Title: Quantum Scientific Computation

Speakers: Jin-Peng Liu

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Abstract: Quantum computers are expected to dramatically outperform classical computers for certain computational problems. While there has been extensive previous work for linear dynamics and discrete models, for more complex realistic problems arising in physical and social science, engineering, and medicine, the capability of quantum computing is far from well understood. One fundamental challenge is the substantial difference between the linear dynamics of a system of qubits and real-world systems with continuum, stochastic, and nonlinear behaviors. Utilizing advanced linear algebra techniques and nonlinear analysis, I attempt to build a bridge between classical and quantum mechanics, understand and optimize the power of quantum computation, and discover new quantum speedups over classical algorithms with provable guarantees. In this talk, I would like to cover quantum algorithms for scientific computational problems, including topics such as linear, nonlinear, and stochastic differential equations, with applications in areas such as quantum dynamics, biology and epidemiology, fluid dynamics, and finance.

Reference:

Quantum spectral methods for differential equations, Communications in Mathematical Physics 375, 1427-1457 (2020), https://arxiv.org/abs/1901.00961

High-precision quantum algorithms for partial differential equations, Quantum 5, 574 (2021), https://arxiv.org/abs/2002.07868

Efficient quantum algorithm for dissipative nonlinear differential equations, Proceedings of the National Academy of Sciences 118, 35 (2021), https://arxiv.org/abs/2011.03185

Quantum-accelerated multilevel Monte Carlo methods for stochastic differential equations in mathematical finance, Quantum 5, 481 (2021), https://arxiv.org/abs/2012.06283

Quantum Scientific Computation

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Dec 6th, 2021

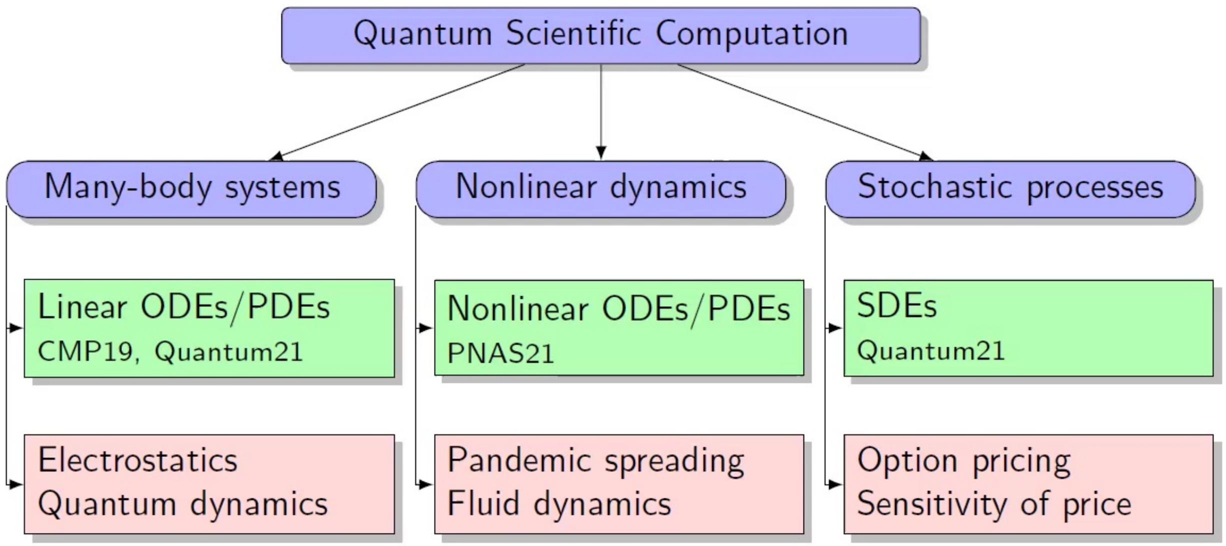
Quantum Scientific Computation

Quantum computers are expected to address complex realistic problems in aeronautics, climate, finance, economy, epidemic...

