Title: Dynamics of spinning compact binaries: synergies between post-Newtonian and self-force approaches

Speakers: Mohammed Khalil

Series: Strong Gravity

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Abstract: Accurate waveform models are crucial for gravitational-wave (GW) data analysis, and since numerical-relativity waveforms are computationally expensive, it is important to improve the analytical approximations for the binary dynamics. The post-Newtonian (PN) approximation is most suited for describing the inspiral of comparable-mass binaries, which are the main sources for ground-based GW detectors. In this talk, I discuss a method for deriving PN results valid for arbitrary mass ratios from first-order self-force results, by exploiting the simple mass dependence of the scattering angle in the post-Minkowskian expansion. I present results for the spin-orbit dynamics up to the fourth-subleading PN order (5.5PN) and the spin-spin dynamics up to the third-subleading PN order (5PN). I also discuss implications for the first law of binary mechanics.

Zoom Link: https://pitp.zoom.us/j/92861625861?pwd=cHpXUlM1d01pc09mNGhhQVZxRHBiQT09

Dynamics of spinning compact binaries: synergies between post-Newtonian and self-force approaches

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Perimeter Institute
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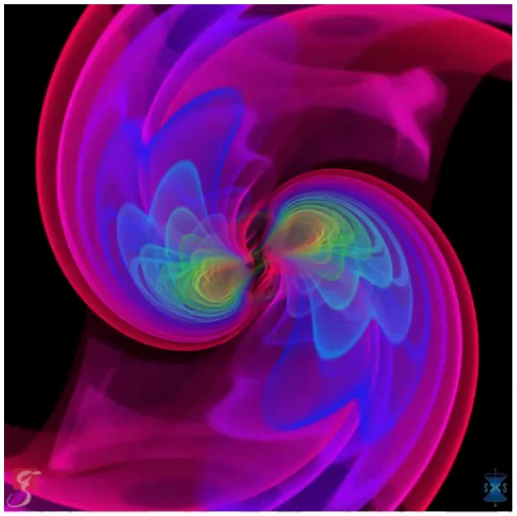
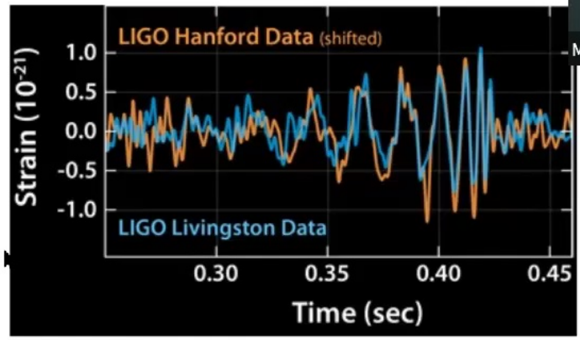
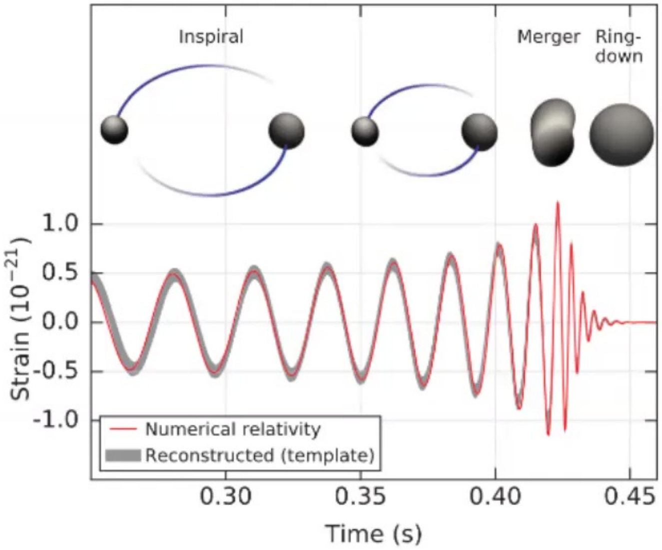
Introduction

- Accurate waveform models are crucial in searching for gravitational-wave signals and inferring their parameters.
- For example, measuring spin magnitude and tilt helps in identifying formation channels.



[Ossokine, Buonanno, SXS]





[Abbott+ 2016]

Analytical approximation methods

- Post-Newtonian (PN)

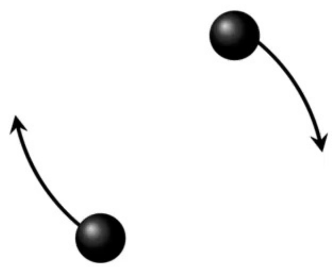
$$\frac{v^2}{c^2} \sim \frac{GM}{rc^2} \ll 1$$

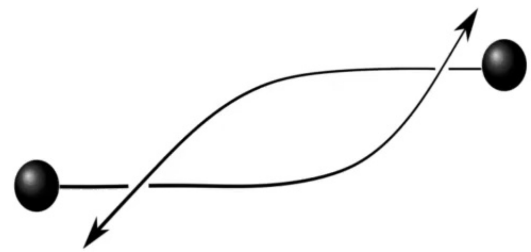
Post-Minkowskian (PM)

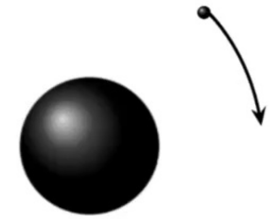
$$\frac{GM}{rc^2} \ll 1$$

Self-Force (SF)

$$\frac{m_1}{m_2} \ll 1$$







- Combining these methods allows deriving PN results for arbitrary-mass ratios from self-force results at first order in the mass ratio. (The "Tutti Frutti" method)
  - 5PN (except 2 coefficients)
    [Bini, Damour, Geralico 1909.02375, 2003.11891]
  - 6PN (except 4 coefficients)
    [Bini, Damour, Geralico 2004.05407, 2007.11239]

Contents of this talk

- Overview of the method, and the mass dependence of the scattering angle
- Spin-orbit (SO) dynamics up to fourth-subleading PN order (5.5PN)
- Spin_1-spin_2 and spin-squared dynamics at third-subleading PN order (5PN)
- First law of binary mechanics with spin quadrupole

Based on
Antonelli, Kavanagh, **MK**, Steinhoff, Vines 2003.11391, 2010.02018
MK 2110.12813
MK, Kavanagh, Steinhoff, Vines (in prep.)

