Title: Discretizing 2d conformal field theories: the lattice action of the conformal algebra

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Series: Quantum Matter

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Abstract: Conformal field theories (CFTs) are ubiquitous in theoretical physics as fixed points of renormalization, descriptions of critical systems and more. In these theories the conformal symmetry is a powerful tool in the computation of correlation functions, especially in 2 dimensions where the conformal algebra is infinite. Discretization of field theories is another powerful tool, where the theory on the lattice is both mathematically well-defined and easy to put on a computer. In this talk I will outline how these are combined using a discrete version of the 2d conformal algebra that acts in lattice models. I will also discuss recent work on convergence of this discretization, as well as on applications to non-unitary CFTs that appear in descriptions of problems of interest in condensed matter physics such as polymers, percolation and disordered systems.

Zoom Link: https://pitp.zoom.us/j/95048143778?pwd=N1hhVHlsZThVYzBWTy9CNlBTUHIydz09

Discretizing $2d$ conformal field theories: the lattice action of the conformal algebra

Linnéa Gräns Samuelsson

Based on work together with:
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Plan of the talk

- Part I: Background and motivation
- Part II: Introducing the discretized conformal algebra
- Part III: Applications to non-unitary CFT
- Part IV: Results about convergence
- Part V: Results in the loop model and the 6-vertex model

Main references:

LGS, L. Liu, Y. He, J. L. Jacobsen, H. Saleur, arXiv:2007.11539

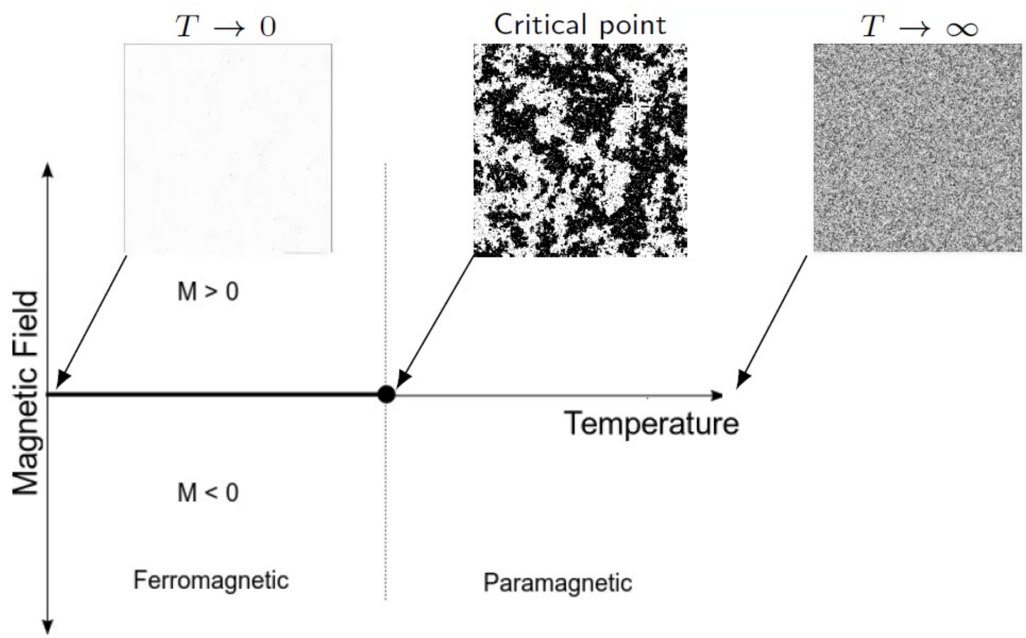
LGS, J.L. Jacobsen, H. Saleur, arXiv:2010.12819

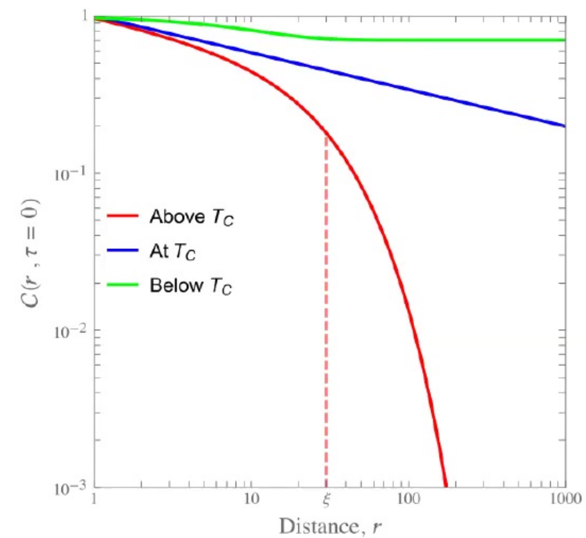
Conformal field theories: *field theories invariant under conformal (angle-preserving) transformations, such as scaling.*

Why conformal field theories

- Fixed points of renormalization group flow $\Leftrightarrow$ scale invariance, which typically extends to conformal invariance.
  – we typically expand QFTs around RG fixed points (most common example: free field theories), since we can more easily find solutions at these points. Thus CFTs play an important role in the general understanding of QFTs
- String theory, AdS/CFT, ...
- Critical systems (liquid/gas, ferromagnetic/paramagnetic, ...)

Example: Criticality in the Ising model. Spins ↑↓ with
$\mathcal{H} = -J\sum_{\langle ij\rangle}\sigma_i\sigma_j - \mu\sum_i \sigma_j h_j$