Challenge

From the linear dynamics of a system of qubits to real-world systems with behaviors of many-body (infinity-body) interaction, nonlinearity, and stochastic volatility.

My Research


Quantum Scientific Computation
Many-body systems
Nonlinear dynamics
Stochastic processes
Linear ODEs/PDEs
CMP19, Quantum21
Nonlinear ODEs/PDEs
PNAS21
SDEs
Quantum21
Electrostatics
Quantum dynamics
Pandemic spreading
Fluid dynamics
Option pricing
Sensitivity of price

Outline

1. Linear Differential Equations and Quantum Dynamics
2. Nonlinear Differential Equations
3. Stochastic Differential Equations

Hamiltonian Simulation

Simulating quantum physics†

Given a description of an s-sparse $n \times n$ Hamiltonian system

$$i \frac{\mathrm{d}}{\mathrm{d}t} |\psi(t)\rangle = H(t)|\psi(t)\rangle, \qquad |\psi(0)\rangle = |\psi_{\text{in}}\rangle, \tag{1}$$

produce the final state

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle \approx U_n \cdots U_2 U_1 |\psi(0)\rangle. \tag{2}$$

Complexity: $\mathrm{poly}(\log n, \log(1/\epsilon))^{\ddagger}$.

† [Feynman 81]
‡ [LLoyd 96; Berry et al. 15; Low, Chuang 17]

Quantum Linear System Problem

Linear system

Given a description of an s-sparse $n \times n$ linear system

$$Ax = b, \tag{3}$$

produce a quantum encoding of the solution proportion to $x = A^{-1}b$.

Complexity: $\kappa \,\mathrm{poly}(\log n, \log(1/\epsilon))^{\dagger}$, $\kappa = \|A\| \cdot \|A^{-1}\|$.

† [Harrow, Hassidim, Lloyd 09; Ambainis 12; Childs, Kothari, Somma 15]

Quantum Linear ODE Problem

Linear ordinary differential equations (ODEs)

Given a description of an s-sparse $n \times n$ linear ODE system

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} - A(t)u(t) = f(t), \qquad \alpha u(0) + \beta u(T) = \gamma, \tag{4}$$

produce a quantum encoding of the solution proportion to $u(T^*)$, $0 \leq T^* \leq T$.

A linear ODE system can be approximated by a linear system, for which we can apply QLSA.

Previous Quantum Algorithms for Linear ODEs

Quantum linear multistep method[†]

- First-order Euler method

$$u(t+\Delta t) \approx u(t) + \Delta t(A(t)u(t) + f(t)). \tag{5}$$

  High-order linear multistep methods are known. In general, if $\epsilon = O((\Delta t)^r)$, then the cost is $T/\Delta t = \Omega(1/\epsilon^{1/r})$.
- Complexity is $\mathrm{poly}(\log n, 1/\epsilon)$, even using QLSA with $\mathrm{poly}(\log n, \log(1/\epsilon))$.

[†] [Berry 14]

Previous Quantum Algorithms for Linear ODEs

Quantum algorithm by truncated Taylor series[†]

- For a time-independent ODE $\frac{\mathrm{d}u(t)}{\mathrm{d}t} = Au(t) + b$, the closed-form solution is approximated by

$$
\begin{aligned}
u(t) &= \exp(At)u(0) + [\exp(At) - I]A^{-1}b \\
&\approx \sum_{k=0}^{N} \frac{(At)^k}{k!} u(0) + \sum_{k=0}^{N} \frac{(At)^{k-1}}{k!} tb,
\end{aligned}
\tag{6}
$$

  which is a combination of A and b.
- Complexity is $\mathrm{poly}(\log n, \log(1/\epsilon))$ by QLSA. But it did not cover time-dependent cases.

[†] [Berry, Childs, Ostrander, Wang 17]

High-precision Quantum Algorithm for Linear ODEs

Quantum Spectral Methods†

- Efficiently discretize the differential equation:
  Spectral Method.
- Efficiently solve the sparse linear system:
  Quantum Linear System Algorithm.
- Bound the global error, condition number, and success probability.

