Title: Friendship in the Axiverse

Speakers: David Cyncynates

Series: Particle Physics

Date: November 30, 2021 - 1:00 PM

URL: https://pirsa.org/21110044

Abstract: A generic low-energy prediction of string theory is the existence of a large collection of axions, commonly known as a string axiverse. String axions can be distributed over many orders of magnitude in mass, and are expected to interact with one another through their joint potential. In this talk, I will show how non-linearities in this potential lead to a new type of resonant energy transfer between axions with nearby masses. This resonance generically transfers energy from axions with larger decay constants to those with smaller decay constants, leading to a multitude of signatures. These include enhanced direct detection prospects for a resonant pair comprising even a small subcomponent of dark matter, and boosted small-scale structure if the pair is the majority of DM. Near-future iterations of experiments such as ADMX and DM Radio will be sensitive to this scenario, as will astrophysical probes of DM substructure.

Friendship in the Axiverse

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Based on arXiv:2109.09755
with Tudor Giurgica-Tiron, Olivier Simon, Jed Thompson

Perimeter Institute

30 November 2021

Background & Motivation
Axions

- Axions are well-motivated extensions of the SM
  - QCD axion
  - String axions
- Non-perturbative effects give rise to a naturally small potential/mass

$$V \sim \Lambda^4(1 - \cos \phi/f)$$

- Naturally produced in the early universe

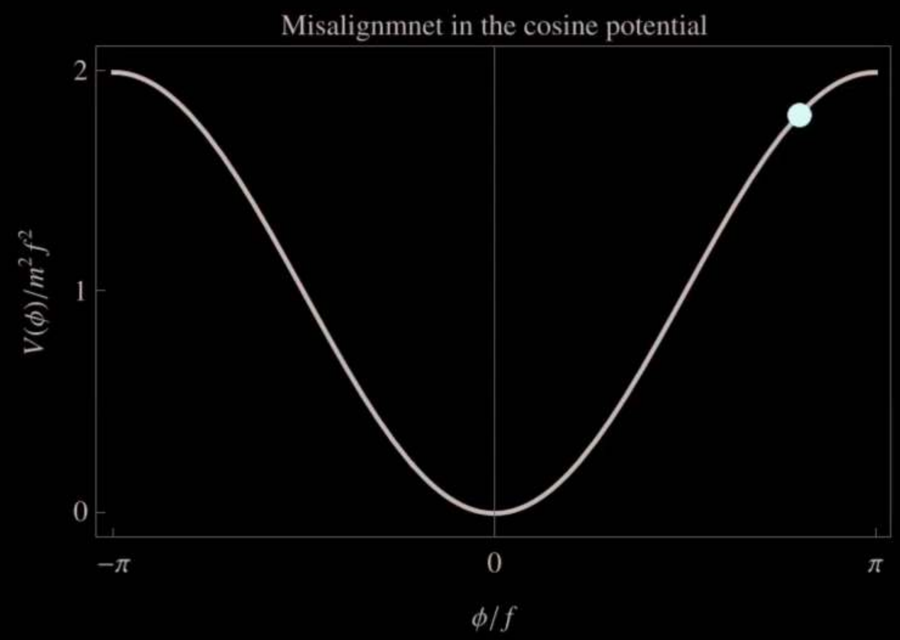
Background & Motivation

The usual misalignment story

- Define $\theta \equiv \phi/f$:
  - homogeneous $\theta(t,x) = \Theta(t)$
  - random $\Theta(0) \in [-\pi, \pi)$.
    - Initial energy density $\rho \sim m^2 f^2$
- $\ddot{\Theta} + 3H\dot{\Theta} + m^2 \sin\Theta = 0$
  - Dilutes like Cold Matter
    $\rho \sim m^2 f^2 (mt)^{-3/2}$