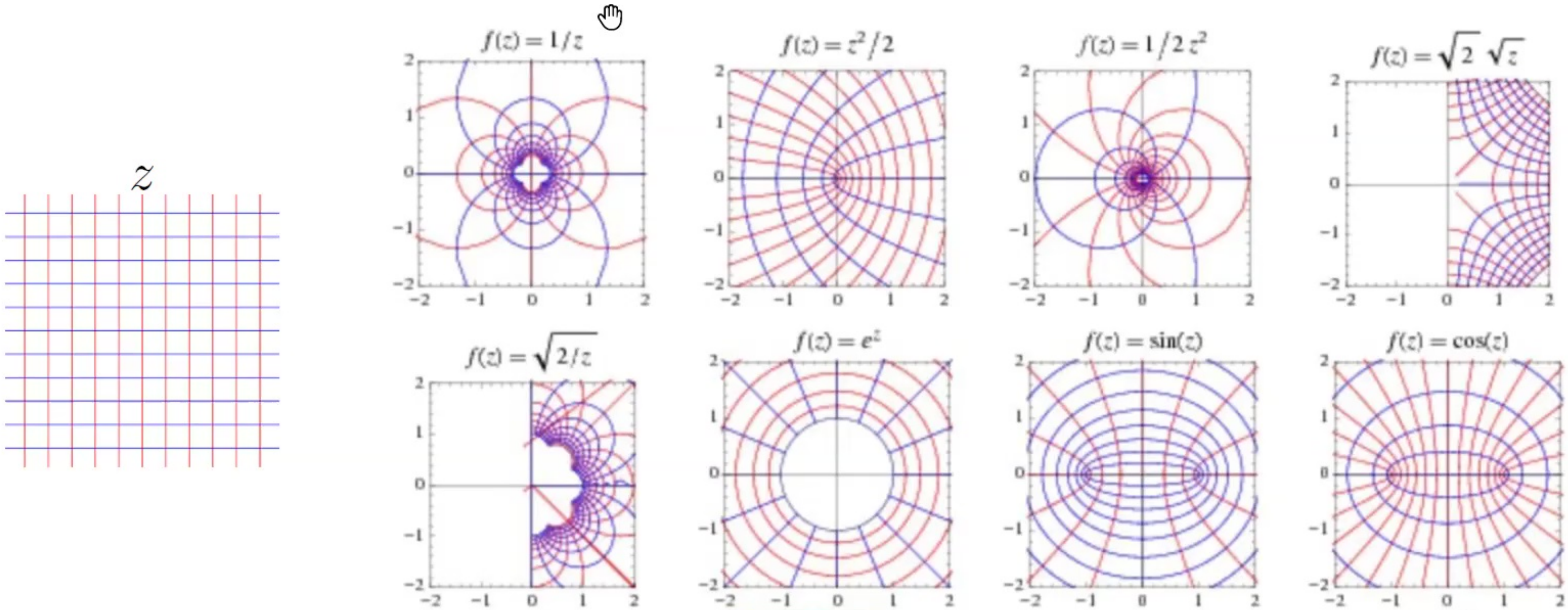
$\langle\sigma_0\sigma_r\rangle_{T_c} \sim r^{-\eta}$
with $\eta = 2\Delta_\sigma = 1/4$

- Typically, correlation functions decay exponentially
- At critical point, correlation length diverges and we find a power law. General goal: find with what power a given correlation function decays.

Conformal symmetry

Invariance under conformal (angle-preserving) transformations. Metric the same up to local scale factor. In general: translation, rotation, dilation, special conformal transformation.

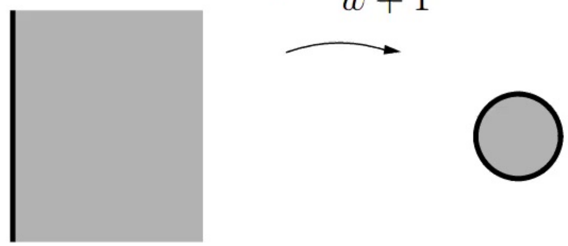
In d=2: any holomorphic function gives a conformal map.



2d CFT beginnings: A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov (1984)

Conformal mappings powerful tool in many contexts. For instance: mapping domains when solving Laplace equation with boundary conditions,

$$\zeta = \frac{w-1}{w+1}$$



Similarly in CFT, conformal symmetry is a powerful tool in the computation of correlation functions.

Lattice discretizations of field theories is another powerful tool: they are mathematically well defined, and easy to put on a computer. Can we combine these tools? *Can we discretize the conformal symmetry?*

Symmetry algebra of $2d$ CFT: the Virasoro algebra

Consider infinitesimal coordinate transformations $z \to z + \epsilon z^{n+1}$.
Generated by $l_n = -z^{n+1}\frac{\partial}{\partial z}$, which obey $[l_m, l_n] = (m-n)l_{m+n}$.

The Virasoro algebra:

$$\underbrace{[L_m, L_n] = (m-n)L_{m+n}}_{\text{The algebra generated by } -z^{n+1}\frac{\partial}{\partial z}} + \underbrace{\frac{c}{12}m(m^2-1)\delta_{n+m,0}}_{\text{central term}}$$

Central charge c:

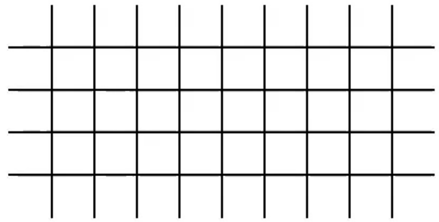
- The quantum anomaly (central term) is proportional to c.
- c measures the degrees of freedom of the system. E.g. theory with n free scalar fields has $c = n$.
- c appears in the 2-point function of the stress-energy tensor: $\langle TT \rangle \propto c/2$.

Asking if we can discretize $2d$ conformal symmetry means asking: **can we discretize the Virasoro algebra and have it act in lattice systems?**

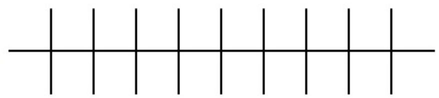
Part II: Introducing the discretized conformal algebra

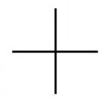
Lattice models:

Transfer matrix $\mathbf{T}$ builds the system row by row,

$\mathbf{T} \uparrow$ 

with a transfer matrix

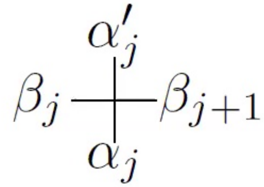
$\mathbf{T} =$ 

We are interested in a type of $2d$ lattice models where the transfer matrix in turn built out of local operators  that are expressed in terms of a lattice algebra: the Temperley-Lieb algebra.

Different representations of the Temperley-Lieb algebra will correspond to different lattice models.

General ref. for the relevant lattice models: "Exactly Solved Models in Statistical Mechanics" by R.J. Baxter.