† [Childs, Liu 19]

Spectral Method

We approximate the solution by a truncated Fourier/Chebyshev series

$$u_i(t) = \sum_{k=0}^{N} c_{i,k}\phi_k(t), \quad i \in [n]. \tag{7}$$

We then interpolate the ODEs with quadrature nodes $\{t_l\}_{l=0}^{N}$

$$\frac{\mathrm{d}u(t_l)}{\mathrm{d}t} = A(t_l)u(t_l) + f(t_l), \quad l \in [N]_0 \tag{8}$$

to obtain a linear system for coefficients $c_{i,k}$ solved by QLSA.

- Error exponentially decreases if $u(t)$ smooth: $\epsilon = O(1/N^N)$.
- $n(N+1) \times n(N+1)$ linear system with sparsity $O(Ns)$.

High-precision Quantum Algorithms for Linear ODEs

Theorem (quantum spectral method†)

Thm 1-2. of [Childs, Liu 19] *Let* $q := \max_t \|u(t)\|/\|u(T)\|$*. There is a quantum algorithm that outputs a normalized state* $|u(T^*)\rangle$ *proportion to* $u(T^*)$ *within* ϵ*, with complexity*

- *initial value problems (IVPs):* $s\|A\|Tq \ \mathrm{poly}(\log(1/\epsilon))$.
- *boundary value problems (BVPs):* $s\|A\|^4T^4q \ \mathrm{poly}(\log(1/\epsilon))$.

An exponential speedup in ϵ for IVPs of time-dependent ODEs; the first quantum algorithm for BVPs.

† [Childs, Liu 19]

Generalization: Linear PDEs

Poisson equation in electrostatics

$$\nabla^2 u(\boldsymbol{x}) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathcal{D} \subset \mathbb{R}^d. \tag{9}$$

- n spatial grids in each coordinate, giving a n^d-dim ODE system.
- Previous quantum algorithms can achieve $\mathrm{poly}(d, 1/\epsilon)^\dagger$.

Theorem (quantum spectral method‡)

Thm 1-2. of [Childs, Liu, Ostrander 21] *There is a quantum algorithm that outputs a normalized state* $|u(\boldsymbol{\chi})\rangle$ *proportion to* $u(\boldsymbol{\chi})$ *on spatial grids* $\{\boldsymbol{\chi}\} \subset \mathbb{R}^d$ *within* ϵ*, with complexity* $d\ \mathrm{poly}(\log(1/\epsilon))$.

An exponential speedup in ϵ; the best known scaling in d.

† [Cao et al. 13; Montanaro, Pallister 16]
‡ [Childs, Liu, Ostrander 21]

Generalization: Real-space Quantum Dynamics

η-particle time-dependent Schrödinger equation

$$i\frac{\partial}{\partial t}\Psi(\boldsymbol{x},t) = \Big[-\frac{1}{2}\nabla^2 + f(\boldsymbol{x},t)\Big]\Psi(\boldsymbol{x},t), \quad \boldsymbol{x} \in \mathcal{D} \subset \mathbb{R}^{3\eta}. \tag{10}$$

We denote $\widetilde{\Psi}(\boldsymbol{x},t)$ as the solution of discretized Hamiltonian system.

- Applications: chemical dynamics, uniform electron gas.
- Most previous work only focus on $\widetilde{\Psi}$ after Galerkin discretization. The orbital number N implicitly relies on η, T, ϵ.
- Previous real-space simulation based on the finite difference can achieve $O(\eta^7 T^3/\epsilon^2)$[†].

† [Kivlichan et al. 17]

High-precision Real-space Quantum Simulation

Fourier spectral method gives a more compact real-space Hamiltonian system than the finite difference.