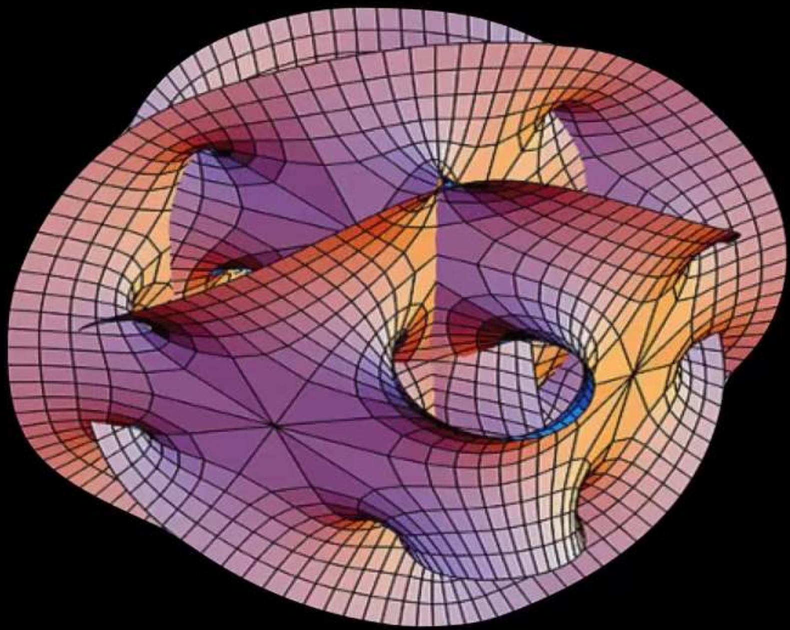
Single Instanton Potential:

$$V(\theta) = m^2 f^2 (1 - \cos\theta)$$



Background & Motivation
Axiverse

- String theory→large number of axions
  - Axiverse
- $V(\phi) \sim \Lambda^4(1 - \cos \phi/f)$
- $\Lambda^4 \sim M_{\rm UV}^4 e^{-S}$
- Each string axion can be produced through misalignment mechanism
  - Expectation: $\rho_{\rm Final} \propto m^{1/2} f^2$



More-realistic potential

$$V(\phi_1, \ldots, \phi_N) = \sum_{i=1}^{M} \Lambda_i^4 \left[1 - \cos\left(\sum_{j=1}^{N} Q_{ij} \frac{\phi_j}{f_j} + \delta_i \right) \right]$$

- $\Lambda_i^4 \sim M_{\mathrm{UV}}^4 e^{-S_i}$
- Some masses may be close to one another: "Friendly axions"

Friendly axions:

➡ Similar dynamical timescales

➡ Dynamical Resonance

➡ Enhanced observational prospects

Take-home message:

Axion friendship leads to enhanced signatures

Outline

- Background & Motivation
- Homogeneous dynamics of friendly axions
- Growth of density perturbations
- Signatures

More-realistic potential

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Two-Axion Model

$$V(\phi_S, \phi_L) = \Lambda_1^4\left(1 - \cos\left(\frac{\phi_S}{f_S} + \frac{\phi_L}{f_L}\right)\right) + \Lambda_2^4\left(1 - \cos\left(\frac{\phi_L}{f_L}\right)\right)$$

$$V(\theta_S, \theta_L) = m^2 f^2 \left[\left(1 - \cos\left(\theta_S + \theta_L\right)\right) + \mu^2 F^2 \left(1 - \cos\theta_L\right)\right]$$