Consider variables α, β on the edges. The local operator

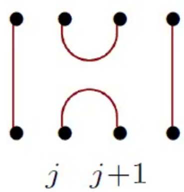
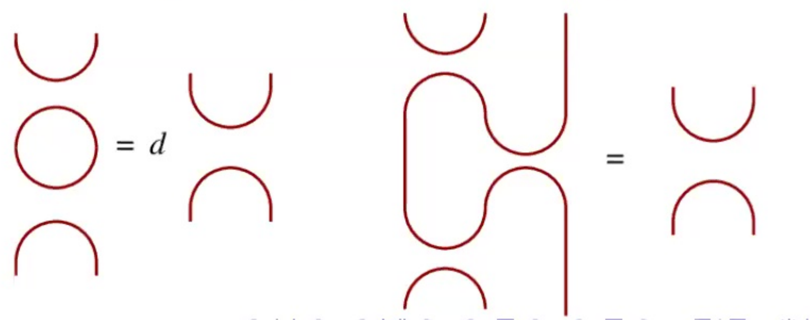


will be a matrix $R_j = R_{(\alpha_j,\beta_j);(\alpha'_j,\beta_{j+1})}$. We constrain it to be on the form $R_j \sim \mathbf{1} + (const)e_j$ with e_j fulfilling the Temperley-Lieb relations:

$$e_j^2 = de_j, \quad e_j e_{j\pm1} e_j = e_j$$

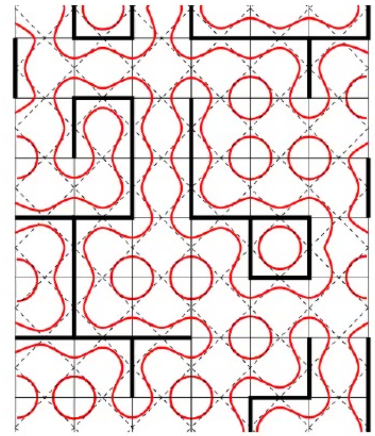
The lattice model will have the property of integrability.

As a diagram algebra, the Temperley-Lieb algebra connects two rows of N points. Multiplication: stacking diagrams vertically.

$e_j = \dots$  $\dots$ with 

Example 1: Loop model. Weight d per loop.

Appears e.g. when considering boundaries of clusters in the Ising model or the more general Q-state Potts model, with applications to percolation.



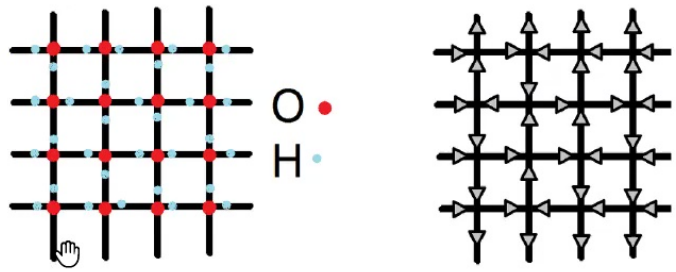
Link-state representation: states are half-diagrams, e.g. 

Build the lattice configurations row by row. $R_j \sim \mathbf{1} + (const)e_j$, with:

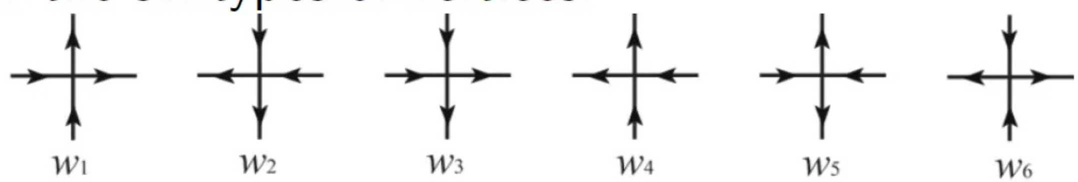
$\mathbf{1} =$  $e_j =$ 

Varying the loop weight we obtain a family of continuum limit CFT's with $c \leq 1$.

Example 2: Ice model. Historically: understanding residual entropy in ice.



Generalize to ice-type model, also called 6-vertex model: different weights for the six types of vertices.



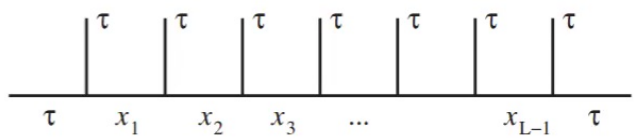
Variables $\uparrow, \downarrow$ on edges. States are spin states, e.g. $\uparrow\downarrow\downarrow$.
e_j will be a particular combination of Pauli matrices that fulfils the Temperley-Lieb relations.

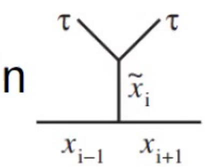
With different choices of weights w, we obtain again a family of continuum limit CFT's with $c \leq 1$, such as the Ising model with $c = 1/2$.

Rephrasing the $2d$ Euclidean lattice models as $(1+1)d$ quantum spin chains, the latter have Hamiltonians $\mathcal{H} \sim -\sum_j e_j$.

- From the 6-vertex model we obtain the familiar

$$\mathcal{H}_{XXZ} \sim \frac{1}{2}\sum_{j=1}^{N}\left[\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \underbrace{\Delta}_{\substack{\text{anisotropy} \\ \Delta=\cos\gamma}}(\sigma_j^z\sigma_{j+1}^z - 1)\right]$$

- In an anyon chain, e.g.  ,

  e_i assigns an energy gain for having $\tilde{x}_i = 1$ in 

  (corresp. $2d$ lattice model: the RSOS model)

We take the models to be at a critical point, so that the continuum limit is described by a CFT.

We consider periodic boundary conditions $\rightarrow$ bulk CFT. $Vir \times Vir$ symmetry (L_n and $\bar{L}_n$).

Goal: find discrete versions of $\overset{(-)}{L}_n$ on the form $\overset{(-)}{\mathcal{L}}_n(\{e_j\})$ acting in the spin chains.

