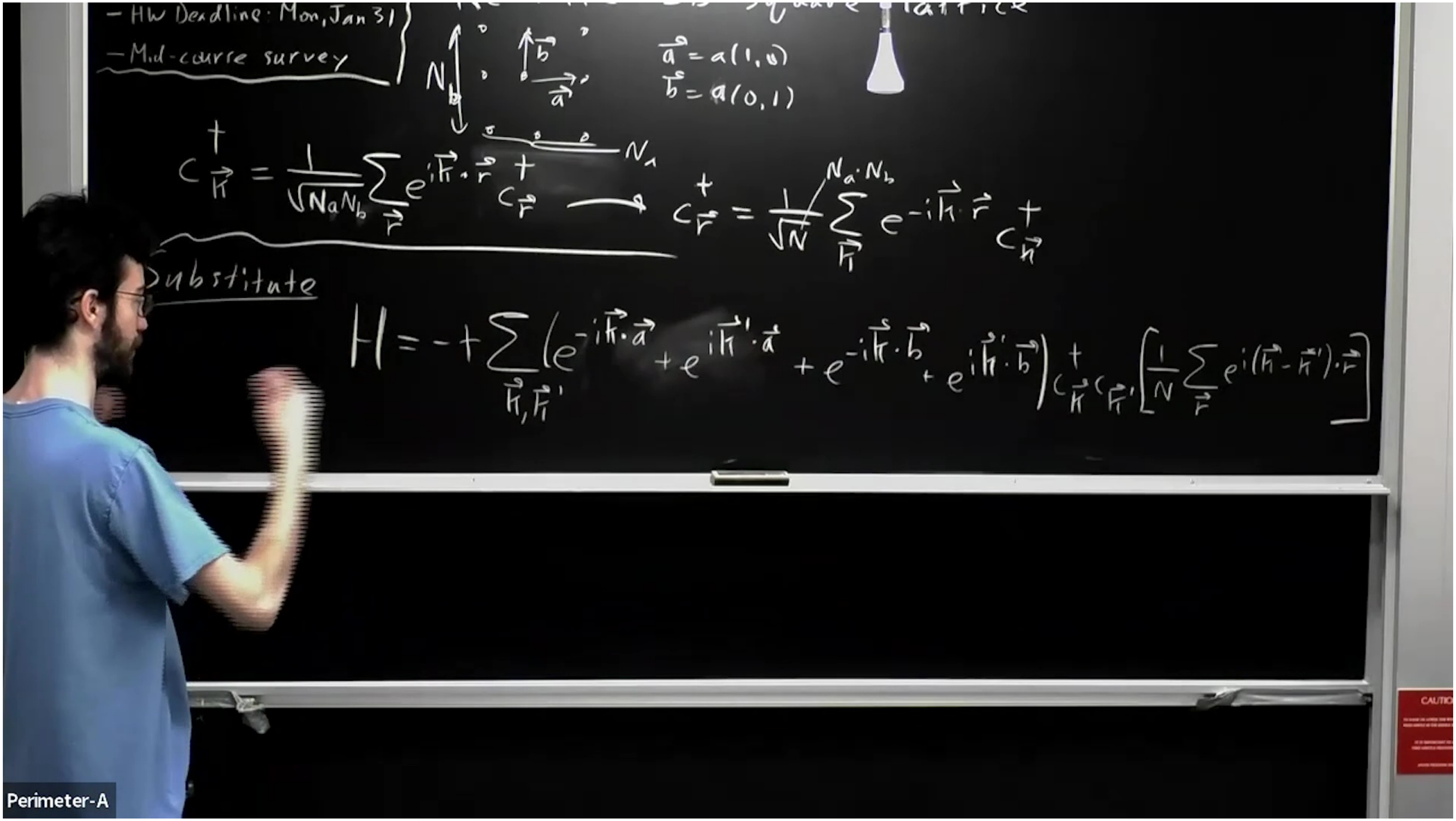
Speakers: Aaron Szasz

Collection: PSI Lecture - Condensed Matter

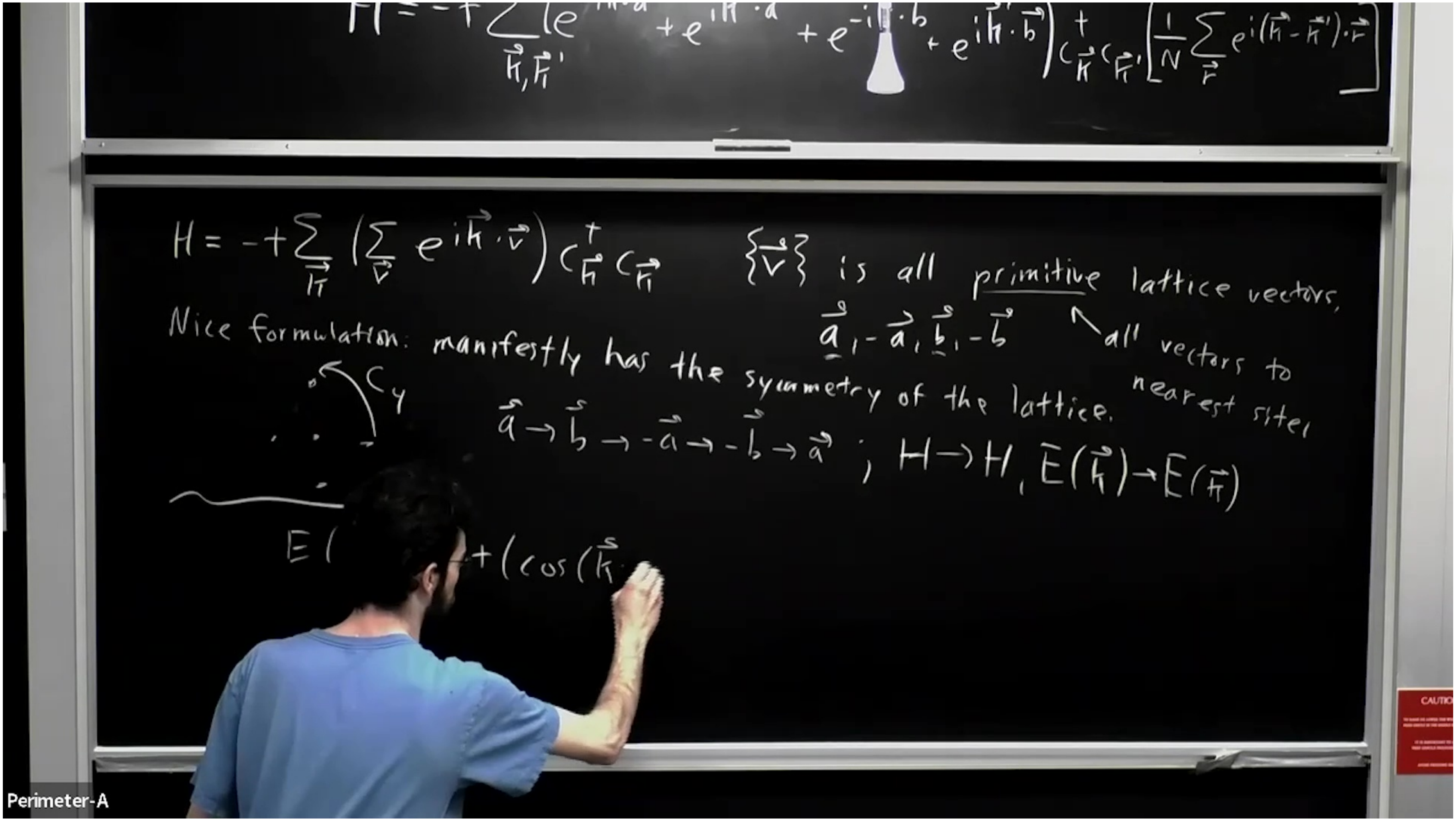
Date: January 25, 2022 - 11:30 AM

URL: https://pirsa.org/22010019


Re-write 2D square ttice
$\vec{a} = a(1,0)$
$\vec{b} = a(0,1)$
$H = -t \sum_{\vec{r}} c^{\dagger}_{\vec{r}+\vec{a}} c_{\vec{r}} + c^{\dagger}_{\vec{r}} c_{\vec{r}+\vec{a}}$
$+ c^{\dagger}_{\vec{r}+\vec{b}} c_{\vec{r}} + c^{\dagger}_{\vec{r}} c_{\vec{r}+\vec{b}}$



- HW Deadline: Mon, Jan 31
- Mid-course survey
N_b
$\vec{b}$
$\vec{a}$
$\vec{a} = a(1,0)$
$\vec{b} = a(0,1)$
N_a
$c^\dagger_{\vec{k}} = \frac{1}{\sqrt{N_a N_b}} \sum_{\vec{r}} e^{i\vec{k}\cdot\vec{r}} c^\dagger_{\vec{r}} \longrightarrow c^\dagger_{\vec{r}} = \frac{1}{\sqrt{N}} \sum_{\vec{k}}^{N_a \cdot N_b} e^{-i\vec{k}\cdot\vec{r}} c^\dagger_{\vec{k}}$
Substitute
$H = -t \sum_{\vec{k},\vec{k}'} \left(e^{-i\vec{k}\cdot\vec{a}} + e^{i\vec{k}'\cdot\vec{a}} + e^{-i\vec{k}\cdot\vec{b}} + e^{i\vec{k}'\cdot\vec{b}}\right) c^\dagger_{\vec{k}} c_{\vec{k}'} \left[\frac{1}{N} \sum_{\vec{r}} e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}\right]$