PN results for the conservative dynamics

PN order	0	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5	6	6.5
no spin	N	1PN		2PN		3PN		4PN		5PN		6PN	
spin-orbit			LO SO		NLO SO		N2LO SO		N3LO SO		N4LO SO		
spin^2				LO S2		NLO S2		N2LO S2		N3LO S2			
spin^3							LO S3		NLO S3		N2LO S3		
spin^4								LO S4		NLO S4			
spin^5											LO S5		NLO S5
spin^6												LO S6	

need up to

- 1PM
- 2PM
- 3PM
- 4PM
- 5PM
- 6PM
- 7PM

credit: Justin Vines

PN results for the conservative dynamics

PN order	0	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5	6	6.5
no spin	N	1PN		2PN		3PN		4PN		5PN		6PN	
spin-orbit			LO SO		NLO SO		N2LO SO		N3LO SO		N4LO SO		
spin^2				LO S2		NLO S2		N2LO S2		N3LO S2			
spin^3							LO S3		NLO S3		N2LO S3		
spin^4								LO S4		NLO S4			
spin^5											LO S5		NLO S5
spin^6												LO S6	

need up to

- 1PM
- 2PM
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- 5PM
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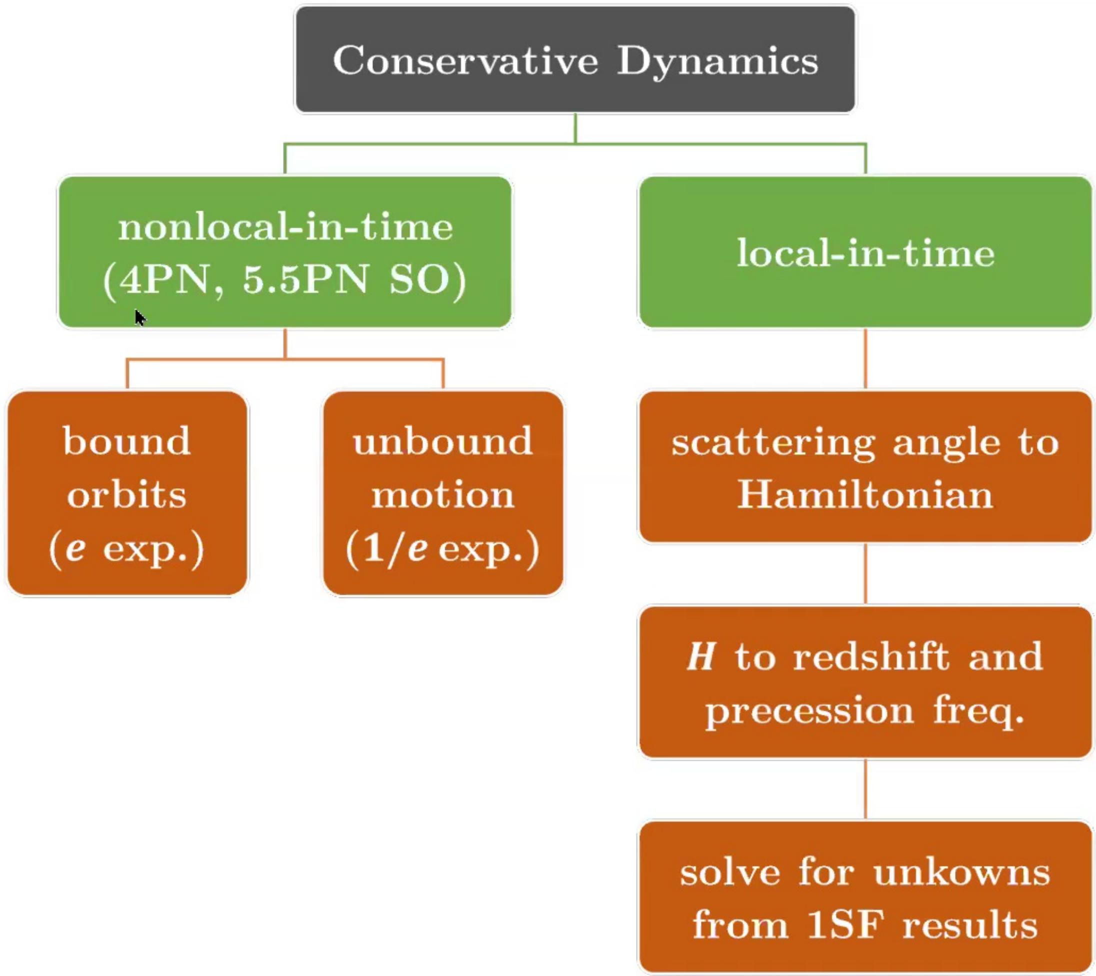
credit: Justin Vines

Overview of the Tutti Frutti method



Nonlocal part of the harmonic-coordinate Hamiltonian

- Total action split into local and nonlocal-in-time pieces $S_{\text{tot}}^{\text{nPN}} = S_{\text{loc}}^{\text{nPN}} + S_{\text{nonloc}}^{\text{nPN}}$

[Damour, Jaranowski, Schäfer 1401.4548, 1502.07245]

$$S_{\text{nonloc}}^{\text{LO}} = \frac{GM}{c^3} \int \mathrm{d}t \, \text{Pf}_{2s/c} \int \frac{\mathrm{d}t'}{|t-t'|} \mathcal{F}^{\text{LO}}(t,t') \equiv - \int \mathrm{d}t \, \delta H_{\text{nonloc}}^{\text{LO}}(t)$$

- Time-symmetric GW energy flux

$$\mathcal{F}^{\text{LO}}(t,t') = \frac{G}{c^5} \left[\frac{1}{5} I_{ij}^{(3)}(t) I_{ij}^{(3)}(t') + \frac{16}{45c^2} J_{ij}^{(3)}(t) J_{ij}^{(3)}(t') \right]$$

- Nonlocal Hamiltonian (define $\tau \equiv t' - t$)

$$\delta H_{\text{nonloc}}^{\text{LO}}(t) = -\frac{GM}{c^3} \text{Pf}_{2s/c} \int \frac{\mathrm{d}\tau}{|\tau|} \mathcal{F}^{\text{LO}}(t, t+\tau) + 2\frac{GM}{c^3} \mathcal{F}^{\text{LO}}(t,t) \ln\left(\frac{r}{s}\right)$$

- The length scale s is chosen to be the harmonic coordinates r to remove $\ln(r)$.

- Integral is computed in small/large eccentricity expansion for bound/unbound orbits.

[Bini, Damour, Geralico 2003.11891, 2007.11239], [**MK** 2110.12813]