Theorem (quantum spectral method with interaction picture†)

Thm 3 of [Childs, Leng, Li, Liu, Zhang in prep.] *There is a quantum algorithm that outputs a wave function* $|\Psi(\boldsymbol{x}, T)\rangle$ *within* ϵ*, with complexity* $\eta^3 T\ \mathrm{poly}(\log(1/\epsilon))$.

An exponential speedup in ϵ; significant speedups in η and T.

† [Childs, Leng, Li, Liu, Zhang in prep.]

Outline

Nonlinearity is Difficult in Quantum Computing

How to tackle nonlinear differential equations by a quantum computer was a long-standing open problem.

PNAS Proceedings of the National Academy of Sciences of the United States of America

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RESEARCH ARTICLE

Efficient quantum algorithm for dissipative nonlinear differential equations

Jin-Peng Liu, Herman Øie Kolden, Hari K. Krovi, Nuno F. Loureiro, Konstantina Trivisa, ...
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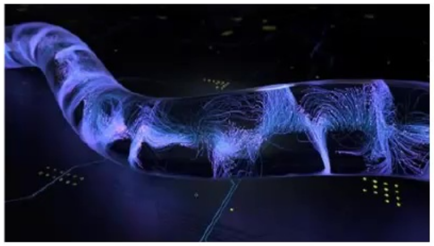

ABSTRACTIONS BLOG

New Quantum Algorithms Finally Crack Nonlinear Equations

By MAX G. LEVY | 4

Two teams found different ways for quantum computers to process nonlinear systems by first disguising them as linear ones.

Nonlinearity is Difficult in Quantum Computing

Inefficient quantum algorithm

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = \sum_{j,k=1}^{n} \alpha_{jk}^{(i)} u_j u_k \approx \frac{u_i(t+\Delta t) - u_i(t)}{\Delta t}. \tag{11}$$

Consider $|\phi_t\rangle = \sum_j u_j |j\rangle$, and use $|\phi_t\rangle|\phi_t\rangle = \sum_{jk} u_j u_k |jk\rangle$ to generate $|\phi_{t+\Delta t}\rangle$ in one iteration. By the no-cloning theorem, it needs to maintain totally $2^{O(T)}$ multiple copies of $|\phi_0\rangle$ for one $|\phi_T\rangle^{\dagger}$.

Quantum lower bound

- Nonlinear quantum mechanics can imply poly-time solution for NP-complete and $\#P$ problems[‡].
- For Gross-Pitaevskii equation $\frac{\mathrm{d}}{\mathrm{d}t}\langle x|\psi\rangle = g|\langle x|\psi\rangle|^2 \langle x|\psi\rangle$ with $|g| > 1$, quantum algorithms have worst-case time complexity $2^{\Omega(T)}$[§].

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† [Leyton, Osborne 08]

‡ [Abrams, Lloyd 98; Aaronson 05]

§ [Childs, Young 16]

Quantum Nonlinear ODE Problem

Quadratic ODEs

$$\frac{\mathrm{d}u}{\mathrm{d}t} = F_2 u^{\otimes 2} + F_1 u + F_0(t), \qquad u(0) = u_{\mathrm{in}}. \tag{12}$$

$u = [u_1, \ldots, u_n]^T \in \mathbb{R}^n$, $u^{\otimes 2} = [u_1^2, u_1 u_2, \ldots, u_n u_{n-1}, u_n^2]^T \in \mathbb{R}^{n^2}$, $F_2 \in \mathbb{R}^{n \times n^2}$, $F_1 \in \mathbb{R}^{n \times n}$, $F_0(t) \in \mathbb{R}^n$ are s-sparse, F_1 is diagonalizable, and eigenvalues λ_j of F_1 satisfy $\mathrm{Re}\,(\lambda_n) \leq \cdots \leq \mathrm{Re}\,(\lambda_1) < 0$.

Inspired by Reynolds number, we define

$$R := \frac{1}{|\,\mathrm{Re}\,(\lambda_1)|}\left(\|u_{\mathrm{in}}\|\|F_2\| + \frac{\|F_0\|}{\|u_{\mathrm{in}}\|}\right). \tag{13}$$

It quantifies nonlinear and inhomogeneous strengths relative to dissipation.