$m \equiv m_S$

$f \equiv f_S$

$\mu^2 F^2 \gtrsim 1$

$$f_L / f_S = F \qquad\qquad m_L / m_S = \mu$$

$F \gtrsim 3$ (or much larger): $S =$ Short, $L =$ Long

$0.5 \lesssim \mu < 1$: nearby masses = Friendly

Two-Axion Model

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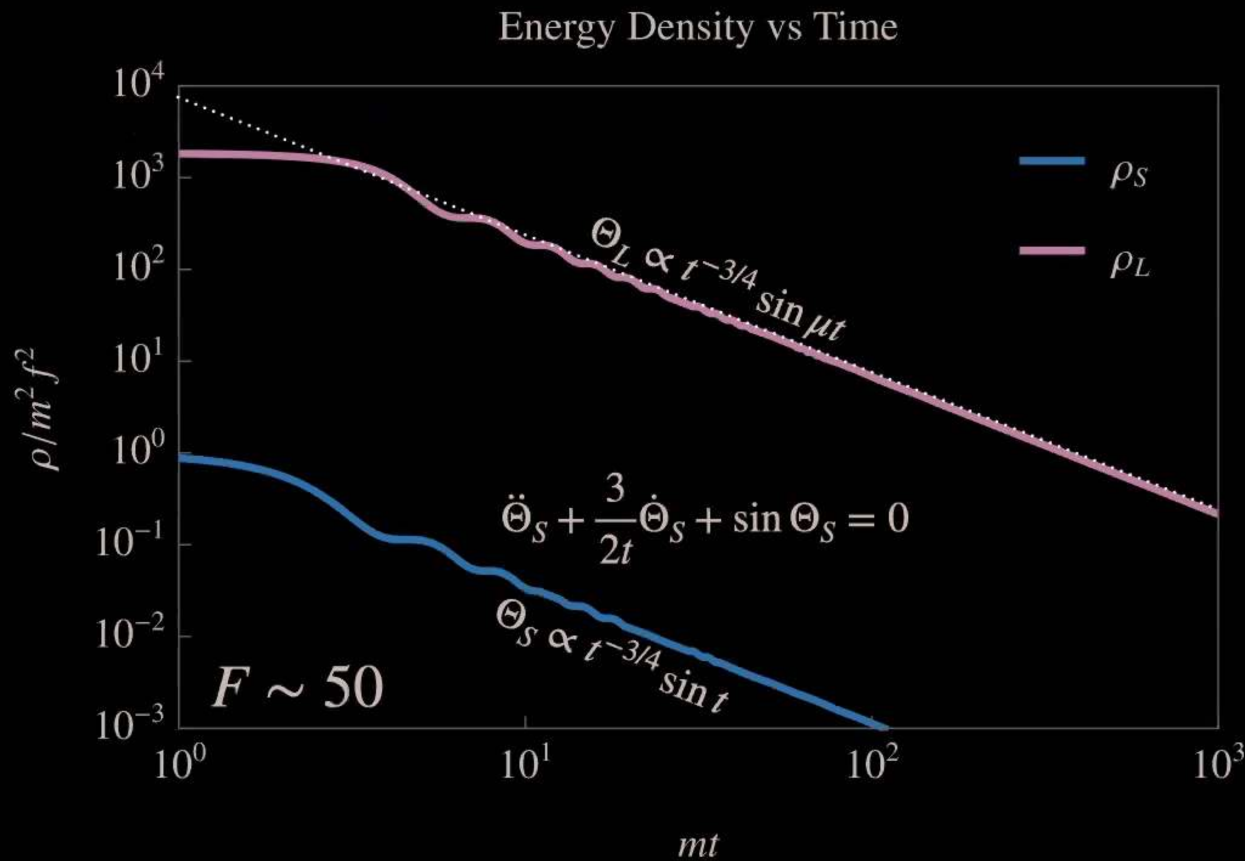
Two-Axion Model : Uncoupled Expectation