Nonlocal part of the harmonic-coordinate Hamiltonian

$$
\begin{aligned}
\langle \delta H^{\text{LO SO}}_{\text{nonloc}} \rangle = \frac{\nu^2 \delta \chi_A}{a_r^{13/2}} \Bigg\{ & \frac{584}{15} \ln a_r - \frac{64}{5} - \frac{464}{3} \ln 2 - \frac{1168}{15} \gamma_E \\
& + e_t^2 \left[\frac{2908}{5} \ln a_r - \frac{5816}{5} \gamma_E + \frac{2172}{5} - \frac{3304}{15} \ln 2 - \frac{10206}{5} \ln 3 \right] \\
& + e_t^4 \left[\frac{43843}{15} \ln a_r - \frac{87686}{15} \gamma_E + \frac{114991}{30} - \frac{201362}{5} \ln 2 + \frac{48843}{4} \ln 3 \right] \\
& + e_t^6 \left[\frac{55313}{6} \ln a_r - \frac{55313}{3} \gamma_E + \frac{961807}{60} + \frac{6896921}{45} \ln 2 - \frac{3236031}{160} \ln 3 - \frac{24296875}{288} \ln 5 \right] \\
& + e_t^8 \left[\frac{134921}{6} \ln a_r - \frac{134921}{3} \gamma_E + \frac{135264629}{2880} - \frac{94244416}{135} \ln 2 + \frac{12145234375}{27648} \ln 5 - \frac{1684497627}{5120} \ln 3 \right] \Bigg\} \\
+ \frac{\nu^2 \chi_S}{a_r^{13/2}} \Bigg\{ & -\frac{64}{5} + \frac{32\nu}{5} + \left(\frac{896\nu}{15} - \frac{1168}{15} \right) \gamma_E + \left(\frac{576\nu}{5} - \frac{464}{3} \right) \ln 2 + \left(\frac{584}{15} - \frac{448\nu}{15} \right) \ln a_r \\
& + e_t^2 \Bigg[\frac{2172}{5} - \frac{4412\nu}{15} + \left(\frac{4216\nu}{5} - \frac{5816}{5} \right) \gamma_E + \left(\frac{5192\nu}{15} - \frac{3304}{15} \right) \ln 2 + \left(\frac{6561\nu}{5} - \frac{10206}{5} \right) \ln 3 \\
& \quad + \left(\frac{2908}{5} - \frac{2108\nu}{5} \right) \ln a_r \Bigg] + e_t^4 \Bigg[\frac{114991}{30} - \frac{38702\nu}{15} + \left(\frac{62134\nu}{15} - \frac{87686}{15} \right) \gamma_E \\
& \quad + \left(\frac{386414\nu}{15} - \frac{201362}{5} \right) \ln 2 + \left(\frac{48843}{4} - \frac{28431\nu}{4} \right) \ln 3 + \left(\frac{43843}{15} - \frac{31067\nu}{15} \right) \ln a_r \Bigg] \\
& + e_t^6 \Bigg[\frac{961807}{60} - \frac{215703\nu}{20} + \left(\frac{193718\nu}{15} - \frac{55313}{3} \right) \gamma_E + \left(\frac{6896921}{45} - \frac{12343118\nu}{135} \right) \ln 2 \\
& \quad + \left(\frac{3768201\nu}{320} - \frac{3236031}{160} \right) \ln 3 + \left(\frac{92421875\nu}{1728} - \frac{24296875}{288} \right) \ln 5 + \left(\frac{55313}{6} - \frac{96859\nu}{15} \right) \ln a_r \Bigg] \\
& + e_t^8 \Bigg[\frac{135264629}{2880} - \frac{45491177\nu}{1440} + \left(\frac{93850\nu}{3} - \frac{134921}{3} \right) \gamma_E + \left(\frac{118966123\nu}{270} - \frac{94244416}{135} \right) \ln 2 \\
& \quad + \left(\frac{537837489\nu}{2560} - \frac{1684497627}{5120} \right) \ln 3 + \left(\frac{12145234375}{27648} - \frac{3790703125\nu}{13824} \right) \ln 5 \\
& \quad + \left(\frac{134921}{6} - \frac{46925\nu}{3} \right) \ln a_r \Bigg] \Bigg\} + \mathcal{O}\left(e_t^{10}\right),
\end{aligned}
$$

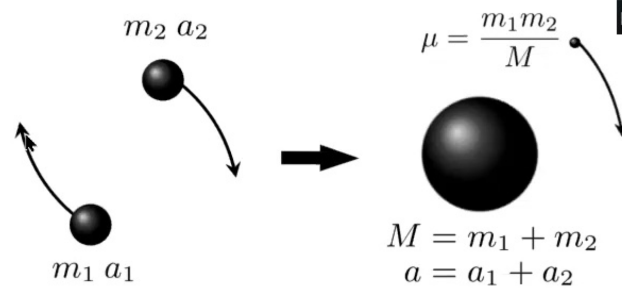
[MK 2110.12813]

Nonlocal part of the EOB Hamiltonian

- Effective-one-body (EOB) Hamiltonian

$$H_{\text{EOB}} = M\sqrt{1 + 2\nu\left(\frac{H_{\text{eff}}}{\mu} - 1\right)}$$

[Buonanno, Damour 9811091]



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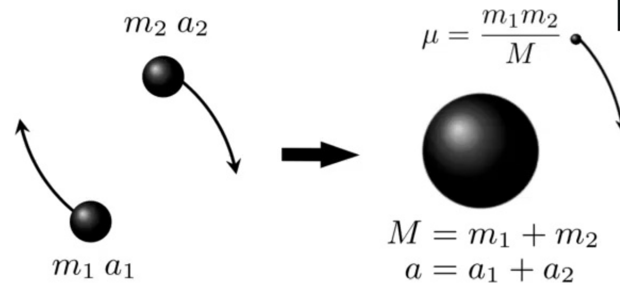


- SO part of the effective Hamiltonian

$$H_{\text{eff}}^{\text{SO}} = \frac{1}{c^3 r^3}\boldsymbol{L}\cdot\left[g_S(r,p_r)\boldsymbol{S} + g_{S^*}(r,p_r)\boldsymbol{S}^*\right]$$

$$g_S = 2 + \cdots + \frac{1}{c^8}\left(g_S^{\text{5.5PN,loc}} + g_S^{\text{5.5PN,nonloc}}\right)$$

$$g_{S^*} = \frac{3}{2} + \cdots + \frac{1}{c^8}\left(g_{S^*}^{\text{5.5PN,loc}} + g_{S^*}^{\text{5.5PN,nonloc}}\right)$$

- At SO level, aligned spins $\leftrightarrow$ generic spins
- Write nonlocal part of the gyro-gravitomagnetic factors with unknown coefficients in p_r expansion, and match the average

$$\left\langle \delta H_{\text{nonloc}}^{\text{harmonic}} \right\rangle = \left\langle \delta H_{\text{nonloc}}^{\text{EOB}} \right\rangle$$

Nonlocal part of the EOB Hamiltonian

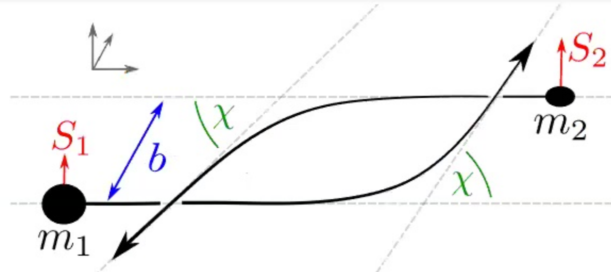
$$g_S^{5.5\text{PN,nonloc}} = 2\nu\left\{\left(\frac{292}{15}\ln r - \frac{32}{5} - \frac{584}{15}\gamma_E - \frac{232}{3}\ln 2\right)\frac{1}{r^4} + \left(\frac{12782}{15} - 104\gamma_E + \frac{32744}{15}\ln 2 - \frac{11664}{5}\ln 3 + 52\ln r\right)\frac{p_r^2}{r^3}\right.$$
$$+\left(\frac{12503}{15} - \frac{635456}{9}\ln 2 + \frac{218943}{5}\ln 3\right)\frac{p_r^4}{r^2} + \left(\frac{38246}{25} + \frac{176799232}{225}\ln 2 - \frac{2517237}{10}\ln 3 - \frac{3015625}{18}\ln 5\right)\frac{p_r^6}{r}$$
$$\left.+\left(\frac{503099}{350} - \frac{898982848}{189}\ln 2 + \frac{6352671875}{3024}\ln 5 - \frac{31129029}{400}\ln 3\right)p_r^8 + \mathcal{O}\left(p_r^{10}\right)\right\},$$
$$g_{S^*}^{5.5\text{PN,nonloc}} = \frac{3}{2}\nu\left\{\left(16\ln r - \frac{32}{5} - 32\gamma_E - \frac{2912}{45}\ln 2\right)\frac{1}{r^4} + \left(\frac{35024}{45} - \frac{1024\gamma_E}{15} + \frac{93952}{45}\ln 2 - \frac{10692}{5}\ln 3 + \frac{512}{15}\ln r\right)\frac{p_r^2}{r^3}\right.$$
$$+\left(\frac{9232}{15} - \frac{2978624}{45}\ln 2 + \frac{206064}{5}\ln 3\right)\frac{p_r^4}{r^2} + \left(\frac{33048}{25} + \frac{1497436672}{2025}\ln 2 - \frac{1199934}{5}\ln 3 - \frac{12593750}{81}\ln 5\right)\frac{p_r^6}{r}$$
$$\left.+\left(\frac{651176}{525} - \frac{9076395968}{2025}\ln 2 + \frac{2226734375}{1134}\ln 5 - \frac{697653}{14}\ln 3\right)p_r^8 + \mathcal{O}\left(p_r^{10}\right)\right\}.$$

$$g_S = 2 + \cdots + \frac{1}{c^8}\left(g_S^{\text{5.5PN,loc}} + g_S^{\text{5.5PN,nonloc}}\right)$$
$$g_{S^*} = \frac{3}{2} + \cdots + \frac{1}{c^8}\left(g_{S^*}^{\text{5.5PN,loc}} + g_{S^*}^{\text{5.5PN,nonloc}}\right)$$