Virasoro generators L_n and the stress-energy tensor

In any $d > 1$: local stress-energy tensor $T^{\mu\nu}$ as conserved current corresponding to the conformal symmetry.
In $d = 2$, $T_{z\bar{z}} = \frac{1}{4}(T_{xx} + T_{yy}) = 0$ (traceless) while

$$T(z) \equiv T_{zz} = \frac{1}{4}\left(T_{xx} - T_{yy} - 2iT_{xy}\right)$$

$$\bar{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}} = \frac{1}{4}\left(T_{xx} - T_{yy} + 2iT_{xy}\right)$$

Virasoro generators appear as modes of $T(z)$. On the cylinder:

$$T(z) = -\sum e^{inz} L_n + \frac{c}{24}$$

To find discrete $\mathcal{L}_n(\{e_j\})$ we look for discrete $\mathcal{T}(\{e_j\})$ and define

$$\mathcal{L}_n(\{e_j\}) = \frac{N}{2\pi} \sum_{j=1}^{N} e^{inj2\pi/N} \mathcal{T}(\{e_j\}) + \frac{c}{24}\delta_{n,0}$$

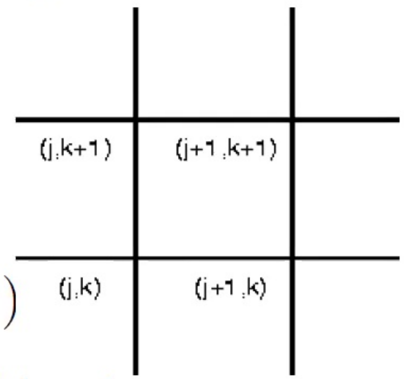
$\mathcal{T}(\{e_j\})$ from lattice Ward identities

L.P. Kadanoff and H. Ceva (1971), W.M.Koo and H.Saleur(1993)

Ward identity in CFT:

$$\langle \int T_{xx}\, \phi_1 ... \phi_N \, \mathrm{d}x\mathrm{d}y \rangle = \pi \sum_{i=1}^{N} \underbrace{\left(-x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i} \right)}_{\text{straining}} \langle \phi_1 ... \phi_N \rangle$$

Consider Ising model on a square lattice, with different couplings in x and y direction:



$$\mathcal{H} = -\sum_{jk} \left[K_x \, \delta(\sigma_{j,k}, \sigma_{j+1,k}) + K_y \, \delta(\sigma_{j,k}, \sigma_{j,k+1}) \right)$$

Look for lattice operator $\mathcal{O}$ giving a lattice Ward identity

$$\langle (\mathcal{O} - \langle \mathcal{O} \rangle) \sigma_{j_1 k_1} \sigma_{j_2 k_2} \rangle = \left(-j_1 \frac{\partial}{\partial j_1} + k_1 \frac{\partial}{\partial k_1} - j_2 \frac{\partial}{\partial j_2} + k_2 \frac{\partial}{\partial k_2} \right) \langle \sigma_{j_1 k_1} \sigma_{j_2 k_2} \rangle$$

We can find $-j\frac{\partial}{\partial j} + k\frac{\partial}{\partial k}$ in terms of a variable S: At the critical point, large distance behaviour depends only on weighted distance $\sqrt{j^2/S^2 + S^2k^2}$ with S a function of the coupling constants K_x, K_y in x and y direction at the self-dual point:

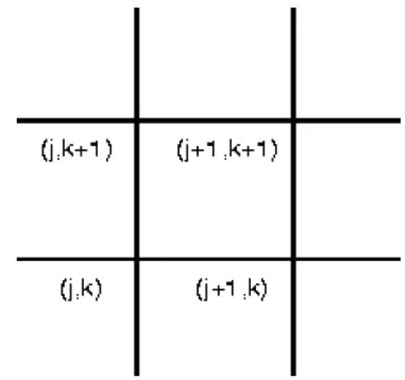
$$S^2 = \tan(\Theta/2)$$

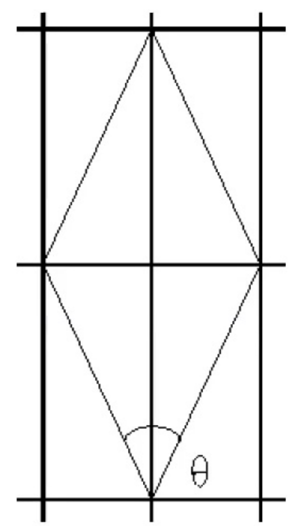
with $\Theta = \Theta(K_x(u), K_y(u))$, and u the spectral parameter.

Thus:

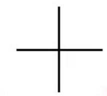
$$-j\frac{\partial}{\partial j} + k\frac{\partial}{\partial k} = S\frac{\partial}{\partial S}$$

and derivatives w.r.t. S give in turn derivatives w.r.t. coupling constants K_x, K_y.





In the Hamiltonian limit: the derivatives w.r.t. coupling constants yield nice expressions in terms of the Temperley-Lieb algebra.

(Recall the operators that govern the interactions, , which make up the transfer matrix.)

> The resulting $\mathcal{O}$ is a sum of local operators, $\mathcal{O} = \sum_{jk} t_{xx}(j,k)$. By comparison with the CFT identity we identify t_{xx} as the lattice version of $T_{xx} = -T_{yy}$. Look for t_{xy} in a similar way.

Recall CFT identity: $\langle \int T_{xx}\, \phi_1 \dots \phi_N \, \mathrm{d}x\mathrm{d}y \rangle = \pi \sum_{i=1}^{N} \left(-x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i} \right) \langle \phi_1 \dots \phi_N \rangle$

vs lattice: $\langle (\mathcal{O} - \langle \mathcal{O} \rangle) \sigma_{j_1 k_1} \sigma_{j_2 k_2} \rangle = \left(-j_1 \frac{\partial}{\partial j_1} + k_1 \frac{\partial}{\partial k_1} - j_2 \frac{\partial}{\partial j_2} + k_2 \frac{\partial}{\partial k_2} \right) \langle \sigma_{j_1 k_1} \sigma_{j_2 k_2} \rangle$

Result: find $\overset{(-)}{\mathcal{T}}(\overset{(-)}{z}) = \frac{1}{2}(t_{xx} \mp i t_{xy})$ with

$$t_{xx} = -2(const)(e_{2j} + e_{2j-1} - 2\epsilon_\infty)$$
$$t_{xy} = 2(const)^2([e_{2j-1}, e_{2j}] + [e_{2j}, e_{2j+1}])$$

For a spin chain with a Temperley-Lieb Hamiltonian, we see that these correspond to energy density and lattice momentum density:

$$\mathcal{H} = -(const)\sum_{j=1}^{N}(e_j - \epsilon_\infty) \quad \Rightarrow \quad \begin{cases} h_j = -(const)(e_j - \epsilon_\infty) \\ p_j = -i(const)^2[e_j, e_{j+1}] \end{cases}$$

and we have $\overset{(-)}{\mathcal{T}}(\overset{(-)}{z}) \propto h_j \pm p_j$. See also A. Milsted and G. Vidal, `arXiv:1706.01436`.