Efficient Quantum Algorithm for Nonlinear ODEs

Theorem (efficient quantum algorithm[†])

Thm 1. of [Liu et al. 21] *Assume* $R < 1$. *Let* $q := \|u_{\text{in}}\|/\|u(T)\|$. *There is a quantum algorithm that outputs a normalized state* $|u(T)\rangle$ *proportion to* $u(T)$ *within* ϵ, *with complexity*

$$\frac{sT^2q}{\epsilon}\,\text{poly}(\log T, \log n, \log \frac{1}{\epsilon}). \tag{14}$$

Math contribution

We improve the previous best known convergence analysis of Carleman linearization. Our results have been applied in classical nonlinear control theories[‡] and classical computational fluid dynamics[§].

[†] [Liu et al. 21]
[‡] [Foret, Schilling 21]
[§] [Itani, Succi 21]

Carleman Linearization

Considering a 1-dim quadratic ODE $\frac{\mathrm{d}u}{\mathrm{d}t} = au^2 + bu + c$.
Naïve linearization $\frac{\mathrm{d}u}{\mathrm{d}t} \approx au(0)u + bu + c$ doesn't work in long time.

Embedding and truncation

- 1st equation: $\frac{\mathrm{d}u}{\mathrm{d}t} = au^2 + bu + c$.
- 2nd equation: $\frac{\mathrm{d}}{\mathrm{d}t}u^2 = 2u\frac{\mathrm{d}u}{\mathrm{d}t} = 2au^3 + 2bu^2 + 2cu$.
- $\cdots\cdots$
- Nth equation: $\frac{\mathrm{d}}{\mathrm{d}t}u^N \approx Nbu^N + Ncu^{N-1}$.
- Give a linear ODE with variables $y_j \approx u^j$ for $j \in [N]$.

High-dimensional generalization

A system of n-dim nonlinear ODEs is embedded to a system of linear ODEs with truncation order N, where the dimension is $n + n^2 + \cdots + n^N$.
We rigorously prove the exponential convergence of N when $R < 1$.

Carleman Linearization

We give a linear ODEs $\frac{\mathrm{d}\hat{y}}{\mathrm{d}t} = A(t)\hat{y} + b(t)$ with $\hat{y}(0) = \hat{y}_{\text{in}}$, by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \vdots \\ \hat{y}_N \end{pmatrix} = \begin{pmatrix} A_1^1 & A_2^1 & & \\ A_1^2 & A_2^2 & \ddots & \\ & \ddots & \ddots & A_N^{N-1} \\ & & A_{N-1}^N & A_N^N \end{pmatrix} \begin{pmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \vdots \\ \hat{y}_N \end{pmatrix} + \begin{pmatrix} F_0(t) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad (15)$$

where $\hat{y}_j \approx u^{\otimes j} \in \mathbb{R}^{n^j}$, $\hat{y}_{\text{in}} = [u_{\text{in}}; u_{\text{in}}^{\otimes 2}; \ldots; u_{\text{in}}^{\otimes N}]$, and

$$A_{j+i-1}^j = F_i \otimes I^{\otimes j-1} + I \otimes F_i \otimes I^{\otimes j-2} + \cdots + I^{\otimes j-1} \otimes F_i. \qquad (16)$$

We apply the Euler method and QLSA to estimate $\hat{y}(T)$ and measure $\hat{y}_1(T) \approx u(T)$ with rigorous complexity analysis.

Quantum Lower Bound for Nonlinear ODEs

Theorem (quantum lower bound[†])

Thm 2. of [Liu et al. 21] *Assume $R \geq \sqrt{2}$. Then there is an instance of the quantum quadratic ODE problem such that any quantum algorithm must have worst-case time complexity exponential in T.*

Basic ideas

- Hardness of distinguishing nonorthogonal quantum states.
- Butterfly effect: a small initial difference results in a large violation.