- At SO level, aligned spins $\leftrightarrow$ generic spins
- Write nonlocal part of the gyro-gravitomagnetic factors with unknown coefficients in p_r expansion, and match the average

$$\left\langle \delta H_{\text{nonloc}}^{\text{harmonic}} \right\rangle = \left\langle \delta H_{\text{nonloc}}^{\text{EOB}} \right\rangle$$

Mass dependence of the impulse



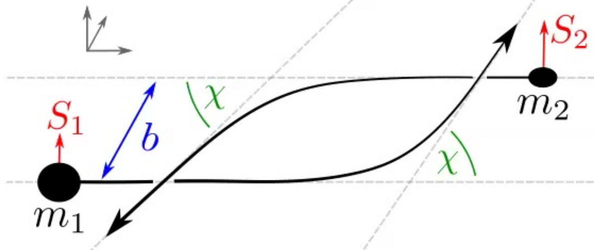
- Based on the structure of the PM expansion, Poincaré symmetry, and dimensional analysis, the magnitude of the impulse has the mass dependence [Damour 1912.02139]
  see also [Vines, Steinhoff, Buonanno 1812.00956], [Bern+ 1908.01493], [Kälin, Porto 1910.03008]

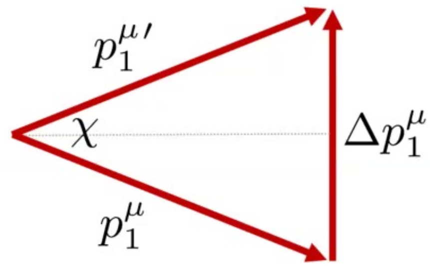
$$
\begin{aligned}
\mathsf{Q} &\equiv (\Delta p_{1\mu} \Delta p_1^\mu)^{1/2} \\
&= \frac{2G m_1 m_2}{b} \Big[\mathsf{Q}^{\rm 1PM}(\gamma) \\
&\quad + \frac{G}{b} \left(m_1 \mathsf{Q}^{\rm 2PM}_{m_1} + m_2 \mathsf{Q}^{\rm 2PM}_{m_2} \right) \\
&\quad + \frac{G^2}{b^2} \left(m_1^2 \mathsf{Q}^{\rm 3PM}_{m_1^2} + m_1 m_2 \mathsf{Q}^{\rm 3PM}_{m_1 m_2} + m_2^2 \mathsf{Q}^{\rm 3PM}_{m_2^2} \right) + \dots \Big]
\end{aligned}
$$

- Extending to spin [Antonelli, Kavanagh, **MK**, Steinhoff, Vines 2010.02018]

$$
\mathsf{Q}^{\cdots}_{\cdots}(\gamma) \to \mathsf{Q}^{\cdots}_{\cdots}\left(\gamma, \frac{a_1}{b}, \frac{a_2}{b}\right), \qquad a_i = \frac{S_i}{m_i}
$$

Relation between scattering angle and impulse





- Scattering angle is related to Q

$$\sin\frac{\chi}{2} = \frac{\mathsf{Q}}{2P_{\text{c.m}}}$$

$$P_{\text{c.m}} = \frac{m_1 m_2}{E}\sqrt{\gamma^2 - 1}, \qquad \gamma \equiv -u_{10}\cdot u_{20} = \frac{1}{\sqrt{1-v^2}} = \frac{E_{\text{eff}}}{\mu},$$

$$E = M\sqrt{1 + 2\nu\,(\gamma - 1)}$$

- Scattering angle scaled by $E/m_1 m_2$ has same mass dependence as Q, i.e.

$$\frac{M}{E}\chi \quad \xleftrightarrow[\text{dependence}]{\text{mass}} \quad \frac{\mathsf{Q}}{\mu}$$

Mass-ratio dependence of the scattering angle

- A homogeneous polynomial in $m_1/M,\ m_2/M$ of degree n can be written as a polynomial in the symmetric mass ratio $\nu \equiv \mu/M$ of degree $\lfloor n/2 \rfloor$

3PM: $$cm_1^2 + dm_1m_2 + cm_2^2 = M^2[c + (d-2c)\nu]$$

4PM: $$cm_1^3 + dm_1^2m_2 + dm_1m_2^2 + cm_2^3 = M^3[c + (d-3c)\nu]$$

- Mass-ratio dependence of the scattering angle

$$\begin{aligned}\frac{M}{E}\chi = {} & \frac{GM}{b}\mathsf{X}_1^{\nu^0} + \left(\frac{GM}{b}\right)^2 \mathsf{X}_2^{\nu^0} \\ & + \left(\frac{GM}{b}\right)^3 \left[\mathsf{X}_3^{\nu^0} + \nu \mathsf{X}_3^{\nu}\right] + \left(\frac{GM}{b}\right)^4 \left[\mathsf{X}_4^{\nu^0} + \nu \mathsf{X}_4^{\nu}\right] \\ & + \left(\frac{GM}{b}\right)^5 \left[\mathsf{X}_5^{\nu^0} + \nu \mathsf{X}_5^{\nu} + \nu^2 \mathsf{X}_5^{\nu^2}\right] + \dots\end{aligned}$$

- Spin leads to a dependence on the antisymmetric mass ratio $\delta \equiv (m_2 - m_1)/M$

3PM: $$a_1\left[c_1m_1^2 + c_2m_1m_2 + c_3m_2^2\right] = M^2a_1\left[\mathsf{X}_3^{\nu^0} + \mathsf{X}_3^{\delta}\delta + \mathsf{X}_3^{\nu}\nu\right]$$

4PM: $$a_1\left[c_1m_1^3 + c_2m_1^2m_2 + c_3m_1m_2^2 + c_4m_2^3\right] = M^3a_1\left[\mathsf{X}_4^{\nu^0} + \mathsf{X}_4^{\delta}\delta + \mathsf{X}_4^{\nu}\nu + \mathsf{X}_4^{\nu\delta}\nu\delta\right]$$

Spin-orbit scattering angle through 5.5PN

- Relate local Hamiltonian to scattering angle [Damour 1609.00354]

$$\chi_{\text{loc}} = -2\int_{r_0}^{\infty} \frac{\partial p_r(H_{\text{loc}}, L, r)}{\partial L}\mathrm{d}r - \pi$$