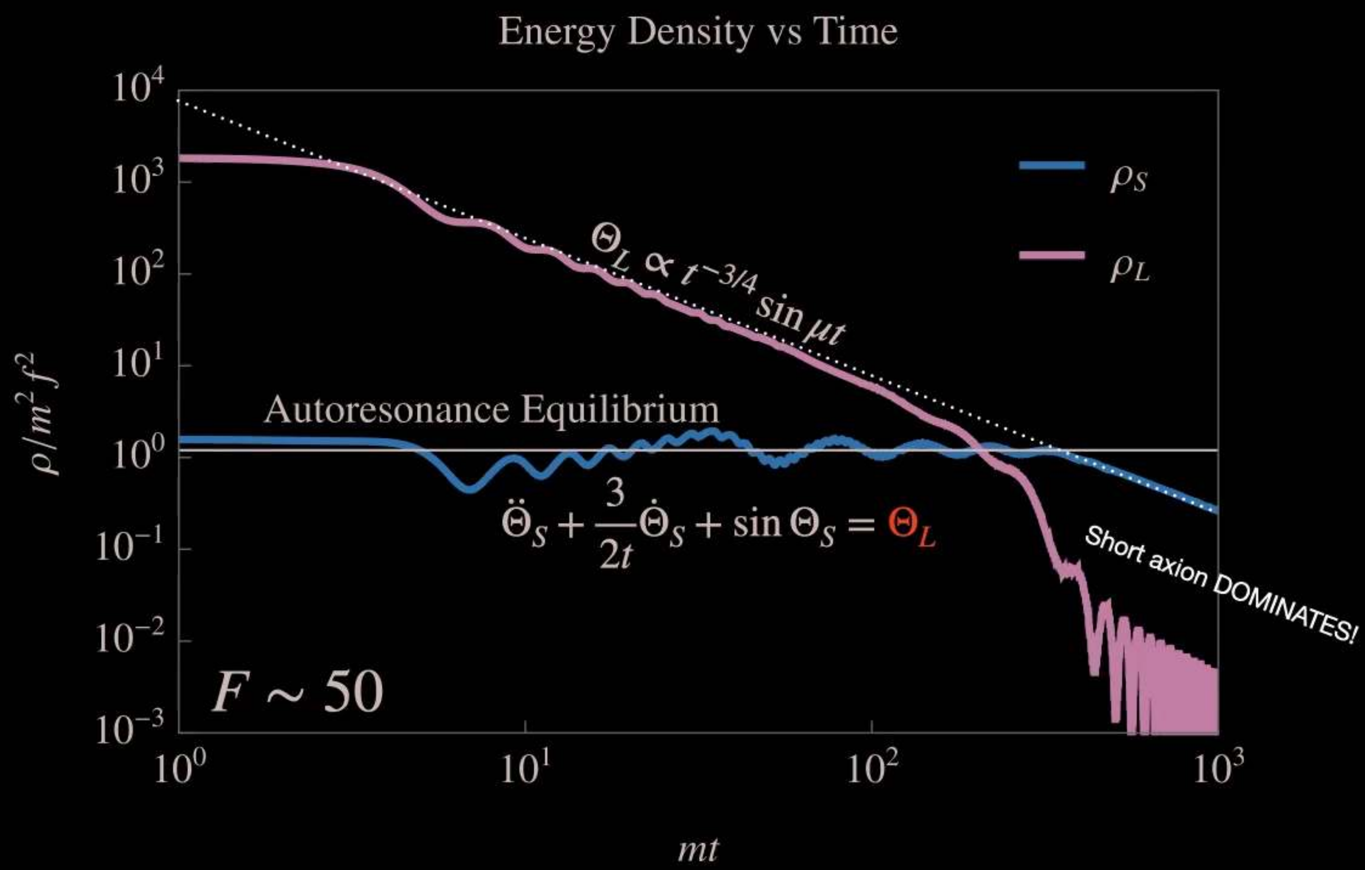
$$V(\theta_S, \theta_L) = m^2 f^2 \left[\left(1 - \cos\left(\theta_S \quad \right) \right) + \mu^2 F^2 \left(1 - \cos\theta_L\right) \right]$$

- Similar mass → both axions dilute like cold matter at roughly the same time
  - Energy density ratio is constant: $\rho_S / \rho_L \sim 1/F^2$

$$V(\theta_S, \theta_L) = m^2 f^2 \left[\left(1 - \cos\left(\theta_S \qquad \right) \right) + \mu^2 F^2 \left(1 - \cos\theta_L\right) \right]$$



$$V(\theta_S, \theta_L) = m^2 f^2 \left[\left(1 - \cos\left(\theta_S + \theta_L\right) \right) + \mu^2 F^2 \left(1 - \cos\theta_L\right) \right]$$