Finally we obtain lattice $\overset{(-)}{\mathcal{L}}_n$ as the modes of $\overset{(-)}{\mathcal{T}}$.

Koo-Saleur generators

$$\overset{(-)}{\mathcal{L}}_n[N] = \frac{N}{2\pi} \underbrace{\sum_{j=1}^{N} e^{\pm inj2\pi/N}}_{\text{take modes}} \underbrace{\frac{1}{2}\left(h_j \pm p_j \right)}_{\text{discrete } \overset{(-)}{T}(\overset{(-)}{z})} + \frac{c}{24}\delta_{n,0}$$

- Virasoro central charge $c = 1 - 6\frac{1}{x(x+1)}$ depends on Temperley-Lieb loop weight $d = 2\cos\gamma$ with $\gamma = \frac{\pi}{x+1}$
- Fields in the CFT correspond to “scaling states” (low-energy states) on the lattice. “Scaling limit”: energy cutoff $\to \infty$ *after* $N \to \infty$

Result: find $\overset{(-)}{\mathcal{T}}(\overset{(-)}{z}) = \frac{1}{2}(t_{xx} \mp i t_{xy})$ with

$$t_{xx} = -2(const)(e_{2j} + e_{2j-1} - 2\epsilon_\infty)$$

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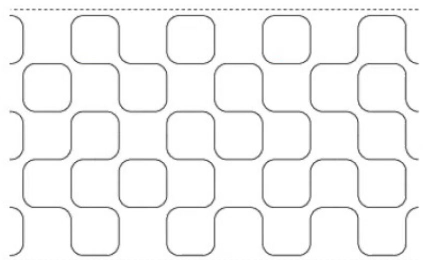
Finally we obtain lattice $\overset{(-)}{\mathcal{L}}_n$ as the modes of $\overset{(-)}{\mathcal{T}}$.

Part III: Applications to non-unitary CFT

Non-unitarity CFT

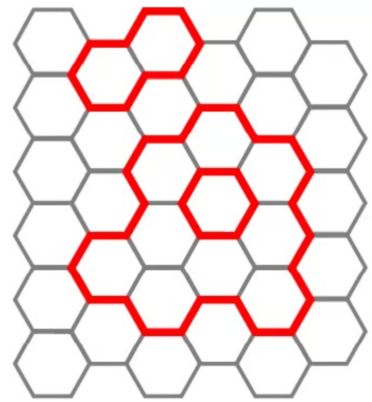
Consider the loop model, with Boltzmann weight d per loop $\rightarrow$ non-local problem.



Correlation functions related to loops are e.g. probability that two points are on the same loop.

Goal: rephrase the problem in terms of local weights.

Turning the loop model local:

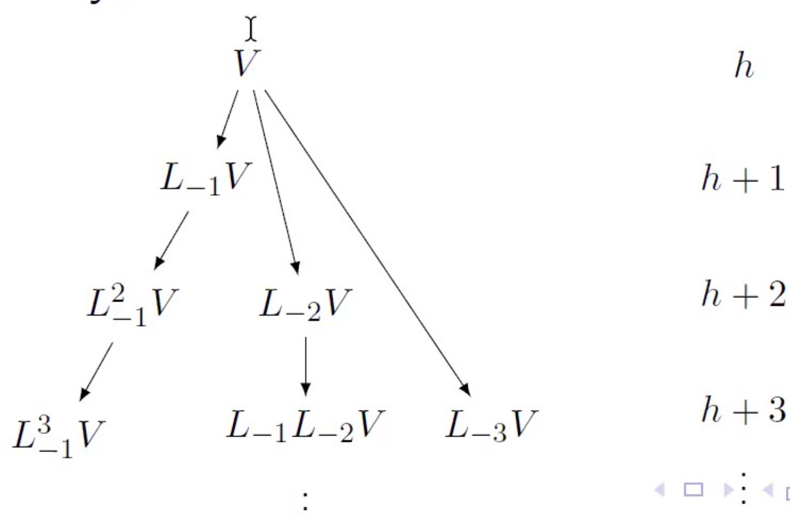


- Can assign a *local* weight e^{iv} (e^{-iv}) for each right (left) turn
- (#left turns − #right turns) $\equiv 0 \mod 6$ for a closed loop
- Sum over both orientations $\Rightarrow$ recover $d = 2\cos 6v$, but with complex local Boltzmann weights

The non-unitarity means that the representation theory of the Virasoro algebra becomes more complicated.

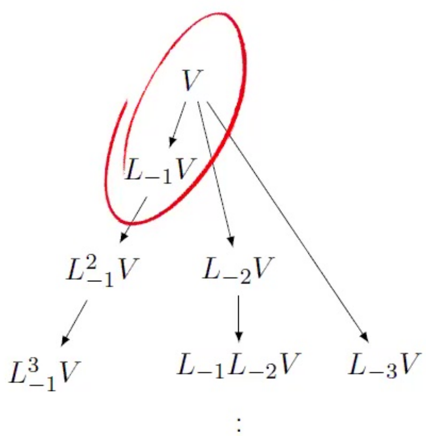
General features of the representation theory:

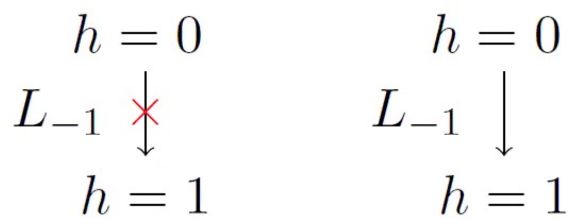
- L_0 plays the role of the Hamiltonian.
- Sort state space into highest-weight (lowest-energy) representations.
- L_0 eigenvalue h (the weight) plays the role of energy
- L_n, $n \neq 0$ play the role of raising and lowering operators.
- V called *primary*, the others *descendants*



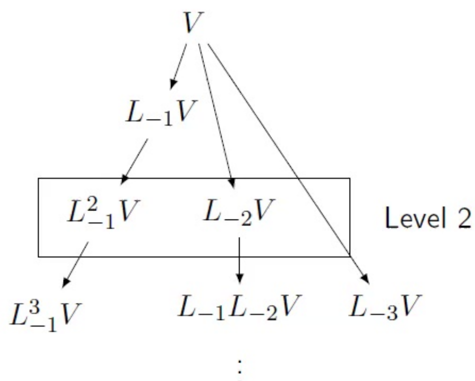
Complication 1:

In non-unitary CFT we cannot identify highest-weight states only by their weight. Example: The identity (vacuum) has conformal weights $h = \bar{h} = 0$. It is annihilated by $L_{-1} = \partial_z$, giving a differential equation for any correlator that involves it. In non-unitary theory: may have another state with $h = \bar{h} = 0$ that is not annihilated by L_{-1}, so that the differential equation does not apply.