- Replace (E, L) with (v, b) [Vines 1709.06016]
- Match scattering angle calculated from H_{loc} to the ansatz

$$\begin{aligned}
\frac{M}{E}\chi_{\text{loc}}^{\text{SO}} = \frac{a_1}{b}\Bigg\{ &\left(\frac{GM}{v^2 b}\right)(-4v) + \pi\left(\frac{GM}{v^2 b}\right)^2\left[\cdots + \left(-\frac{21}{4} + \frac{3\delta}{4}\right)v^3\right] \\
&+ \left(\frac{GM}{v^2 b}\right)^3\left[\cdots + (-50 + 10\delta + \boldsymbol{X_{35}^{\nu}}\nu)v^5 + \boldsymbol{X_{37}^{\nu}}\nu v^7 + \boldsymbol{X_{39}^{\nu}}\nu v^9\right] \\
&+ \pi\left(\frac{GM}{v^2 b}\right)^4\left[\cdots + \left(-\frac{1365}{32} + \frac{315\delta}{32} + \boldsymbol{X_{47}^{\nu}}\nu + \boldsymbol{X_{47}^{\delta\nu}}\delta\nu\right)v^7 + \left(\boldsymbol{X_{49}^{\nu}}\nu + \boldsymbol{X_{49}^{\delta\nu}}\delta\nu\right)v^9\right] \\
&+ \left(\frac{GM}{v^2 b}\right)^5\left[\cdots + \left(-336 + 84\delta + \boldsymbol{X_{59}^{\nu}}\nu + \boldsymbol{X_{59}^{\delta\nu}}\delta\nu + \boldsymbol{X_{59}^{\nu^2}}\nu^2\right)v^9\right]\Bigg\} + 1 \leftrightarrow 2
\end{aligned}$$

(...) known from test scattering angle [Bini ,Geralico, Vines 1707.09814], and lower PM orders,
1 unknown at 3.5PN, **3 unknowns at 4.5PN**, **6 unknowns at 5.5PN**

Redshift and spin-precession frequency

- First law of binary mechanics for eccentric orbits, aligned spins, to linear order in s

$$\mathrm{d}E = \Omega_r \mathrm{d}I_r + \Omega_\phi \mathrm{d}L + \textstyle\sum_i \left(z_\mathrm{i} \mathrm{d}m_\mathrm{i} + \Omega_{S_\mathrm{i}} \mathrm{d}S_\mathrm{i}\right)$$

[Le Tiec, Blanchet, Whiting 1111.5378], [Blanchet, Buonanno, Le Tiec 1211.1060]
[Le Tiec 1506.05648], [Antonelli, Kavanagh, **MK**, Steinhoff, Vines 2010.02018]

- Redshift and spin-precession frequency from (local + nonlocal) Hamiltonian

$$z_\mathrm{i} = \left\langle \frac{\partial H}{\partial m_\mathrm{i}} \right\rangle, \qquad \Omega_{S_\mathrm{i}} = \left\langle \frac{\partial H}{\partial S_\mathrm{i}} \right\rangle$$

- Express z_1 and Ω_{S_1} in terms of gauge-independent variables, expand to linear order in the mass ratio, then express in terms of Kerr-geodesic variables (e, u_p).

- Match PN to 1SF results and check that all logarithms and Euler gamma cancel

$$z_1 = \dots + q \left[\dots + s(\dots) + a(\dots + e^2(\dots) + e^4(\dots))\right]$$
$$\Omega_{S_1} = \dots + q \left[\dots + e^2(\dots)\right]$$

q: mass ratio, s: spin of the small body, a: background Kerr spin, e: eccentricity

[Akcay, Barack, Bini, Damour, Dempsey, Detweiler, Dolan, Geralico, Harte, Hopper, Kavanagh, Le Tiec, Munna, Nolan, Ottewill, Pound, Sago, Shah, van de Meent, Warburton, Wardell,...]

Determining unknowns from self-force results

- Scattering angle ansatz

$$\frac{M}{E}\chi_{\mathsf{SO}}^{\mathsf{loc}} = \frac{a_1}{b}\Bigg\{\left(\frac{GM}{v^2 b}\right)(-4v) + \pi\left(\frac{GM}{v^2 b}\right)^2\left[\cdots + \left(-\frac{21}{4} + \frac{3\delta}{4}\right)v^3\right]$$
$$+\left(\frac{GM}{v^2 b}\right)^3\left[\cdots + (-50 + 10\delta + X_{35}^{\nu}\nu)v^5 + X_{37}^{\nu}\nu v^7 + X_{39}^{\nu}\nu v^9\right]$$
$$+\pi\left(\frac{GM}{v^2 b}\right)^4\left[\cdots + \left(-\frac{1365}{32} + \frac{315\delta}{32} + X_{47}^{\nu}\nu + X_{47}^{\delta\nu}\delta\nu\right)v^7 + \left(X_{49}^{\nu}\nu + X_{49}^{\delta\nu}\delta\nu\right)v^9\right]$$
$$+\left(\frac{GM}{v^2 b}\right)^5\left[\cdots + \left(-336 + 84\delta + X_{59}^{\nu}\nu + X_{59}^{\delta\nu}\delta\nu + X_{59}^{\nu^2}\nu^2\right)v^9\right]\Bigg\} + 1 \leftrightarrow 2$$

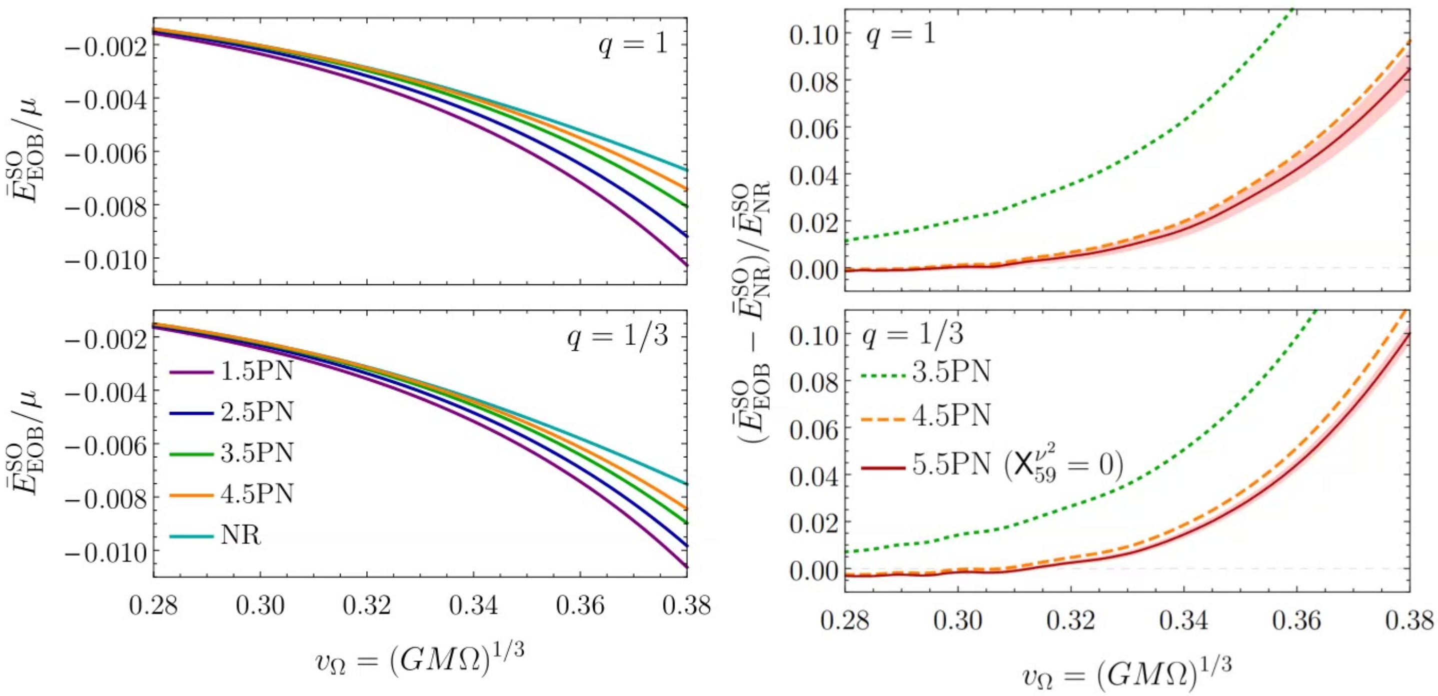
- Need both redshift and spin precession to solve for the unknowns