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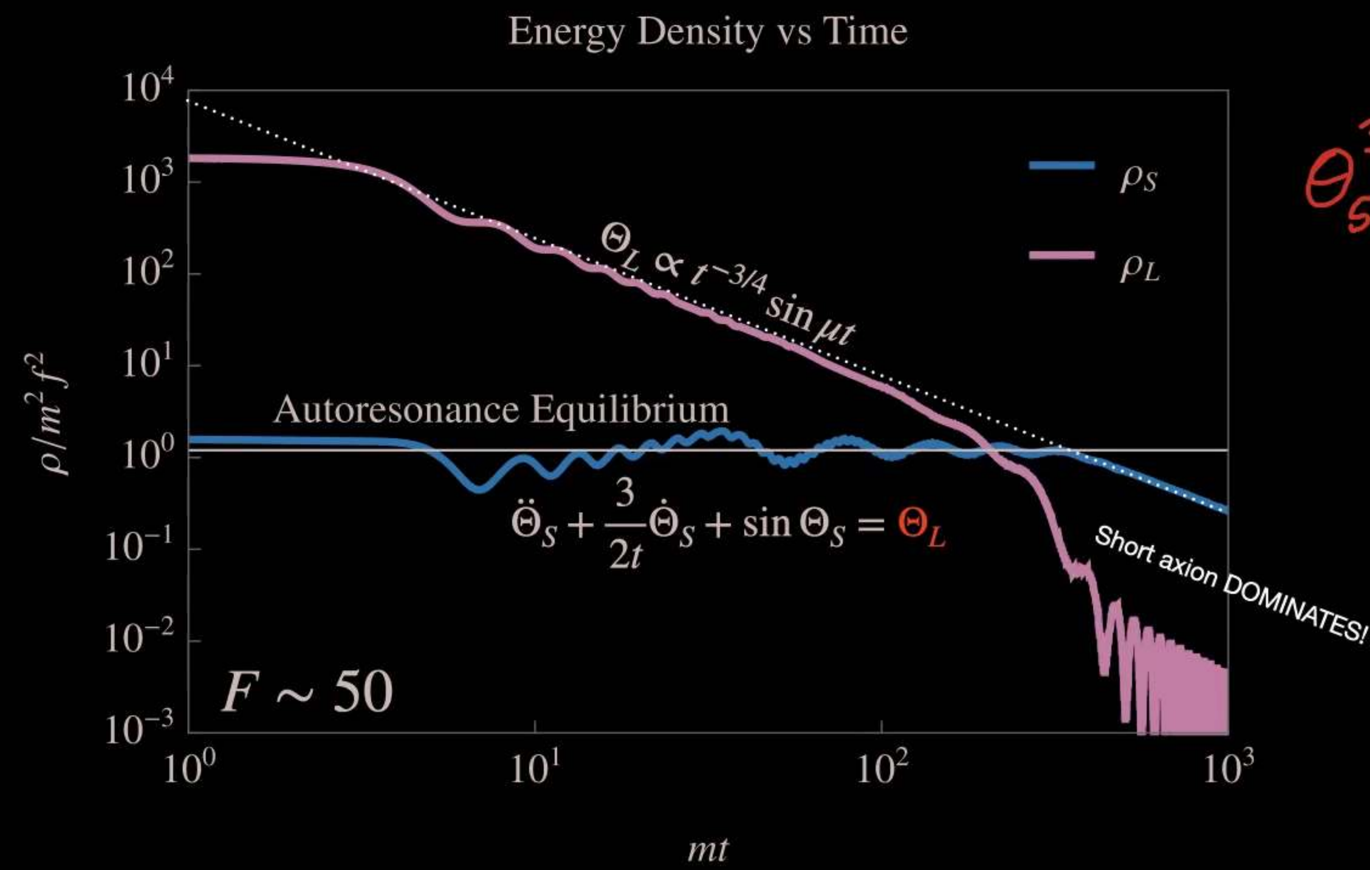
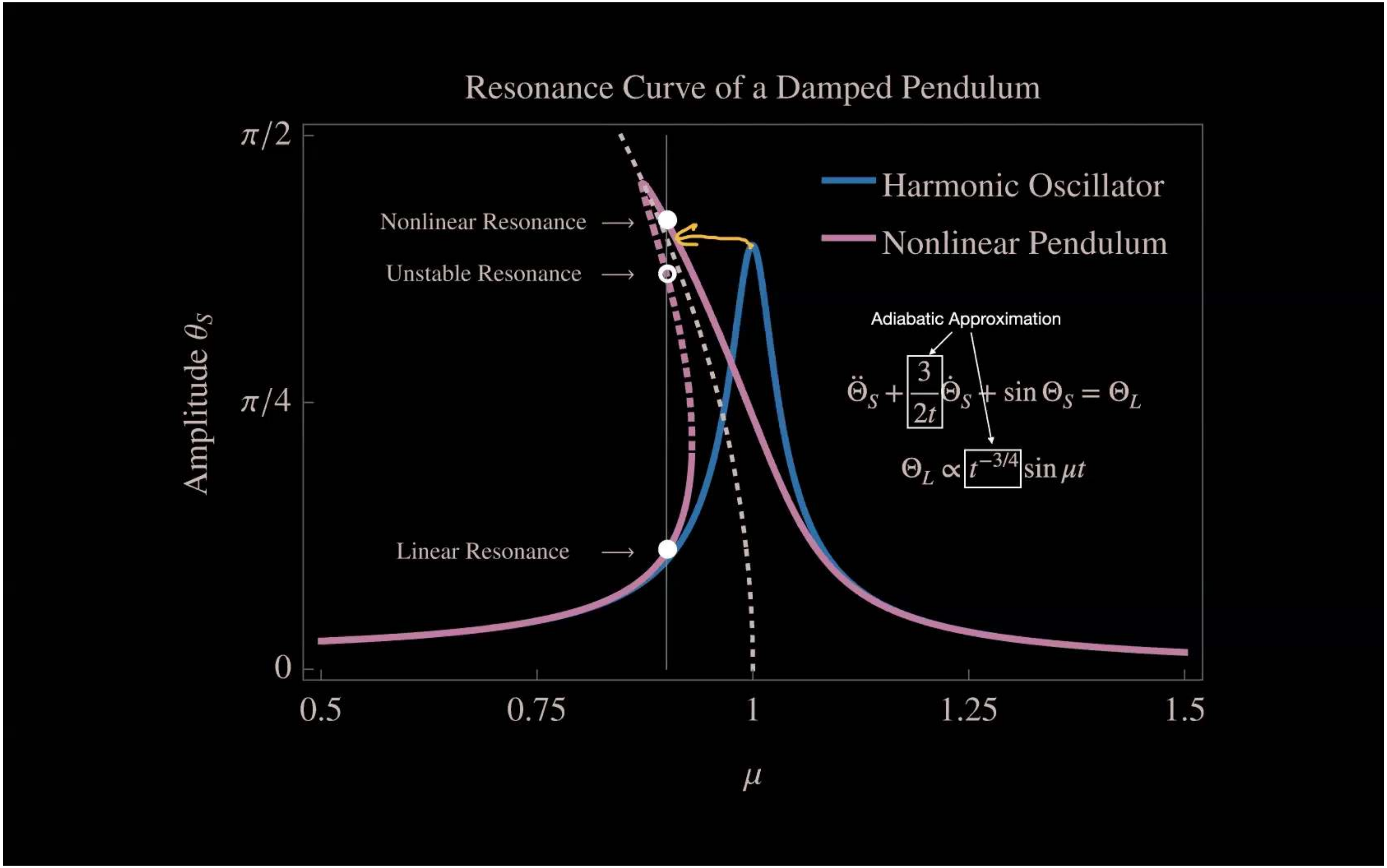