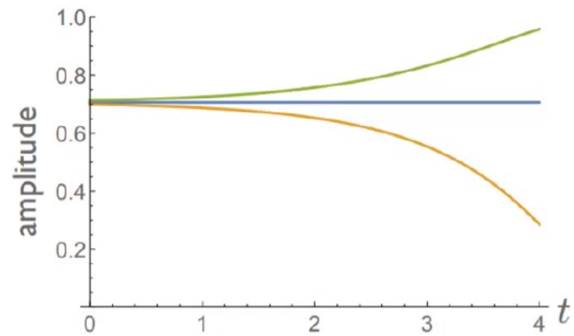
Quantum computers cannot efficiently capture chaotic behaviors.

[†] [Liu et al. 21]

Quantum Lower Bound for Nonlinear ODEs

- Let $|\psi\rangle, |\phi\rangle$ be states of a qubit with $|\langle\psi|\phi\rangle| = 1 - \epsilon$. Then any bounded-error protocol for determining whether the state is $|\psi\rangle$ or $|\phi\rangle$ must take time $\Omega(1/\epsilon)$.
- We construct a quadratic ODE with $R \geq \sqrt{2}$ that evolves $|\psi\rangle, |\phi\rangle$ to have a small overlap after evolution time $T = O(\log(1/\epsilon))$.

Any quantum algorithm must need $2^{\Omega(T)}$ time for the quadratic ODE.



Computational Epidemiology

Susceptible-Exposed-Infectious-Recovered (SEIR) model

$$\begin{aligned}
\frac{\mathrm{d}P_S}{\mathrm{d}t} &= -\Lambda\frac{P_S}{P} - r_{\mathrm{vac}}P_S + \Lambda - r_{\mathrm{vac}}P_S\frac{P_I}{P} \\
\frac{\mathrm{d}P_E}{\mathrm{d}t} &= -\Lambda\frac{P_E}{P} - \frac{P_E}{T_{\mathrm{lat}}} + r_{\mathrm{vac}}P_S\frac{P_I}{P} \\
\frac{\mathrm{d}P_I}{\mathrm{d}t} &= -\Lambda\frac{P_I}{P} + \frac{P_E}{T_{\mathrm{lat}}} - \frac{P_I}{T_{\mathrm{inf}}} \\
\frac{\mathrm{d}P_R}{\mathrm{d}t} &= -\Lambda\frac{P_R}{P} + r_{\mathrm{vac}}P_S + \frac{P_I}{T_{\mathrm{inf}}}.
\end{aligned} \tag{17}$$

- Realistic parameters of rapid vaccination satisfies $R < 1$[†].
- A high-dimensional generalization can model many interacting cities.

[†] [Liu et al. 21]

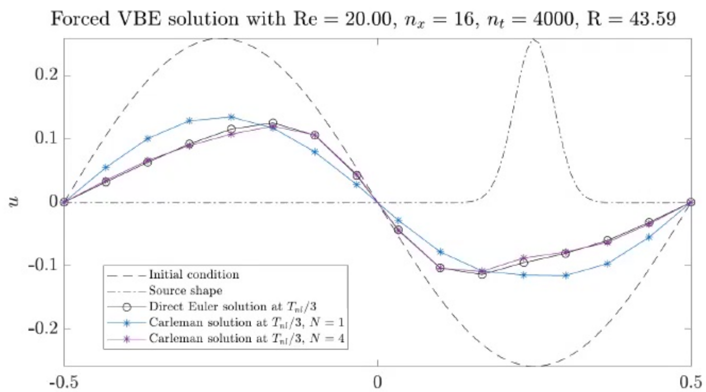
Computational Fluid Dynamics

Forced viscous Burgers equation

$$\partial_t u + u\partial_x u = \nu\partial_x^2 u + f, \tag{18}$$

It is a 1-dim case of Navier-Stokes equation. We apply the central difference in space to obtain a quadratic ODE.

Numerical results show good convergence of Carleman linearization for convective flows[†].