Perimeter-A



$H = -t \sum_{\vec{k}} \left(\sum_{\vec{v}} e^{i\vec{k}\cdot\vec{v}} \right) c^\dagger_{\vec{k}} c_{\vec{k}}$
$\{\vec{v}\}$ is all primitive lattice vectors,
$\vec{a}, -\vec{a}, \vec{b}, -\vec{b}$
all vectors to nearest sites
Nice formulation: manifestly has the symmetry of the lattice.
C_4
$\vec{a} \to \vec{b} \to -\vec{a} \to -\vec{b} \to \vec{a}$; $H \to H$, $E(\vec{k}) \to E(\vec{k})$
Perimeter-A

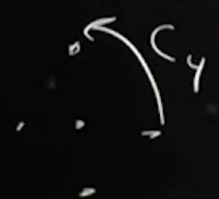
$$H = -t\sum_{\vec{k},\vec{k}'}\left(e^{\dots} + e^{i\vec{k}'\cdot\vec{a}} + e^{-i\vec{k}\cdot\vec{b}} + e^{i\vec{k}\cdot\vec{b}}\right) c^{\dagger}_{\vec{k}} c_{\vec{k}'}\left[\frac{1}{N}\sum_{\vec{r}} e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}\right]$$

$$H = -t\sum_{\vec{k}}\left(\sum_{\vec{v}} e^{i\vec{k}\cdot\vec{v}}\right) c^{\dagger}_{\vec{k}} c_{\vec{k}}$$

$\{\vec{v}\}$ is all primitive lattice vectors, $\vec{a}, -\vec{a}, \vec{b}, -\vec{b}$