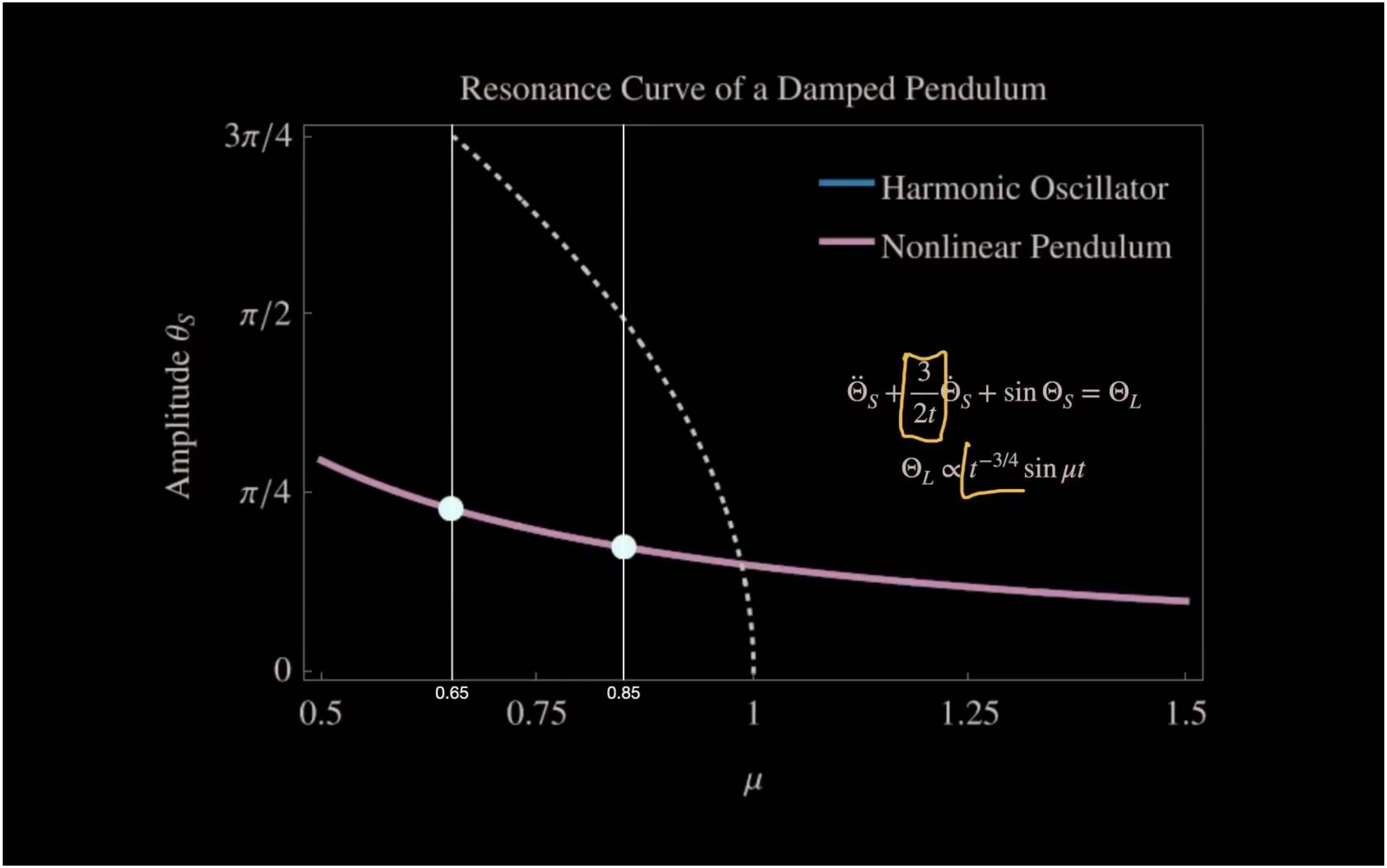