$$z_1 = \cdots + q\left[\cdots + s(\dots) + a\left(X_{39}^{\nu},\ X_{49}^{\nu} - X_{49}^{\delta\nu},\ X_{59}^{\nu} - X_{59}^{\delta\nu}\right)\right]$$
$$\Omega_{S_1} = \cdots + q\left[X_{39}^{\nu},\ X_{49}^{\nu} + X_{49}^{\delta\nu},\ X_{59}^{\nu} + X_{59}^{\delta\nu}\right]$$

$a_1 \to s$: spin of the small body, $a_2 \to a$: background Kerr spin

Binding energy comparison with numerical relativity

- Binding energy $\bar{E} \equiv H - M$, as a function of orbital frequency Ω
- Spin-orbit contribution $\bar{E}^{\mathsf{SO}}(\nu, a, a) \simeq \frac{1}{2}\left[\bar{E}(\nu, a, a) - \bar{E}(\nu, -a, -a)\right]$
- Remaining unknown $\mathsf{X}_{59}^{\nu^2}$ is varied between 500 and -500



NR data from [Ossokine, Dietrich, Foley, Katebi, Lovelace 1712.06533]

From scattering to bound orbits

- Radial action for bound orbits

$$I_r(E,L) = \frac{1}{2\pi}\oint p_r(r,E,L)\mathrm{d}r$$

- Scattering angle to radial action

$$W(E,L) = \frac{1}{2\pi}\mathrm{Pf}\int_{-\infty}^{\infty} p_r(r,E,L)\mathrm{d}r$$

$$\pi + \chi = -2\pi\frac{\partial W}{\partial L}$$

$$I_r(E,L,a_\mathrm{i}) = W(E,L,a_\mathrm{i}) - W(E,-L,-a_\mathrm{i})$$

- Scattering angle to periastron advance [Kälin, Porto 1910.03008, 1911.09130]

$$\Phi = 2\pi + \Delta\Phi = -2\pi\frac{\partial I_r}{\partial L}$$

$$\Delta\Phi(E,L,a_\mathrm{i}) = \chi(E,L,a_\mathrm{i}) + \chi(E,-L,-a_\mathrm{i})$$

- General map from scattering to bound orbits [Saketh, Vines, Steinhoff, Buonanno 2109.05994]

$$\mathrm{O}_\mathsf{bound}(E,L) = 2\int_{r_\mathsf{min}}^{r_\mathsf{max}} f(r,E,L)\mathrm{d}r, \qquad \mathrm{O}_\mathsf{unbound}(E,L) = 2\int_{r_\mathsf{min}}^{\infty} f(r,E,L)\mathrm{d}r$$

$$\mathrm{O}_\mathsf{bound}(E,L) = \mathrm{O}_\mathsf{unbound}(E,L) + \theta(f)\mathrm{O}_\mathsf{unbound}(E,-L)$$

for $\theta(f) = \pm 1$ if f is odd/even in L

5PN spin$_1$-spin$_2$ dynamics for aligned spins

- Scattering angle ansatz

$$\frac{M}{E}\chi_{a_1 a_2} = \frac{a_1 a_2}{b^2}\Bigg\{ \left(\frac{GM}{v^2 b}\right)(4+4v^2) + \pi\left(\frac{GM}{v^2 b}\right)^2\left(3+\frac{45}{2}v^2+\frac{9}{2}v^4\right)$$
$$+\left(\frac{GM}{v^2 b}\right)^3\left[\cdots + \left(440 + X_{34}^{\nu}\nu\right)v^4 + \left(40 + X_{36}^{\nu}\nu\right)v^6\right]$$
$$+\pi\left(\frac{GM}{v^2 b}\right)^4\left[\cdots + \left(\frac{9975}{16} + X_{46}^{\nu}\nu\right)v^6\right]\Bigg\}$$

1 unknown at 4PN, **2 unknowns at 5PN**

- Need 1SF spin precession to $\mathcal{O}(ae^2)$ to solve for unknowns

$$\Omega_{S_1} = \cdots + q\left[\cdots + a\left(\cdots + e^2(\ldots)\right)\right]$$

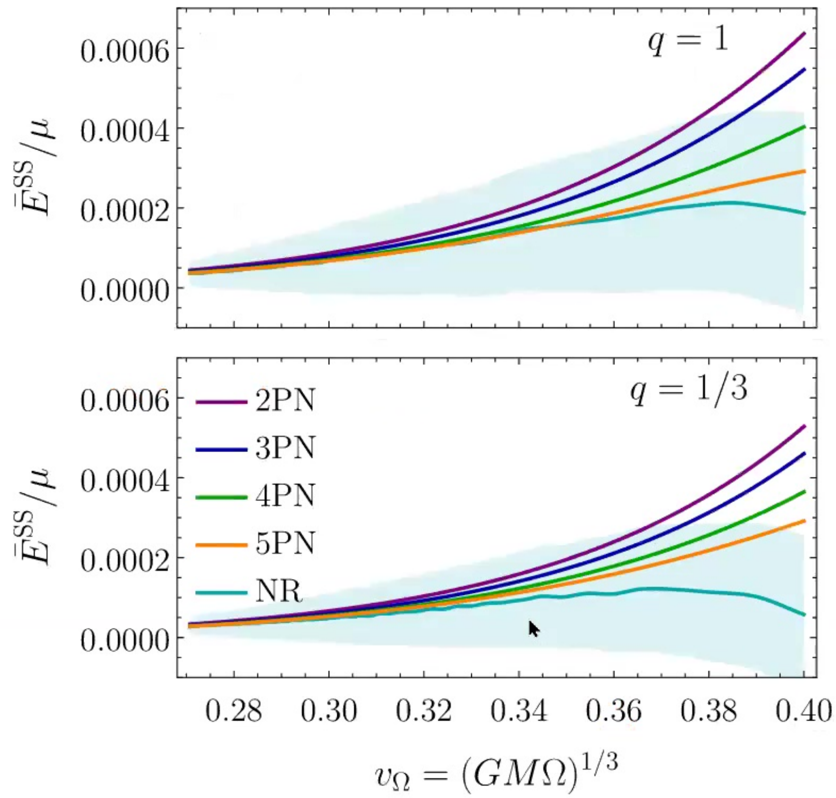
- Solution

$$X_{34}^{\nu} = -66, \qquad X_{36}^{\nu} = -\frac{1093}{5}, \qquad X_{46}^{\nu} = \frac{615}{256}\left(3\pi^2 - 416\right)$$