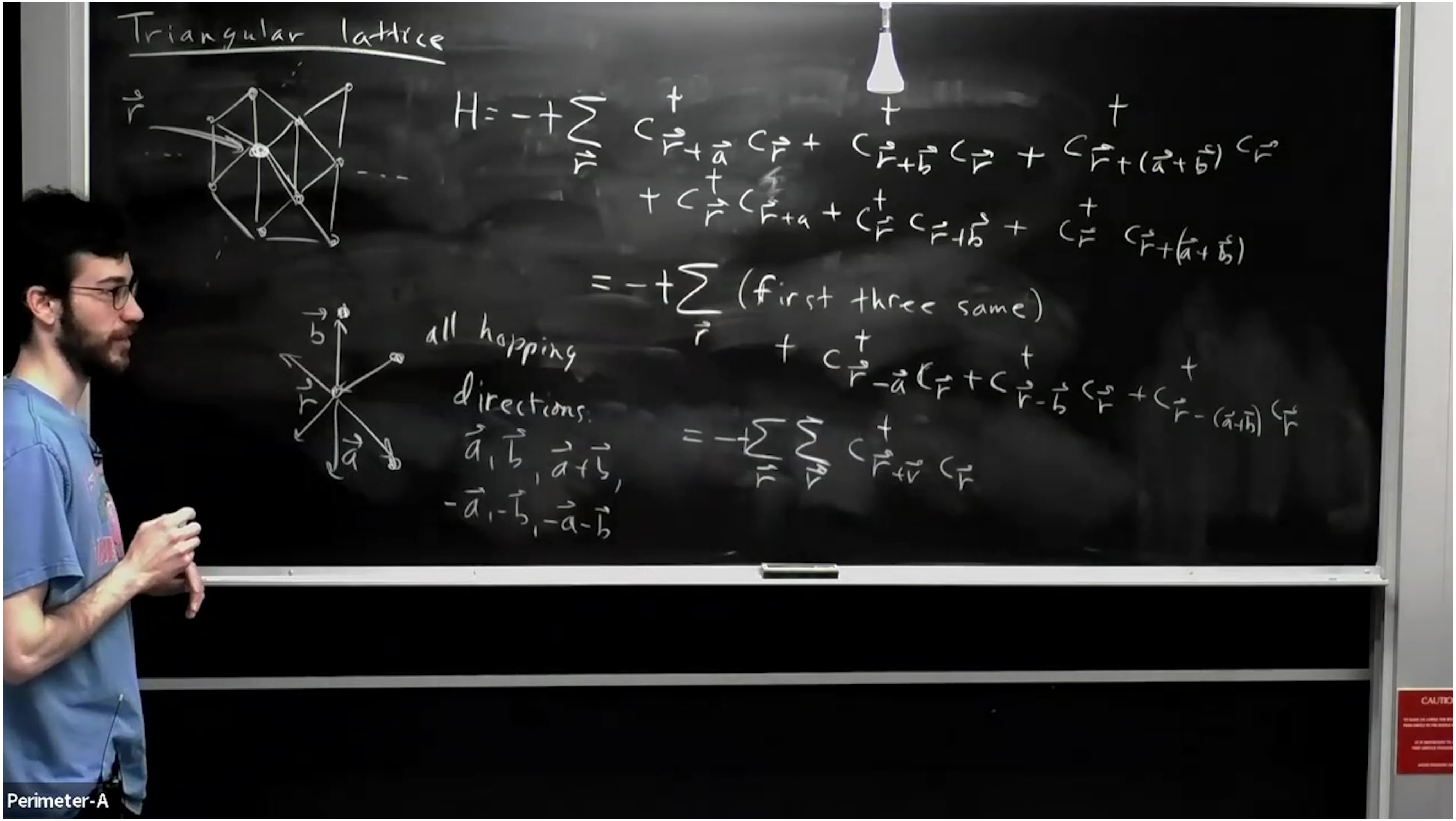
all vectors to nearest sites

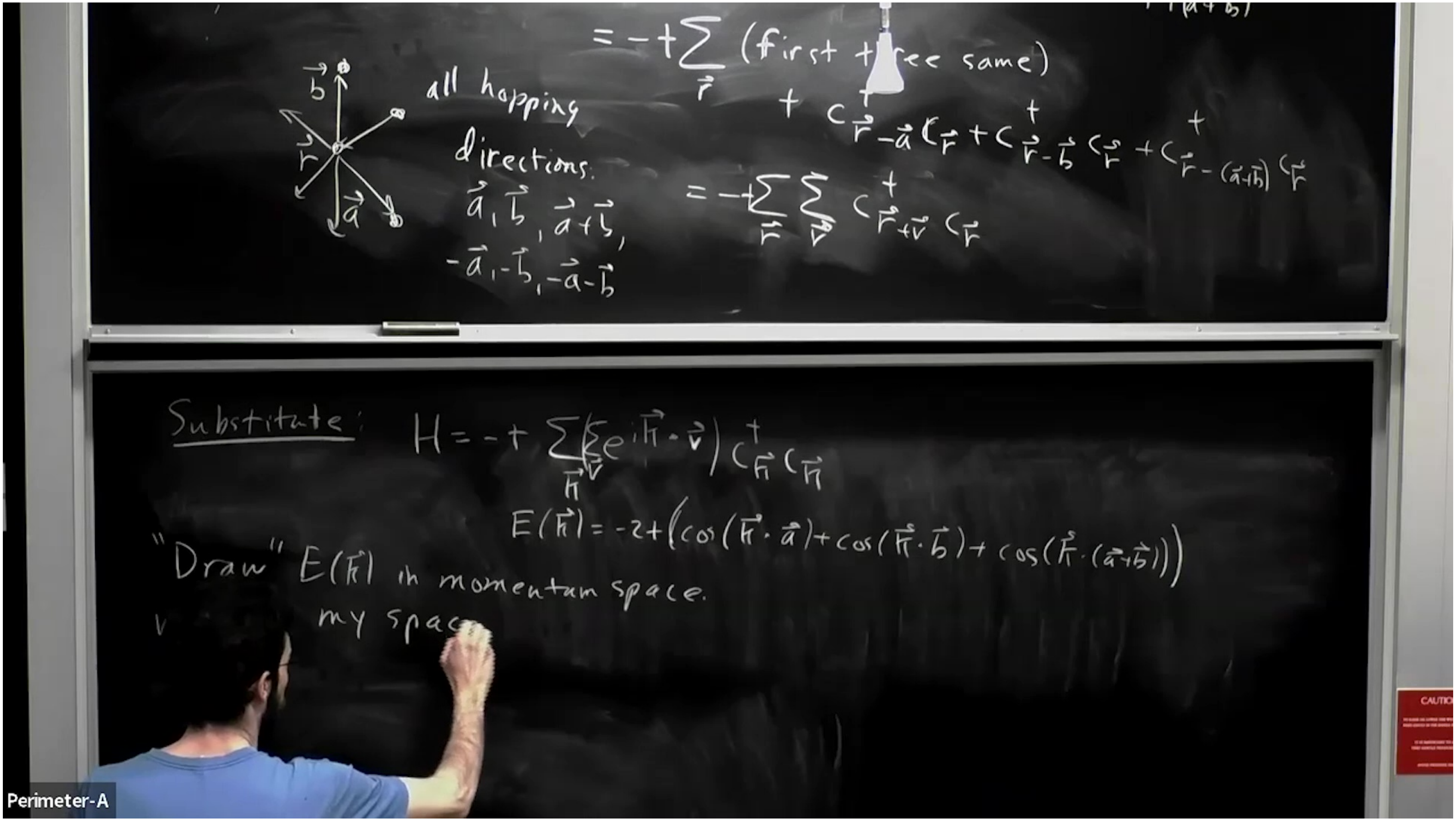
Nice formulation: manifestly has the symmetry of the lattice.