More broadly, for specific values of the weight h of the the highest-weight state there will be relations between level n descendant states.



From a relation between $L_{-1}^n V$ and other descendants at level n we get an n-th order differential equation for correlation functions involving V. In non-unitary CFT we must check if such equations still describe all correlation functions involving states of weight h.

(In CFT parlance: we check if null states are zero.)

Complication 2:

In non-unitary CFT we are sometimes (but not always!) unable to fully diagonalize L_0. Put in Jordan normal form $\rightarrow$ get Jordan blocks with fields that mix under the action of L_0.

When L_0 is diagonalizable: $L_0 = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}$ in a basis of V_1, V_2.

Correlation functions:

$\langle V_1(0)V_1(z)\rangle \sim \frac{1}{z^{2h_1}}$, $\langle V_1(0)V_2(z)\rangle = 0$ and $\langle V_2(0)V_2(z)\rangle \sim \frac{1}{z^{2h_2}}$

Jordan blocks will lead to correlation functions that contain logarithms (this is allowed by scale invariance).

Jordan block of rank 2: $L_0 = \begin{pmatrix} h & 1 \\ 0 & h \end{pmatrix}$ in a basis of V_1, V_2

$$V_1 \xrightarrow[L_0]{} V_2$$

Correlation functions:

$\langle V_1(0)V_1(z)\rangle = 0$, $\langle V_1(0)V_2(z)\rangle \sim \frac{\beta}{z^{2h}}$ and $\langle V_2(0)V_2(z)\rangle \sim \frac{\beta \log(z)}{z^{2h}}$.

Logarithmic CFT beginnings: V. Gurarie (1993).

Example of lattice determination of β: J. Dubail, J. L. Jacobsen, Hubert Saleur, arXiv:1001.1151

We can use the Koo-Saleur generators to distinguish between states in complication 1, and find out if L_0 mixes states in complication 2.

- Find eigenstates of $\mathcal{H}[N]$ at system size N that will correspond to the desired states at $N \to \infty$. Note:

$$\overset{(-)}{\mathcal{L}}_0[N] = \frac{N}{2\pi} \sum_{j=1}^{N} \frac{1}{2}\left(h_j \pm p_j \right) + \frac{c}{24} = \frac{N}{2\pi}(\mathcal{H} \pm \mathcal{P}) + \frac{c}{24}$$

  The conformal weights are thus directly related to energy and lattice momentum.

- Act with $\overset{(-)}{\mathcal{L}}_n[N]$, $n \neq 0$ for increasingly large N to form matrix elements such as $\langle V_A | \mathcal{L}_n | V_B \rangle$, then extrapolate to $N \to \infty$ to deduce the action of the corresponding raising/lowering operator L_n.

To reach large N: Bethe ansatz, Quantum Inverse Scattering Method.

But first...

Before using the Koo-Saleur generators we need to check: do they in fact converge to the Virasoro generators?

Convergence at $N \to \infty$

$$\mathcal{L}_n[N] \xrightarrow{\quad ? \quad} L_n$$

Looking at *matrix elements* of $\mathcal{L}_n[N]$ i.e. we can show at most *weak* convergence. But *do* we have weak convergence in general?

$$\mathcal{L}_n[N] \overset{?}{\dashrightarrow} L_n$$

Artefacts of the lattice discretization and the scaling limit

For a system of finite size N we cannot accommodate arbitrarily large lattice momenta. Conversely, high energy states for a finite-sized lattice will not correspond to states in the continuum theory. *For any given N we want to restrict to low energy states.* ("Scaling states".) This restriction will crucially affect products of $\mathcal{L}_n$, where we need to use a double-limit procedure called the *scaling limit*.

Example: Measuring the central charge c through $\langle TT \rangle \propto$

$T = L_{-2}\mathbf{1}$ and $\langle \mathbf{1}|L_2 L_{-2}|\mathbf{1}\rangle = c/2$. The state $|\mathbf{1}\rangle$ is a scaling state, however