Binding energy comparison with numerical relativity

spin$_1$-spin$_2$ contribution to the binding energy

$$\bar{E}^{\mathrm{SS}}(\nu, a, a) \simeq \bar{E}(\nu, a, 0) + \bar{E}(\nu, 0, -a) - \bar{E}(\nu, a, -a) - \bar{E}(\nu, 0, 0)$$



NR data from [Ossokine, Dietrich, Foley, Katebi, Lovelace 1712.06533]

5PN spin-squared dynamics for aligned spins

$$\frac{M}{E}\chi_{a^2} = \frac{a_1^2}{b^2}\Bigg\{ \left(\frac{GM}{v^2 b}\right)(2+2v^2) + \pi \left(\frac{GM}{v^2 b}\right)^2 \left[\frac{3}{2} + \left(\frac{87}{8} - \frac{21\delta}{8}\right)v^2 + \left(\frac{69}{32} - \frac{21\delta}{32}\right)v^4\right]$$

$$+ \left(\frac{GM}{v^2 b}\right)^3 \left[\cdots + (220 - 80\delta + \boldsymbol{X_{34}^{\nu}}\nu)\, v^4 + (20 - 8\delta + \boldsymbol{X_{36}^{\nu}}\nu)\, v^6\right]$$

$$+ \pi \left(\frac{GM}{v^2 b}\right)^4 \left[\cdots + \left(\frac{20625}{64} - \frac{8775\delta}{64} + \boldsymbol{X_{46}^{\nu}}\nu + \boldsymbol{X_{46}^{\nu\delta}}\nu\delta\right) v^6\right]\Bigg\} + 1 \leftrightarrow 2$$

- 1SF redshift known to $\mathcal{O}(a^2 e^4)$ and $\mathcal{O}(s^2 e^0)$, but not enough to solve for unknowns
  [Bini, Geralico 1907.11080], [Bini, Geralico, Steinhoff 2003.12887]

$$z_1 = \cdots + q\Big[\cdots + s^2 C_{1ES^2}(\ldots) + \underbrace{a^2\left(\cdots + e^2(\ldots) + e^4(\ldots)\right)}_{\boldsymbol{X_{36}^{\nu}},\ \boldsymbol{X_{46}^{\nu}} - \boldsymbol{X_{46}^{\delta\nu}}}\Big]$$

$$\Omega_{S_1} = \cdots + q\Big[\cdots + \underbrace{s\,(\text{unavailable})}_{\boldsymbol{X_{36}^{\nu}},\ \boldsymbol{X_{46}^{\nu}} + \boldsymbol{X_{46}^{\delta\nu}}}\Big]$$

- Assuming Ω_{S_1} is known to $\mathcal{O}(s)$ for circular orbits in Schwarzschild,

$$\psi_1 \equiv \frac{\Omega_{S_1}}{\Omega_\phi} = \cdots + qs\left(y^{3/2} - 3y^{5/2} + 0 + \mathbf{C}y^{9/2}\right)$$

$$\boldsymbol{X_{36}^{\nu}} = -\frac{1041}{10}, \quad \boldsymbol{X_{46}^{\nu}} = \frac{115245\pi^2}{32768} - \frac{35955}{64} - \frac{15\mathbf{C}}{32}, \quad \boldsymbol{X_{46}^{\nu\delta}} = \frac{945\pi^2}{32768} + \frac{2235}{64} - \frac{15\mathbf{C}}{32}$$

First law of binary mechanics to linear order in spin

[Blanchet, Buonanno, Le Tiec 1211.1060], [Antonelli, Kavanagh, **MK**, Steinhoff, Vines 2010.0201

- Lagrangian depends on dynamical variables X_A and constants C_B

$$\mathcal{S} = \int \mathrm{d}t \, \mathcal{L}(X_A, C_B)$$

- Taking the dynamical variables X_A on-shell (fulfilling their equations of motion)

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial C_B}\delta C_B + \underbrace{\frac{\delta\mathcal{L}}{\delta X_A}}_{=0 \text{ (on-shell)}} \delta X_A + (\text{td})$$

- Performing a transformation of the dynamical variables $X_A \to X'_{A'}$

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial C_B}\delta C_B + \left[\underbrace{\frac{\delta\mathcal{L}}{\delta X_A}}_{0}\frac{\delta X_A}{\delta X'_{A'}}\frac{\partial X'_{A'}}{\partial C_B} + (\text{td})\right]\delta C_B + \underbrace{\frac{\delta\mathcal{L}}{\delta X_A}}_{0}\frac{\delta X_A}{\delta X'_{A'}}\delta X'_{A'} + (\text{td})$$

- Allowing for changes of the Lagrangian of the form $\mathcal{L} = \mathcal{L}' + (\text{td})$,

$$\left\langle \left(\frac{\partial\mathcal{L}'}{\partial C_B}\right)_{X'_{A'}} \right\rangle = \left\langle \left(\frac{\partial\mathcal{L}}{\partial C_B}\right)_{X_A} \right\rangle$$

First law of binary mechanics to linear order in spin

[Blanchet, Buonanno, Le Tiec 1211.1060], [Antonelli, Kavanagh, **MK**, Steinhoff, Vines 2010.0201

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- Performing a transformation of the dynamical variables $X_A \to X'_{A'}$

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial C_B}\delta C_B + \left[\underbrace{\frac{\delta\mathcal{L}}{\delta X_A}}_{0} \frac{\delta X_A}{\delta X'_{A'}} \frac{\partial X'_{A'}}{\partial C_B} + (\text{td})\right]\delta C_B + \underbrace{\frac{\delta\mathcal{L}}{\delta X_A}}_{0} \frac{\delta X_A}{\delta X'_{A'}} \delta X'_{A'} + (\text{td})$$

- Allowing for changes of the Lagrangian of the form $\mathcal{L} = \mathcal{L}' + (\text{td})$,

$$\left\langle \left(\frac{\partial\mathcal{L}'}{\partial C_B}\right)_{X'_{A'}} \right\rangle = \left\langle \left(\frac{\partial\mathcal{L}}{\partial C_B}\right)_{X_A} \right\rangle$$

First law of binary mechanics to linear order in spin (cont.)