$\vec{a} \to \vec{b} \to -\vec{a} \to -\vec{b} \to \vec{a}$; $H \to H$, $E(\vec{k}) \to E(\vec{k})$

$$E(\vec{k}) = -2t\left(\cos(\vec{k}\cdot\vec{a}) + \cos(\vec{k}\cdot\vec{b})\right)$$
$$= -2t\left(\cos(k_x a) + \cos(k_y a)\right)$$


Triangular lattice
$\vec r$
$\vec b$
$\vec r$
$\vec a$
all hopping directions
$\vec a, \vec b, \vec a+\vec b,$
$-\vec a, -\vec b, -\vec a-\vec b$
$H = -t\sum_{\vec r} c^\dagger_{\vec r+\vec a} c_{\vec r} + c^\dagger_{\vec r+\vec b} c_{\vec r} + c^\dagger_{\vec r+(\vec a+\vec b)} c_{\vec r}$
$+ c^\dagger_{\vec r} c_{\vec r+a} + c^\dagger_{\vec r} c_{\vec r+\vec b} + c^\dagger_{\vec r} c_{\vec r+(\vec a+\vec b)}$
$= -t\sum_{\vec r}$ (first three same)
$+ c^\dagger_{\vec r-\vec a} c_{\vec r} + c^\dagger_{\vec r-\vec b} c_{\vec r} + c^\dagger_{\vec r-(\vec a+\vec b)} c_{\vec r}$
$= -t\sum_{\vec r}\sum_{\vec v} c^\dagger_{\vec r+\vec v} c_{\vec r}$
Perimeter-A


Perimeter-A
all hopping directions:
$\vec{a}, \vec{b}, \vec{a}+\vec{b}, -\vec{a}, -\vec{b}, -\vec{a}-\vec{b}$
$= -t \sum_{\vec{r}}$ (first three same)
$+ c^\dagger_{\vec{r}-\vec{a}} c_{\vec{r}} + c^\dagger_{\vec{r}-\vec{b}} c_{\vec{r}} + c^\dagger_{\vec{r}-(\vec{a}+\vec{b})} c_{\vec{r}}$
$= -t \sum_{\vec{r}} \sum_{\vec{v}} c^\dagger_{\vec{r}+\vec{v}} c_{\vec{r}}$
Substitute:
$H = -t \sum_{\vec{k}} \left(\sum_{\vec{v}} e^{i\vec{k}\cdot\vec{v}}\right) c^\dagger_{\vec{k}} c_{\vec{k}}$
$E(\vec{k}) = -2t\left(\cos(\vec{k}\cdot\vec{a}) + \cos(\vec{k}\cdot\vec{b}) + \cos(\vec{k}\cdot(\vec{a}+\vec{b}))\right)$
"Draw" $E(\vec{k})$ in momentum space.
my spac
CAUTION

$\vec{a},\ \vec{b},\ \vec{a}+\vec{b},$

$-\vec{a},\ -\vec{b},\ -\vec{a}-\vec{b}$

$\sum_{\vec{r}} \sum_{\vec{v}} c^{\dagger}_{\vec{r}+\vec{v}} c_{\vec{r}}$

$$E(\vec{k}) = -2t\left(\cos(\vec{k}\cdot\vec{a}) + \cos(\vec{k}\cdot\vec{b}) + \cos(\vec{k}\cdot(\vec{a}+\vec{b}))\right)$$

"Draw" $E(\vec{k})$ in momentum space.

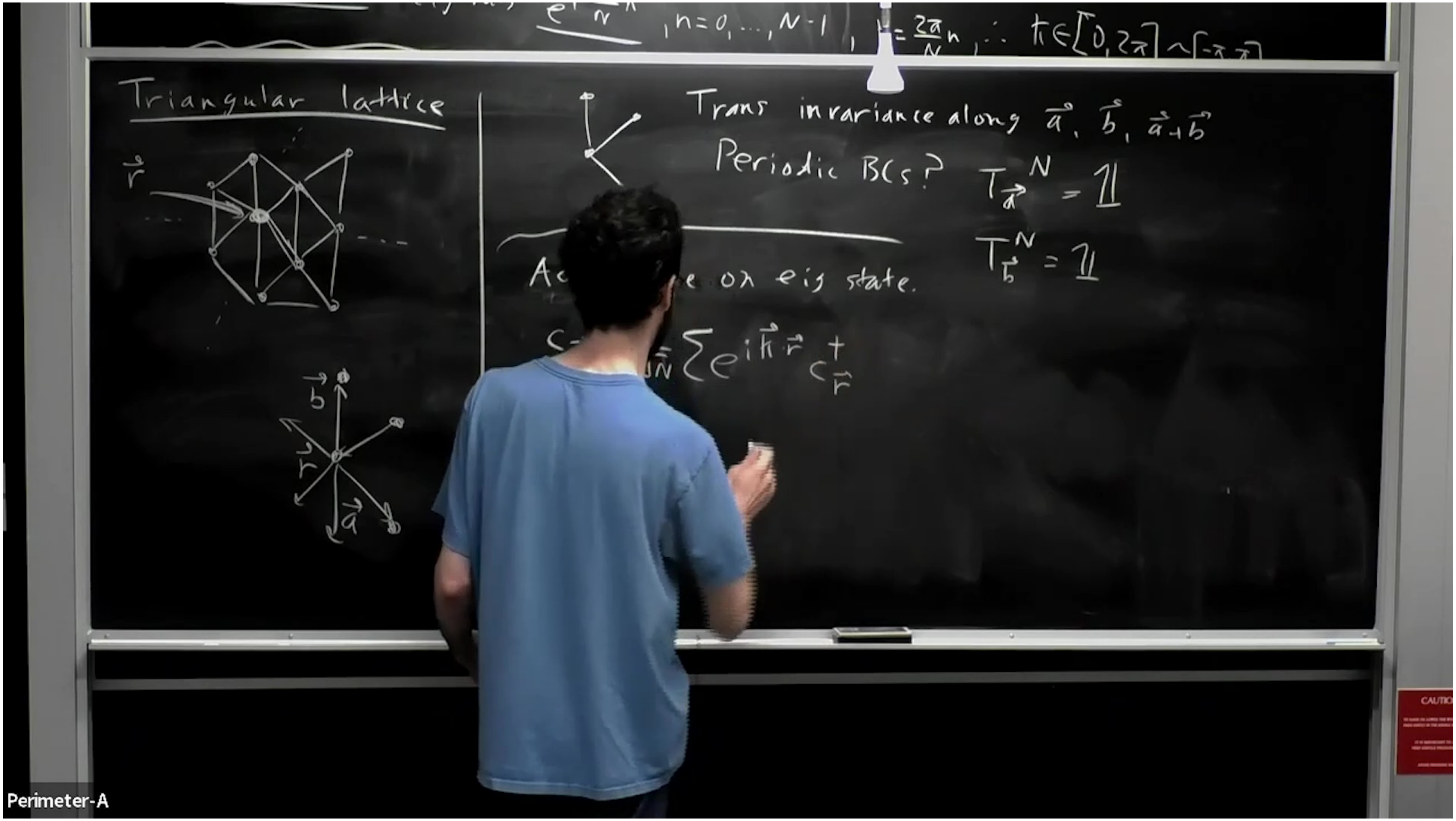
what is my space?