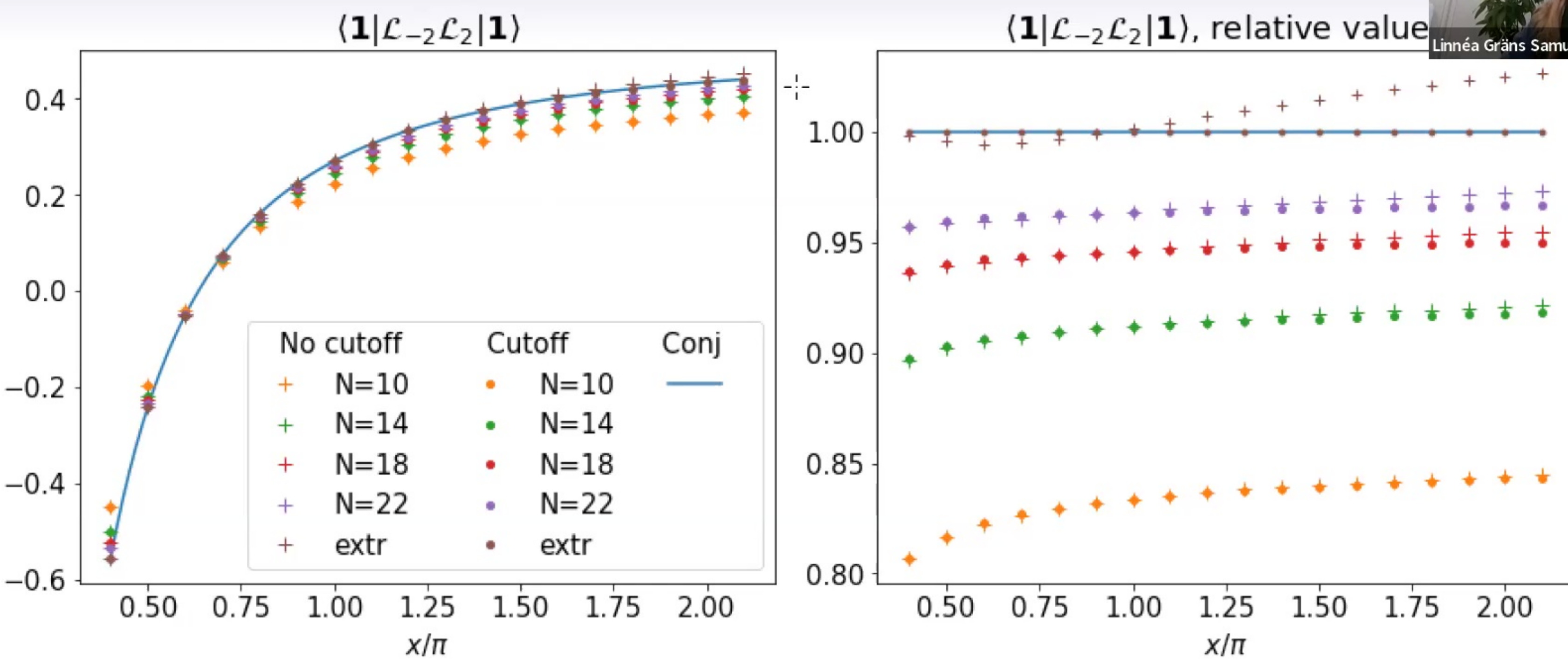
$$\langle \mathbf{1}|\mathcal{L}_2\mathcal{L}_{-2}|\mathbf{1}\rangle = \sum_{j=1}^{\#\text{states}} \langle \mathbf{1}|\mathcal{L}_2|v_{(j)}\rangle\langle v_{(j)}|\mathcal{L}_{-2}|\mathbf{1}\rangle$$

includes intermediate unwanted high-energy states.

On their own, unwanted matrix elements $\langle v_{(j)}|\mathcal{L}_{-2}|\mathbf{1}\rangle$ converge to zero. However, # high energy states grows rapidly and the total unwanted contribution is finite. We consider instead

$$\sum_{j=1}^{\text{cutoff}} \langle \mathbf{1}|\mathcal{L}_2|v_{(j)}\rangle\langle v_{(j)}|\mathcal{L}_{-2}|\mathbf{1}\rangle$$

We can only send cutoff to ∞ *after* $N \to \infty$.



Convergence of $\sum_{j=1}^{\text{cutoff}} \langle \mathbf{1}|\mathcal{L}_2|v_{(j)}\rangle\langle v_{(j)}|\mathcal{L}_{-2}|\mathbf{1}\rangle \to c/2$.

Effect of no cutoff is the largest at large c, disappears at $x = 1, 2, 3$. Same effect for the 6-vertex, loop, and RSOS models (at x integer for RSOS). Same effect with modified version suggested by Shokrian-Zini and Wang in `arXiv:1706.08497`.

Convergence at $N \to \infty$

$$\mathcal{L}_n[N] \xrightarrow{\quad ? \quad} L_n$$

Looking at *matrix elements* of $\mathcal{L}_n[N]$ i.e. we can show at most *weak* convergence. But *do* we have weak convergence in general?

$$\mathcal{L}_n[N] \dashrightarrow^{?} L_n$$

Double-limit procedure $\Rightarrow$ "Scaling-weak convergence"

$$\mathcal{L}_n[N] \cdots\cdots\rightarrow L_n$$

Interestingly, even without the cutoff in the double-limit, results would be "almost right"...

...with no cutoff, commutators only have central term wr

Virasoro: $[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{n+m,0}$

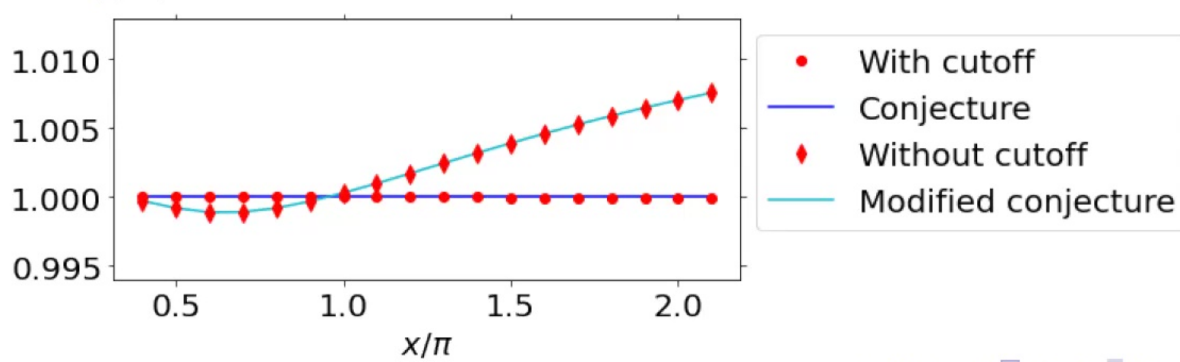
Finding expectation values of Temperley-Lieb operators
$\Rightarrow$ predicting modified relation without cutoff:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3c^* - mc)\delta_{m+n,0}$$

$$c^* = -\frac{24\gamma^3 I_0}{\pi^2 \sin^2\gamma} + \frac{48\gamma^3}{\pi^2}I_1 \text{ with } I_n = \int_{-\infty}^{\infty} t^{2n}\frac{\sinh(\pi-\gamma)t}{\sinh \pi t \cosh \gamma t}\,dt.$$

$c = c^*$ only holds for $x = 1, 2, 3$ ($c = 1 - 6\frac{1}{6x(x+1)}$, $\gamma = \frac{\pi}{x+1}$)

$\langle \mathbf{1}|[\mathcal{L}_2, \mathcal{L}_{-2}]|\mathbf{1}\rangle$ for XXZ spin chains, plotting values divided by $c/2$:

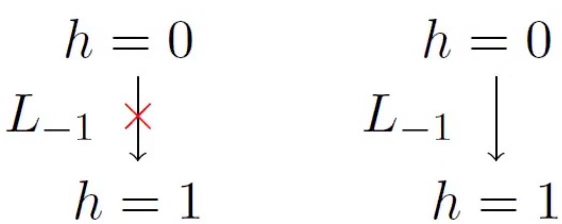


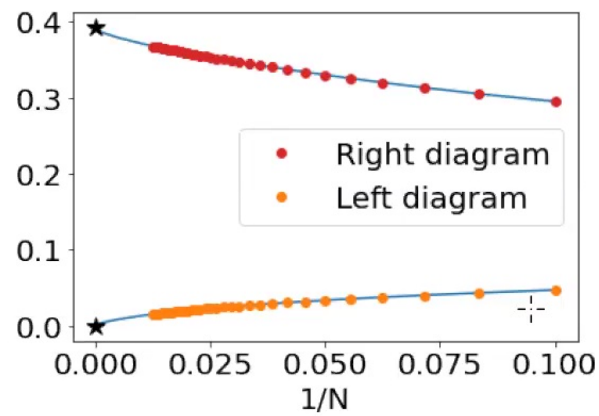
Part V: Results about the loop model and the 6-vertex model

Complication 1: Check if only the identity has $h = \bar{h} = 0$

- Act with $\mathcal{L}_{-1}[N]$ on state with $h = \bar{h} = 0$ (weights of the identity state) and project on state with $h = 1, \bar{h} = 0$.
- Extrapolate $\langle h = 1|\mathcal{L}_{-1}|h = 0\rangle$ to $N \to \infty$.

Possible outcomes:





Loop model: only left diagram is present.
6-vertex model: both are present.

Recalling $L_{-1} = \partial_z$, the applicability of $\partial_z \langle V(z) \prod_i V_i \rangle = 0$ depends only on V having weights $h = \bar{h} = 0$ in the loop model, but not in the 6-vertex model.

Complication 2: Check if L_0 will mix states or not.

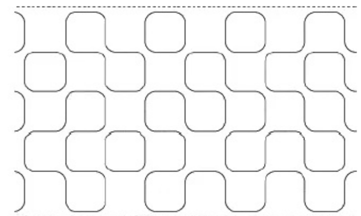
Loop model:

L_0 has rank-2 Jordan blocks. In a basis of V_1, V_2 we find $L_0 = \begin{pmatrix} h & 1 \\ 0 & h \end{pmatrix}$ and we expect

$\langle V_1(0)V_1(z)\rangle = 0$, $\langle V_1(0)V_2(z)\rangle \sim \frac{\beta}{z^{2h}}$

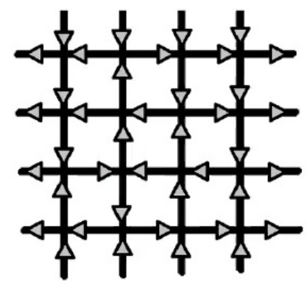
and $\langle V_2(0)V_2(z)\rangle \sim \frac{\beta \log(z)}{z^{2h}}$.

We say that it is a logarithmic CFT.



6-vertex model:

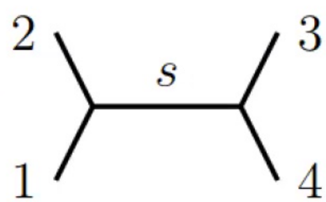
L_0 is diagonalizable. For a state $L_0 V = hV$ we expect $\langle V(0)V(z)\rangle \sim \frac{1}{z^{2h}}$



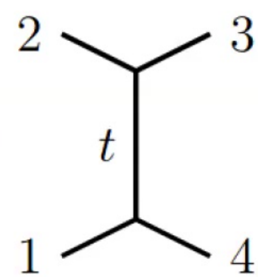
Relevance in bootstrap of Q-state Potts and $O(n)$ models:

The results from the loop representation show that we have logarithmic representations of the Virasoro algebra. We must therefore consider logarithmic conformal blocks in the crossing symmetry equation

$$\sum_{\Delta_s \in S} C_{12s} C_{s34}$$



$$= \sum_{\Delta_t \in S} C_{23t} C_{t41}$$

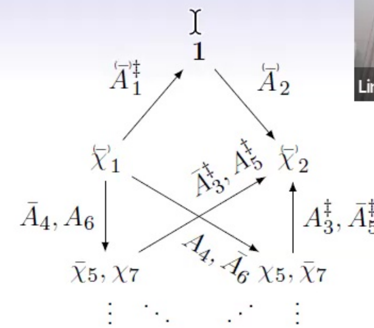


See recent paper:
LGS, R. Nivesvivat; J. L. Jacobsen, S. Ribault, H. Saleur, arXiv:2111.01106

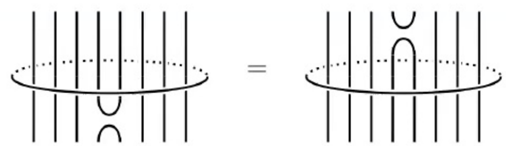
Summary

- Koo-Saleur generators: discretization of the Virasoro generators. Write $\mathcal{L}_n[N]$ as function of generators of the lattice Temperley-Lieb algebra
- Application: non-unitary CFT, where the representation theory of the conformal algebra is more complicated
- "Scaling-weak" convergence: need double-limit procedure with an energy cutoff inside products of $\mathcal{L}_n[N]$, or the central term comes out wrong in commutators
- Both the loop model and the 6-vertex model are non-unitary, yet behave differently. In loop model: L_0 has Jordan blocks, logarithmic CFT. In 6-vertex model: find state with $h = \bar{h} = 0$ that is not the identity (vacuum). (And similarly for other states that have some specific values of $h, \bar{h}$).

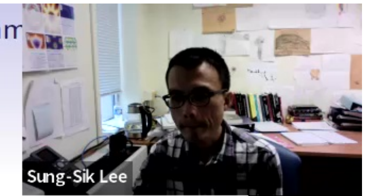
Future directions



- Rational values of c, where the modules are more complicated. Example at $c = 0$:
- RSOS models, anyon chains. Implementation of Koo-Saleur generators in A_n type RSOS models currently under-way.
- Application: identification of topological defects. Simple example: Y



  Y is in the the center of Temperley-Lieb, so $[\mathcal{L}_n[N], Y] = 0$, meaning that it is topological (can be "pulled across" the stress-energy tensor) already on the lattice.
- Better understanding of the results about convergence and the appearance of c^*.



Questions