† [Liu et al. 21]

Efficient Quantum Algorithm for Nonlinear ODEs

Main results

- Propose the first poly-time quantum algorithm when $R < 1$.
- Establish worst-case complexity exponential in time when $R \geq \sqrt{2}$, giving an almost tight classification of quantum complexity.
- Show potential quantum applications in epidemic and fluid dynamics.

† [Liu et al. 21]

Outline

1. Linear Differential Equations and Quantum Dynamics
2. Nonlinear Differential Equations
3. Stochastic Differential Equations

Stochastic Processes

Stochastic differential equations (SDEs)

$$\mathrm{d}X_t = \mu(X_t, t)\mathrm{d}t + \sigma(X_t, t)\mathrm{d}W_t, \tag{19}$$

where μ is a drift, σ is a volatility, and W_t is a standard Brownian motion.

Monte Carlo simulation

Repeat simulating the SDE and average the outcomes to estimate expected quantities.

Stochastic Processes in Finance

- Black-Scholes model

$$dX_t = rX_t dt + \sigma X_t dW_t. \tag{20}$$

- Local Volatility model

$$dX_t = rX_t dt + \sigma(X_t, t)X_t dW_t. \tag{21}$$

- "Greeks" (sensitivity of price of derivatives)

$$\begin{aligned} dX_t &= \mu(X_t)dt + \sigma(X_t)dW_t, \\ dY_t &= \mu'(X_t)Y_t dt + \sigma'(X_t)Y_t dW_t. \end{aligned} \tag{22}$$

Payoff Models in Finance

Given a SDE of stock price $\{X_t\}_0^T$, compute the expected payoff:

$$\mathbb{E}[\mathcal{P}(X_T) \mid X_0]. \tag{23}$$

- European option

$$\mathcal{P}(X_T) = e^{-rT} \max\{X_T - K, 0\}. \tag{24}$$

- Asian option

$$\mathcal{P}(X_T) = e^{-rT} \max\left\{\frac{1}{T}\int_0^T X_t \mathrm{d}t - K, 0\right\}. \tag{25}$$

- Digital option (Cash-or-nothing option)

$$\mathcal{P}(X_T) = e^{-rT}\mathcal{H}(X_T - K). \tag{26}$$

Quantum-accelerated Monte Carlo

Lemma (quantum speedup of MC[†])

Let $\mathcal{A}$ be a (randomized or quantum) algorithm. Let $v(\mathcal{A})$ be the random variable with $\mathbb{V}[v(\mathcal{A})] \leq \sigma^2$, then $\mathbb{E}[v(\mathcal{A})]$ can be estimated within ϵ in $\Omega(1)$ probability with cost $\widetilde{O}(\sigma/\epsilon)$.

- It has been applied to estimate the price of options with $\widetilde{O}(\epsilon^{-1})$ for Black-Scholes model[‡] with access to closed-form solutions.
- However, when we simulate $X_0 \to X_T$ by numerical schemes, each sample takes $\widetilde{O}(\epsilon^{-\alpha})$, and the computational cost is $\widetilde{O}(\epsilon^{-1-\alpha})$.

[†] [Montanaro 15]

[‡] [Rebentrost, Gupt and Bromley 18; Stamatopoulos et al. 19]

Quantum-accelerated Multilevel Monte Carlo

Let $P_0, P_1, \ldots, P_L$ are estimators of a random variable P with increasing accuracy. We use $\sum_l \mathbb{E}[P_l - P_{l-1}] = \mathbb{E}[P_L]$ to estimate $\mathbb{E}[P]$, and denote C_l, V_l, and N_l as cost, variance, and sampling number of $P_l - P_{l-1}$.

MLMC: more coarse-grained samples, fewer fine-grained samples, s.t.

- variance $\sum_{l=0}^{L} N_l^{-1} V_l \approx N^{-1} V$;
- cost $\sum_{l=0}^{L} N_l C_l \ll NC$.