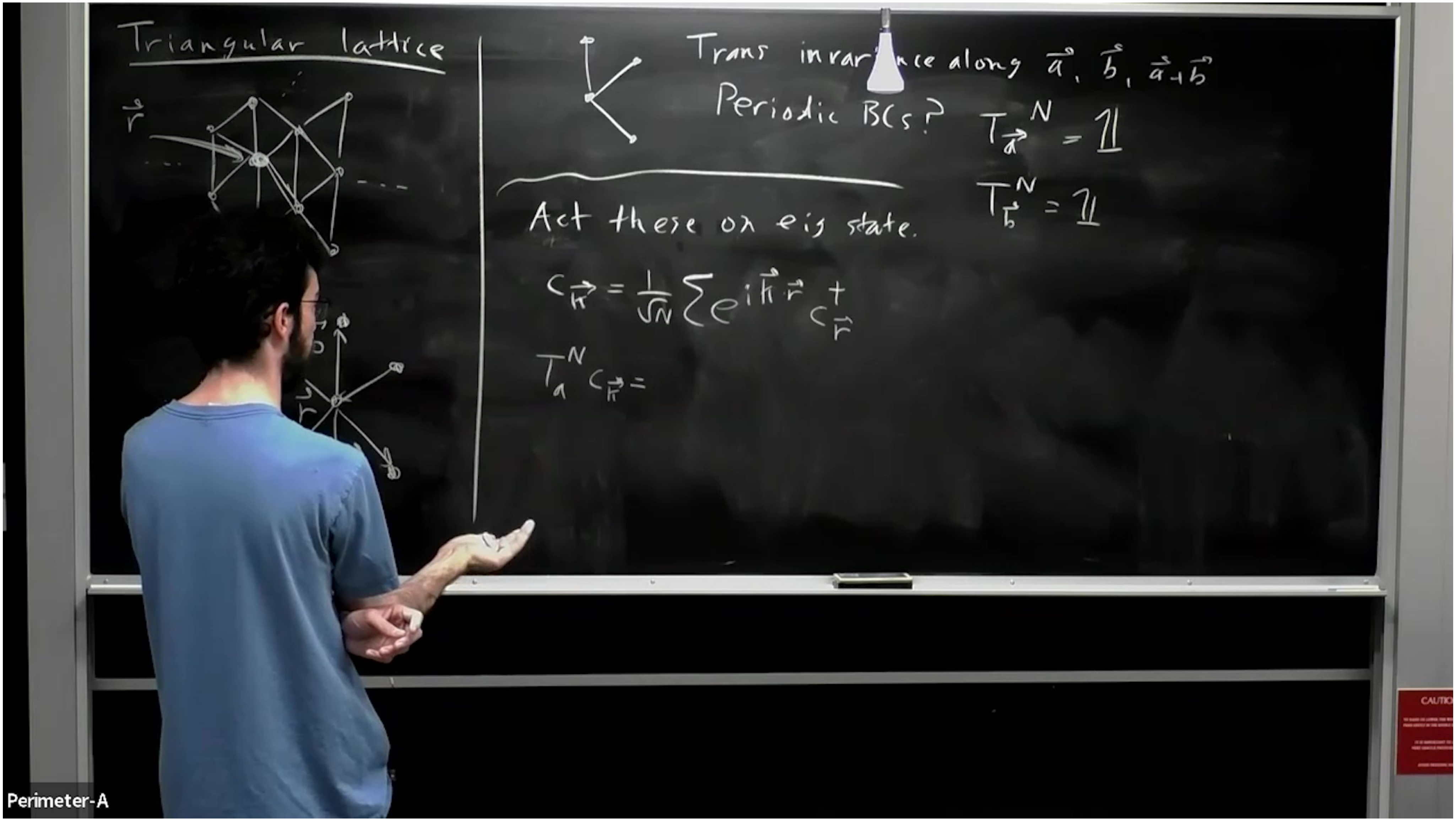
square lattice, we had $\vec{k} \in \left[-\frac{\pi}{a}, \frac{\pi}{a}\right]^2$

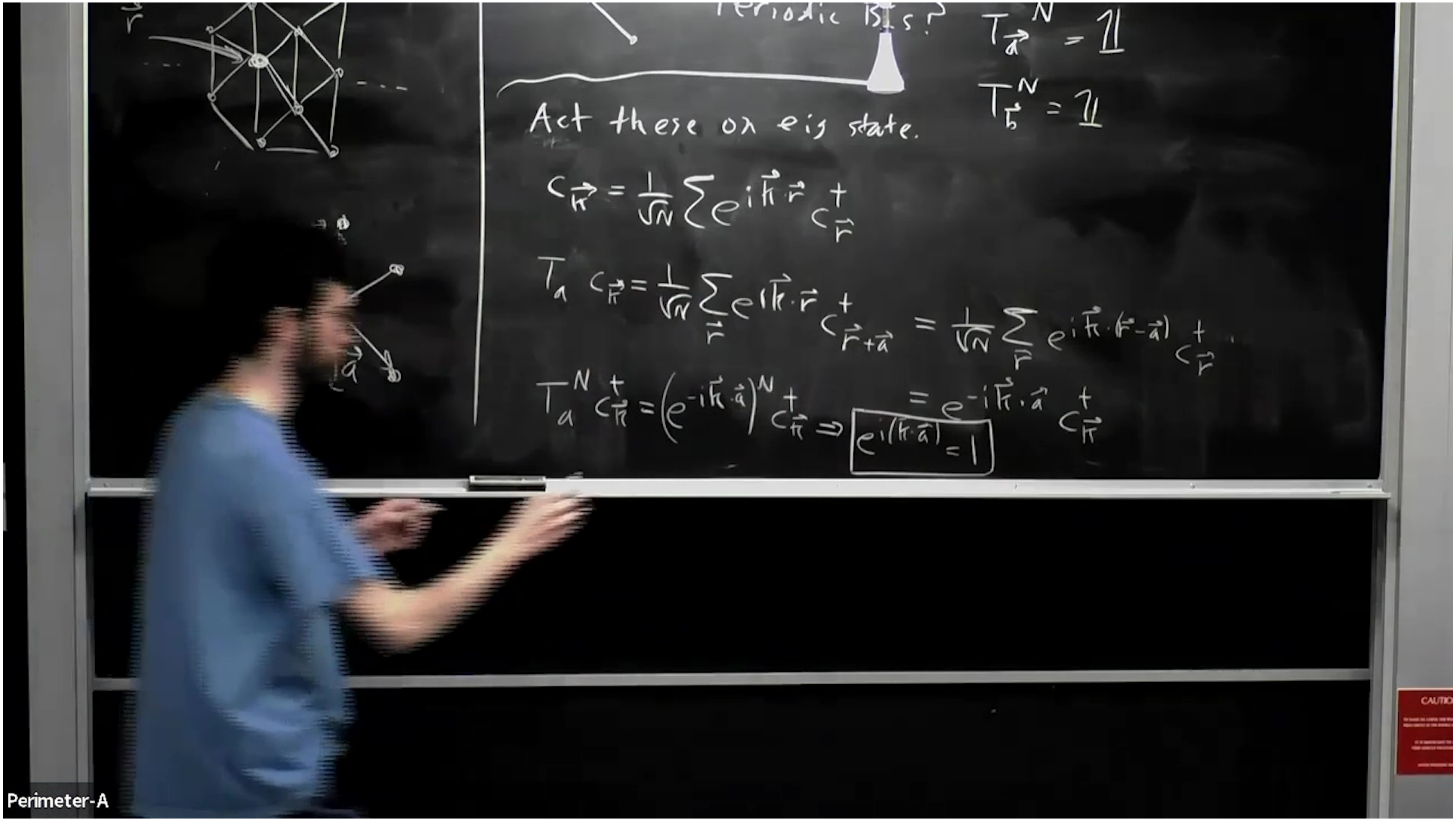
came from: eig vals $e^{i\frac{2\pi}{N}n}$, $n = 0, \dots, N-1$, $k = \frac{2\pi}{N}n$, $\therefore$ $k \in [0, 2\pi] \sim [-\pi, \pi]$