- Redshift to linear order in spin

$$\int \mathrm{d}t\, \mathcal{L} \sim -m_\mathrm{i} \int \mathrm{d}t\, \frac{\mathrm{d}\tau_\mathrm{i}}{\mathrm{d}t} \quad \rightarrow \quad z_\mathrm{i} \equiv \left\langle \frac{\mathrm{d}\tau_\mathrm{i}}{\mathrm{d}t} \right\rangle = -\left\langle \frac{\partial \mathcal{L}}{\partial m_\mathrm{i}} \right\rangle = \left\langle \frac{\partial H}{\partial m_\mathrm{i}} \right\rangle$$

- Precession frequency follows from the EOM for the canonical spin vectors

$$\frac{\mathrm{d}\vec{S}_\mathrm{i}}{\mathrm{d}t} = \vec{\Omega}^{\text{inst}}_{S_\mathrm{i}} \times \vec{S}_\mathrm{i}\,, \qquad \vec{\Omega}^{\text{inst}}_{S_\mathrm{i}} \equiv \frac{\partial H}{\partial \vec{S}_\mathrm{i}}$$

Spin magnitude is constant, and for nonprecessing spins

$$\Omega_{S_\mathrm{i}} \equiv \left\langle \left| \vec{\Omega}^{\text{inst}}_{S_\mathrm{i}} \right| \right\rangle = \left\langle \frac{\partial H}{\partial S_\mathrm{i}} \right\rangle$$

- For a Hamiltonian in terms of action variables, $H'(I_r, I_\phi = L; C_B)$,

$$\Omega_r = \frac{\partial H'}{\partial I_r} = \text{const}\,, \qquad \Omega_\phi = \frac{\partial H'}{\partial L} = \text{const}$$

$$z_\mathrm{i} = \frac{\partial H'}{\partial m_\mathrm{i}}\,, \qquad \Omega_{S_\mathrm{i}} = \frac{\partial H'}{\partial S_\mathrm{i}}$$

First law of binary mechanics for spin quadrupole

- Nonminimal part of the action for spin quadrupole [Levi, Steinhoff 1501.04956]

$$\mathcal{L}_{ES^2} = \frac{C_{ES^2}}{2m} \frac{\mathcal{E}_{\mu\nu}}{\sqrt{-u^2}} S^\mu S^\nu$$

$\mathcal{E}_{\mu\nu} \equiv R_{\mu\alpha\nu\beta} u^\alpha u^\beta$ is the electric component of the Riemann tensor.

- Define the quadrupole $Q \equiv C_{ES^2} S^2/2m$ to remove the mass dependence of the nonminimal part, such that $z = \langle \partial H/\partial m \rangle$.

- For aligned spins $S^\mu \equiv S s^\mu$,

$$\mathcal{L}_{ES^2} = Q \frac{\mathcal{E}_{\mu\nu}}{\sqrt{-u^2}} s^\mu s^\nu$$

- Define the eigenvalues $M^2 \mathcal{E}(u) = \text{diag}[\Lambda_1^E, \Lambda_2^E, -(\Lambda_1^E + \Lambda_2^E)]$, leading to

$$\frac{\partial H}{\partial Q} = -\frac{\partial \mathcal{L}}{\partial Q} = -\frac{z \Lambda_2^E}{M^2} \qquad \text{(aligned spin)}$$

First law of binary mechanics for spin quadrupole (cont.)

$$\mathrm{d}E = \Omega_r \mathrm{d}I_r + \Omega_\phi \mathrm{d}L + \sum_{\mathrm{i}} \left(z_{\mathrm{i}} \mathrm{d}m_{\mathrm{i}} + \Omega_{S_{\mathrm{i}}} \mathrm{d}S_{\mathrm{i}} - \frac{z_{\mathrm{i}} \Lambda_2^E}{M^2} \mathrm{d}Q_{\mathrm{i}} \right), \qquad Q_{\mathrm{i}} \equiv \frac{C_{\mathrm{i}ES^2} S_{\mathrm{i}}^2}{2m_{\mathrm{i}}}$$

- Redshift from PN Hamiltonian agrees with 1SF [Bini,Geralico,Steinhoff 2003.12887]

$$z_1 = \left(\frac{\partial H}{\partial m_1} \right)_{Q_1} = s^0 + s^2 + q\left[s^0 + s + s^2 C_{1ES^2} \right] + \dots$$

- Spin-precession invariant from H agrees with the test spin result $\psi_1^s = s \frac{3y^{5/2}}{\sqrt{1-3y}}$

$$\psi_1 = s^0 + s + q\left[s^0 + s \right] + \dots$$

- Eigenvalue Λ_2^E calculated from H agrees with 1SF results for λ_2^E
[Dolan,Nolan,Ottewill,Warburton,Wardell 1406.4890],[Bini,Damour 1409.6933],[Bini,Geralico 1806.03495]

$$\lambda_2^E = \frac{m_2^2}{M^2} \Lambda_2^E = -\frac{m_2^2}{z_1} \frac{\partial H}{\partial Q_1} = y^3 + 3y^4 + 9y^5 + q\left(-y^3 - \frac{3y^4}{2} - \frac{23y^5}{8} \right)$$

First law of binary mechanics for all spin multipoles

- Spin-induced part of the action [Levi, Steinhoff 1501.04956]

$$L_{\mathsf{SI}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \frac{C_{ES^{2n}}}{m^{2n-1}} D_{\mu_{2n}} \cdots D_{\mu_3} \frac{E_{\mu_1\mu_2}}{\sqrt{-u^2}} S^{\mu_1} S^{\mu_2} \dots S^{\mu_{2n-1}} S^{\mu_{2n}}$$

$$+ \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{C_{BS^{2n+1}}}{m^{2n}} D_{\mu_{2n+1}} \cdots D_{\mu_3} \frac{B_{\mu_1\mu_2}}{\sqrt{-u^2}} S^{\mu_1} S^{\mu_2} \dots S^{\mu_{2n-1}} S^{\mu_{2n}} S^{\mu_{2n+1}}$$

- **Conjecture:**

$$\mathrm{d}E = \Omega_r \mathrm{d}I_r + \Omega_\phi \mathrm{d}L + \sum_{\mathrm{i}} \left(z_{\mathrm{i}} \mathrm{d}m_{\mathrm{i}} + \Omega_{S_{\mathrm{i}}} \mathrm{d}S_{\mathrm{i}} + \sum_{n=2}^{\infty} (-)^{\lceil \frac{n-1}{2} \rceil} \frac{z_{\mathrm{i}} \Lambda^{(n)}}{M^n} dJ_{\mathrm{i}}^{(n)} \right)$$

$$J_{\mathrm{i}}^{(n)} \equiv \frac{C_{\mathrm{i}E/BS^n}}{n! m_{\mathrm{i}}^{n-1}} S_{\mathrm{i}}^n .$$

- $\Lambda^{(3)}$ is an eigenvalue of the octupolar tensor $\mathcal{B}_{\mu\nu\lambda} \equiv R^*_{\mu\alpha\nu\beta;\lambda} u^\alpha u^\beta$.
Agrees with the eigenvalue denoted $\Delta\mathcal{B}_{(222)}$ calculated at 1SF
[Nolan, Kavanagh, Dolan, Ottewill, Warburton 1505.04447]

$$\Delta\mathcal{B}_{(222)} = \frac{m_2^3}{M^3} \Lambda^{(3)} = -6y^{9/2} + 9ay^5 + q(6y^{9/2} - 15ay^5)$$

Conclusions

- Calculated spin-orbit coupling up to 5.5PN (local and nonlocal), except for one coefficient, using information from the PN, PM, and SF.
- Calculated spin-spin coupling at 5PN for aligned spins, except for one coefficient that can be determined from 1SF results.
- Motivated a first law with spin quadrupole that agrees with available results for redshift, spin precession, and the eigenvalues of the tidal tensors.
- Future work:
  - Determine the remaining SO unknown through
    targeted PN calculations [Bini, Damour, Geralico 2107.08896]
    or second-order SF [Warburton, Pound, Wardell, Miller, Durkan 2107.01298]
  - Use new PN results in EOB waveform models
  - Calculate 1SF precession frequency to linear order in spin
  - Derive the first law with spin quadrupole [Ramond, Le Tiec 2005.00602]
  - Extend the spin-spin results to precessing spins