Cost minimization subject to fixed variance

- MC: cost $=$ large $N \times$ large C.
- MLMC†: cost $=$ sum of decreasing $N_l \times$ increasing C_l.

† [Giles 08; Giles 15]

Quantum-accelerated Multilevel Monte Carlo

MLMC[†]

The optimum is $\widetilde{O}\left(\epsilon^{-2}(\sum_{l=0}^{L}\sqrt{V_l C_l})^2\right)$, given by $N_l = \widetilde{O}(\sqrt{V_l/C_l})$.
Furthermore, to cancel out the term $(\sum_{l=0}^{L}\sqrt{V_l C_l})^2$, we require the decrease of V_l is no slower than the increase of C_l (i.e. $V_l C_l = O(1)$).

Theorem (QA-MLMC[‡])

Thm 2. of [An et al. 21] *If half of the decrease of V_l is no slower than the increase of C_l, then there is a quantum algorithm that estimates $\mathbb{E}[P]$ within ϵ in $\Omega(1)$ probability with computational cost $\widetilde{O}(\epsilon^{-1})$.*

The first quantum speedup for computational cost of general SDEs.

[†] [Giles 08; Giles 15]
[‡] [An et al. 21]

Applications in Payoff Models

Recall that we aim to compute $\mathbb{E}[\mathcal{P}(X_T) \mid X_0]$ given a SDE $\{X_t\}_0^T$.

We set $P = \mathcal{P}(X_T)$, and $P_l = \mathcal{P}(\widehat{X}_{n_l})$, where $\widehat{X}_{n_l}$ is simulated by a numerical scheme with $n_l = O(2^l)$ grids on $[0, T]$. Each simulation of $\widehat{X}_{n_l}$ has cost $C_l = O(2^l)$ and variance $V_l = O(2^{-\beta l})$.

- If $\beta \geq 1$, MLMC can achieve computational cost $\widetilde{O}(\epsilon^{-2})$[†].
- If $\beta \geq 2$, QA-MLMC can achieve computational cost $\widetilde{O}(\epsilon^{-1})$[‡].

β relies on the convergence rate of numerical schemes.

We provide high-order stochastic schemes for various option and derivative pricing models[‡].

[†] [Giles 08; Giles 15]
[‡] [An et al. 21]

Numerical Results

- We consider European option and Digital option

$$\mathcal{P}(X_T) = e^{-\mu T}\max\{X_T - K, 0\}, \quad (27)$$

$$\mathcal{P}(X_T) = 5e^{-\mu T}(1 + \mathcal{H}(X_T - K)), \quad (28)$$

with parameters $\mu = 0.05, \sigma = 0.2, T = 1, X_0 = 100, K = 100.$

- We implement high-order stochastic schemes to test β, based on 10^6 to 10^9 independent simulations.

Option	EM	Milstein	TS1.5	TS2	TS3
European	0.976999	1.962848	2.970166	3.964626	5.958417
Digital	0.473426	0.869393	1.452448	1.775679	2.957982

Table: numerical estimates of β for five schemes: Euler Maruyama(EM) scheme, Milstein scheme, Taylor-Stratonovich (TS) schemes of strong order 1.5, 2, and 3.

† [An et al. 21]

Summary

Linear differential equations and quantum dynamics

[CMP19, Quantum21] Optimize the performance of quantum algorithms in terms of ϵ, d, η, T, with applications in classical and quantum systems.

Nonlinear differential equations

[PNAS21] Propose the first poly-time quantum algorithm and an almost tight lower bound, toward applications in epidemic and fluid dynamics.

Stochastic differential equations

[Quantum21] Establish the quantum speedup for computational cost of pricing general options and derivations.

Outlook

Scalable quantum many-body simulation

Real-space quantum dynamics, state preparation, postprocessing.

Efficient quantum algorithms for nonlinear dynamics

Nonlinear systems in control, programming, and game theories.

Quantum speedups for randomized computation

Markov chain Monte Carlo, pricing and hedging, portfolio management.