in eig vect $a_N = a_0 \lambda^N$ b/c $T^N = \mathbb{1}$

Perimeter-A


n=0,...,N-1
Triangular lattice
Trans invariance along a, b, a+b
Periodic BCs?
T_a^N = 1
T_b^N = 1
e on eig state.
Perimeter-A


Triangular lattice
$\vec{r}$
Trans invariance along $\vec{a}$, $\vec{b}$, $\vec{a}+\vec{b}$
Periodic BCs?
$T_{\vec{a}}^{N} = \mathbb{1}$
$T_{\vec{b}}^{N} = \mathbb{1}$
Act these on eig state.
$c_{\vec{k}} = \frac{1}{\sqrt{N}} \sum e^{i\vec{k}\cdot\vec{r}} c^{\dagger}_{\vec{r}}$
$T_{a}^{N} c_{\vec{k}} =$
Perimeter-A


Periodic B.C.s?
$T_{\vec a}^N = \mathbb{1}$
$T_{\vec b}^N = \mathbb{1}$
Act these on eig state.
$c_{\vec k} = \frac{1}{\sqrt N}\sum e^{i\vec k\cdot\vec r} c^\dagger_{\vec r}$
$T_a c_{\vec k} = \frac{1}{\sqrt N}\sum_{\vec r} e^{i\vec k\cdot\vec r} c^\dagger_{\vec r+\vec a} = \frac{1}{\sqrt N}\sum_{\vec r} e^{i\vec k\cdot(\vec r-\vec a)} c^\dagger_{\vec r}$
$= e^{-i\vec k\cdot\vec a} c^\dagger_{\vec k}$
$T_a^N c^\dagger_{\vec k} = \left(e^{-i\vec k\cdot\vec a}\right)^N c^\dagger_{\vec k} \Rightarrow \boxed{e^{i(\vec k\cdot\vec a)} = 1}$
Perimeter-A

$$\frac{1}{\sqrt N}\sum_{\vec r} e^{i\vec k\cdot\vec r}\, c^\dagger_{\vec r+\vec a} = \frac{1}{\sqrt N}\sum_{\vec r} e^{i\vec k\cdot(\vec r-\vec a)}\, c^\dagger_{\vec r}$$

$$= e^{-i\vec k\cdot\vec a}\, c^\dagger_{\vec k}$$

$$T_a^{N_a} c^\dagger_{\vec k} = \left(e^{-i\vec k\cdot\vec a}\right)^{N_a} c^\dagger_{\vec k} \Rightarrow \boxed{e^{i(\vec k\cdot\vec a)N_a} = 1}$$

Quantization condition: $\vec k\cdot\vec a = \frac{2\pi}{N_a} n_a$, $n_a = 0, 1, \ldots, N_a - 1$

(*) $\vec k\cdot\vec b = \frac{2\pi}{N_b} n_b$, $n_b = 0, \ldots, N_b - 1$

Find those $\vec k$?

Easiest way is to define "reciprocal lattice vectors"

$$\vec K_a\cdot\vec a = 2\pi, \quad \vec K_a\cdot\vec b = 0$$

$$\vec K_b\cdot\vec b = 2\pi, \quad \vec K_b\cdot\vec a = 0$$

In that case, let $\vec k = \frac{n_a}{N_a}\vec K_a + \frac{n_b}{N_b}\vec K_b$,

and it will automatizally satisfy (*)

$$T_a c^\dagger_{\vec k} = \frac{1}{\sqrt N}\sum_{\vec r} e^{i\vec k\cdot\vec r}\, c^\dagger_{\vec r+\vec a} = \frac{1}{\sqrt N}\sum_{\vec r} e^{i\vec k\cdot(\vec r-\vec a)}\, c^\dagger_{\vec r} = e^{-i\vec k\cdot\vec a}\, c^\dagger_{\vec k}$$

$$T_a^{N_a} c^\dagger_{\vec k} = \left(e^{-i\vec k\cdot\vec a}\right)^{N_a} c^\dagger_{\vec k} \Rightarrow \boxed{e^{i(\vec k\cdot\vec a)N_a} = 1}$$

Quantization condition: $(*)$ $\vec k\cdot\vec a = \frac{2\pi}{N_a} n_a$, $n_a = 0, 1, \ldots, N_a - 1$

Find these $\vec k$? $\quad \vec k\cdot\vec b = \frac{2\pi}{N_b} n_b$, $n_b = 0, \ldots, N_b - 1$

Easiest way is to define "reciprocal lattice vectors"

$$\vec K_a\cdot\vec a = 2\pi, \quad \vec K_a\cdot\vec b = 0$$
$$\vec K_b\cdot\vec b = 2\pi, \quad \vec K_b\cdot\vec a = 0$$

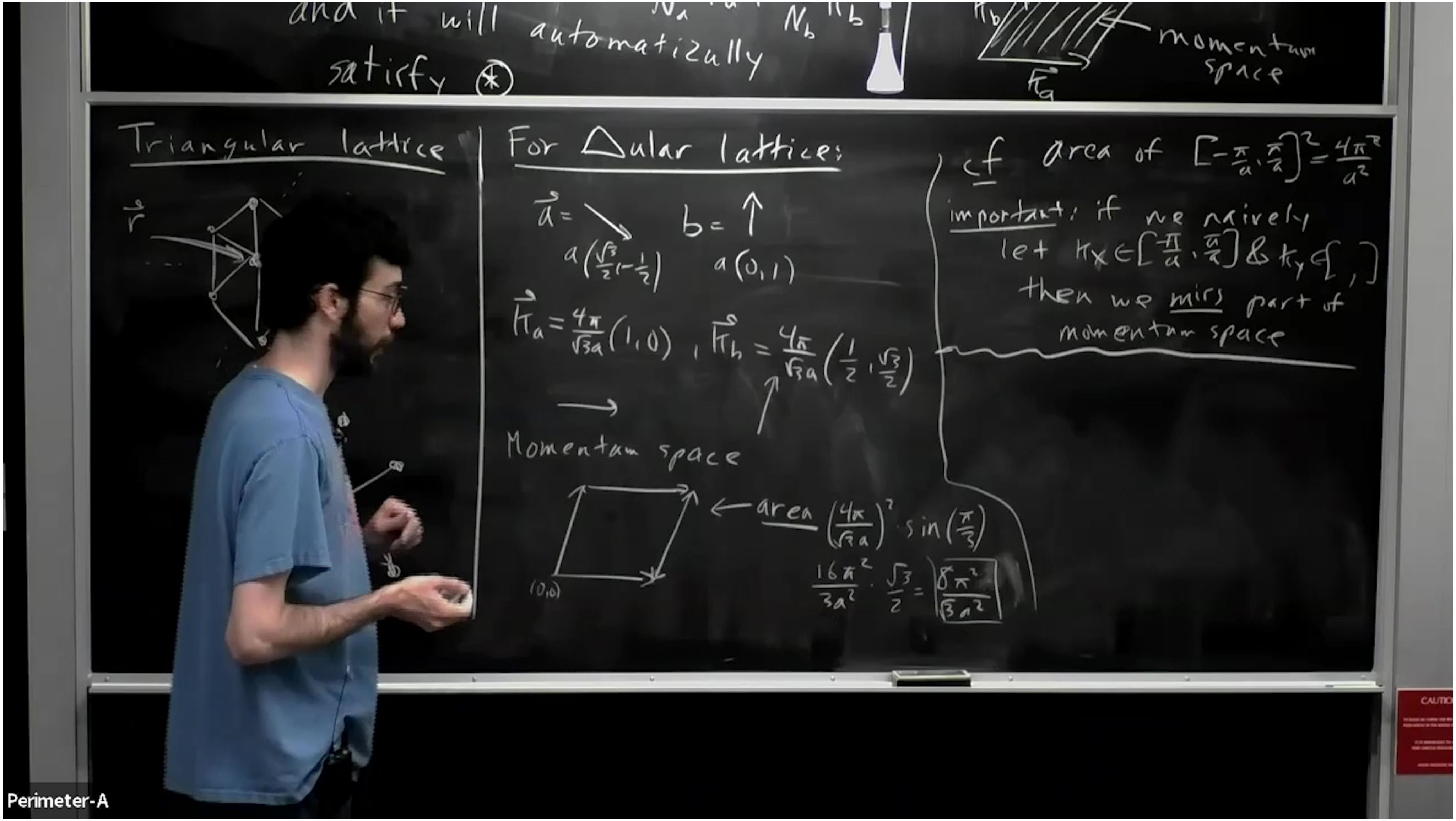
In that case, let $\vec k = \frac{n_a}{N_a}\vec K_a + \frac{n_b}{N_b}\vec K_b$,

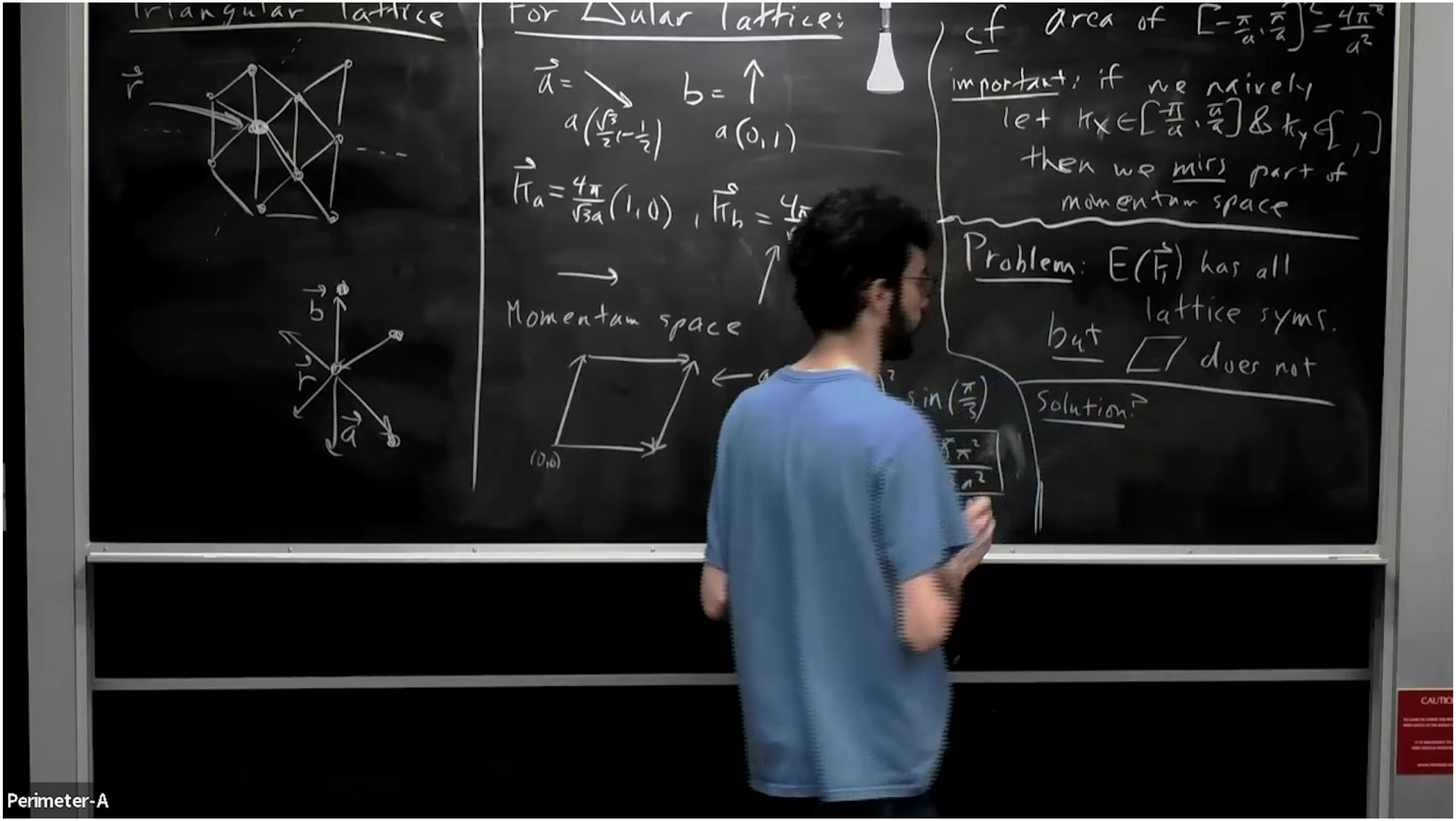
and it will automatizully satisfy $(*)$

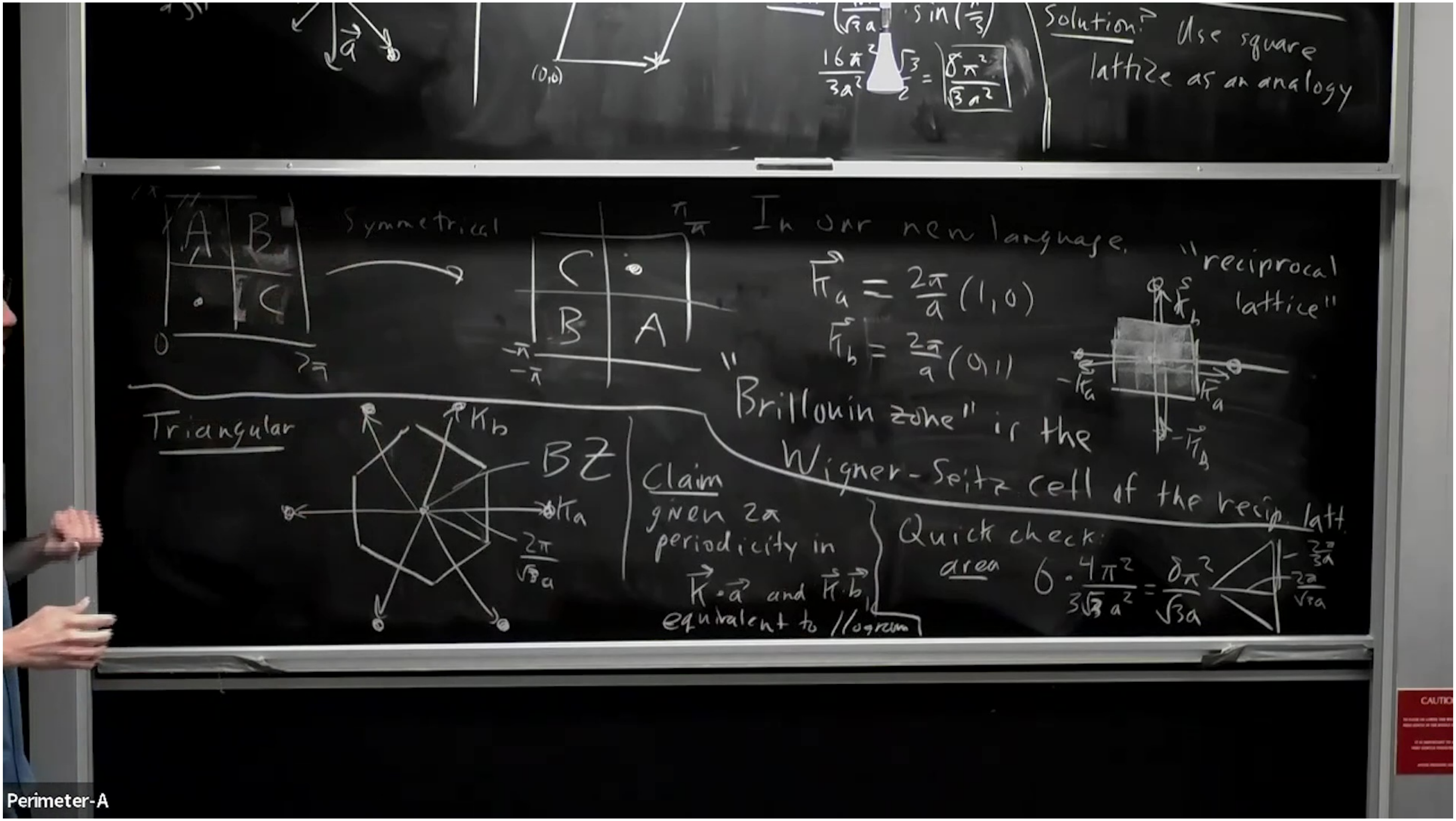
The space of allowed $\vec k$ is the //ogram bounded by $\vec K_a$ and $\vec K_b$

momentum space

Perimeter-A


and it will automatizully
satisfy ⊛
momentum space
Triangular lattice
For Δular lattice:
a = ↘   b = ↑
a(√3/2, −1/2)   a(0,1)
K_a = 4π/(√3a) (1,0), K_b = 4π/(√3a) (1/2, √3/2)
Momentum space
(0,0)
← area (4π/(√3a))² · sin(π/3)
16π²/(3a²) · √3/2 = 8π²/(√3a²)
cf area of [−π/a, π/a]² = 4π²/a²
important: if we naively
let k_x ∈ [−π/a, π/a] & k_y ∈ [,]
then we miss part of
momentum space
Perimeter-A


Triangular lattice
For △ular lattice:
$\vec{a} = \searrow$ $b = \uparrow$
$a(\frac{\sqrt{3}}{2}, -\frac{1}{2})$ $a(0,1)$
$\vec{k}_a = \frac{4\pi}{\sqrt{3}a}(1,0)$, $\vec{k}_b = \frac{4\pi}{}$
Momentum space
(0,0)
$\sin(\frac{\pi}{3})$
cf area of $[-\frac{\pi}{a}, \frac{\pi}{a}]^2 = \frac{4\pi^2}{a^2}$
important: if we naively let $k_x \in [\frac{-\pi}{a}, \frac{\pi}{a}]$ & $k_y \in [\ ,\ ]$ then we miss part of momentum space
Problem: $E(\vec{k})$ has all lattice syms. but ▱ does not
Solution?
Perimeter-A
CAUTION


Solution? Use square lattice as an analogy
Symmetrical
In our new language,
"reciprocal lattice"
"Brillouin zone" is the Wigner-Seitz cell of the recip. latt.
Triangular
BZ
Claim
given 2π periodicity in K·a and K·b, equivalent to //ogram
Quick check:
area
Perimeter-A