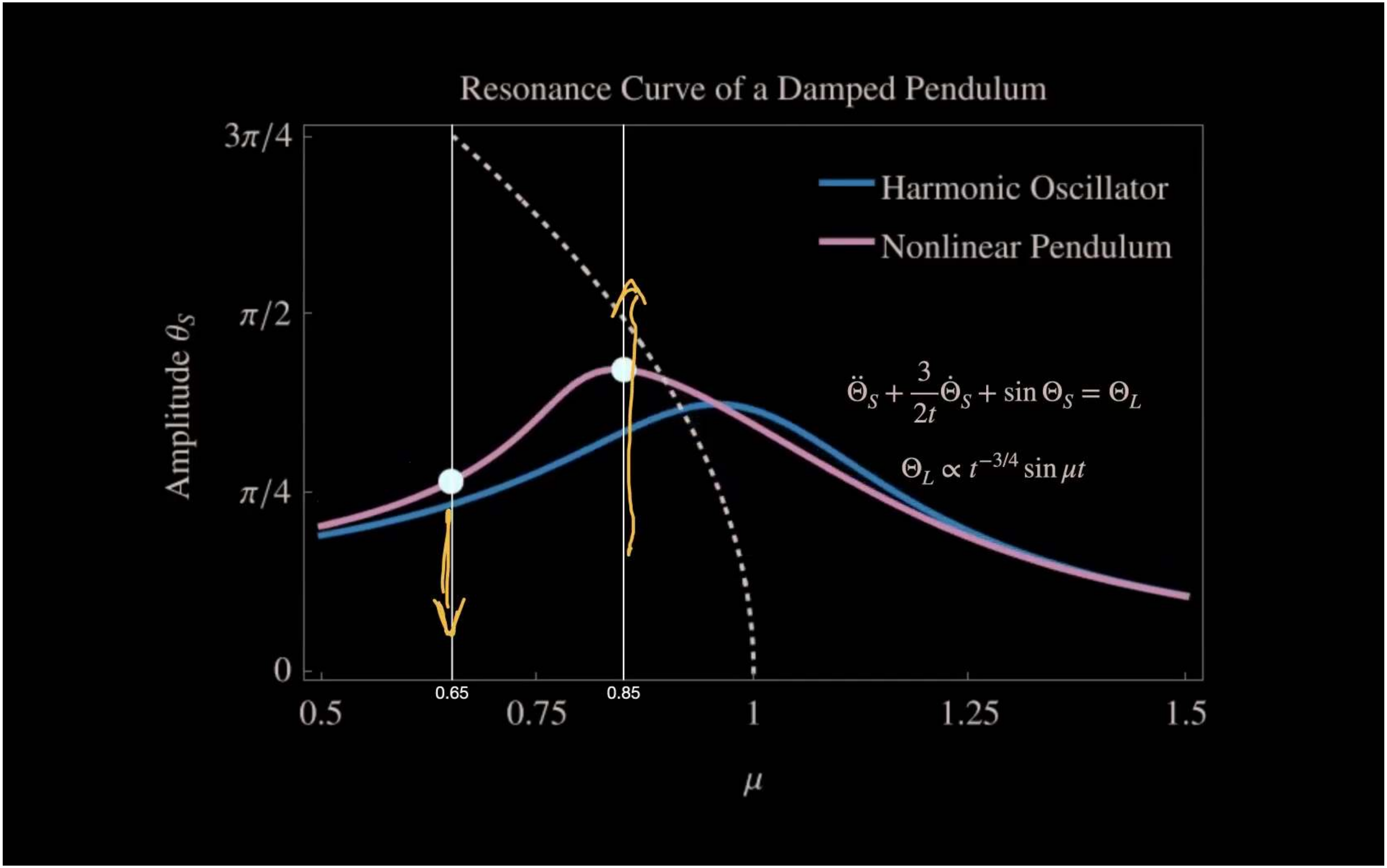
$\theta_S^3 > \theta_L \theta_S^2$

What are the dynamics of autoresonance?

How friendly do axions need to be for autoresonance to be common?


Resonance Curve of a Damped Pendulum
π/2
π/4
0
Amplitude θ_S
Harmonic Oscillator
Nonlinear Pendulum
Nonlinear Resonance ⟶
Unstable Resonance ⟶
Linear Resonance ⟶
Adiabatic Approximation
$\ddot{\Theta}_S + \frac{3}{2t}\dot{\Theta}_S + \sin\Theta_S = \Theta_L$
$\Theta_L \propto t^{-3/4}\sin\mu t$
0.5
0.75
1
1.25
1.5
μ


Resonance Curve of a Damped Pendulum
Harmonic Oscillator
Nonlinear Pendulum
Amplitude θ_S
$3\pi/4$
$\pi/2$
$\pi/4$
0
0.65
0.85
0.5
0.75
1
1.25
1.5
μ
$\ddot{\Theta}_S + \frac{3}{2t}\dot{\Theta}_S + \sin\Theta_S = \Theta_L$
$\Theta_L \propto t^{-3/4}\sin\mu t$



Resonance Curve of a Damped Pendulum
Harmonic Oscillator
Nonlinear Pendulum
$\ddot{\Theta}_S + \frac{3}{2t}\dot{\Theta}_S + \sin\Theta_S = \Theta_L$
$\Theta_L \propto t^{-3/4} \sin \mu t$
Amplitude θ_S
$3\pi/4$
$\pi/2$
$\pi/4$
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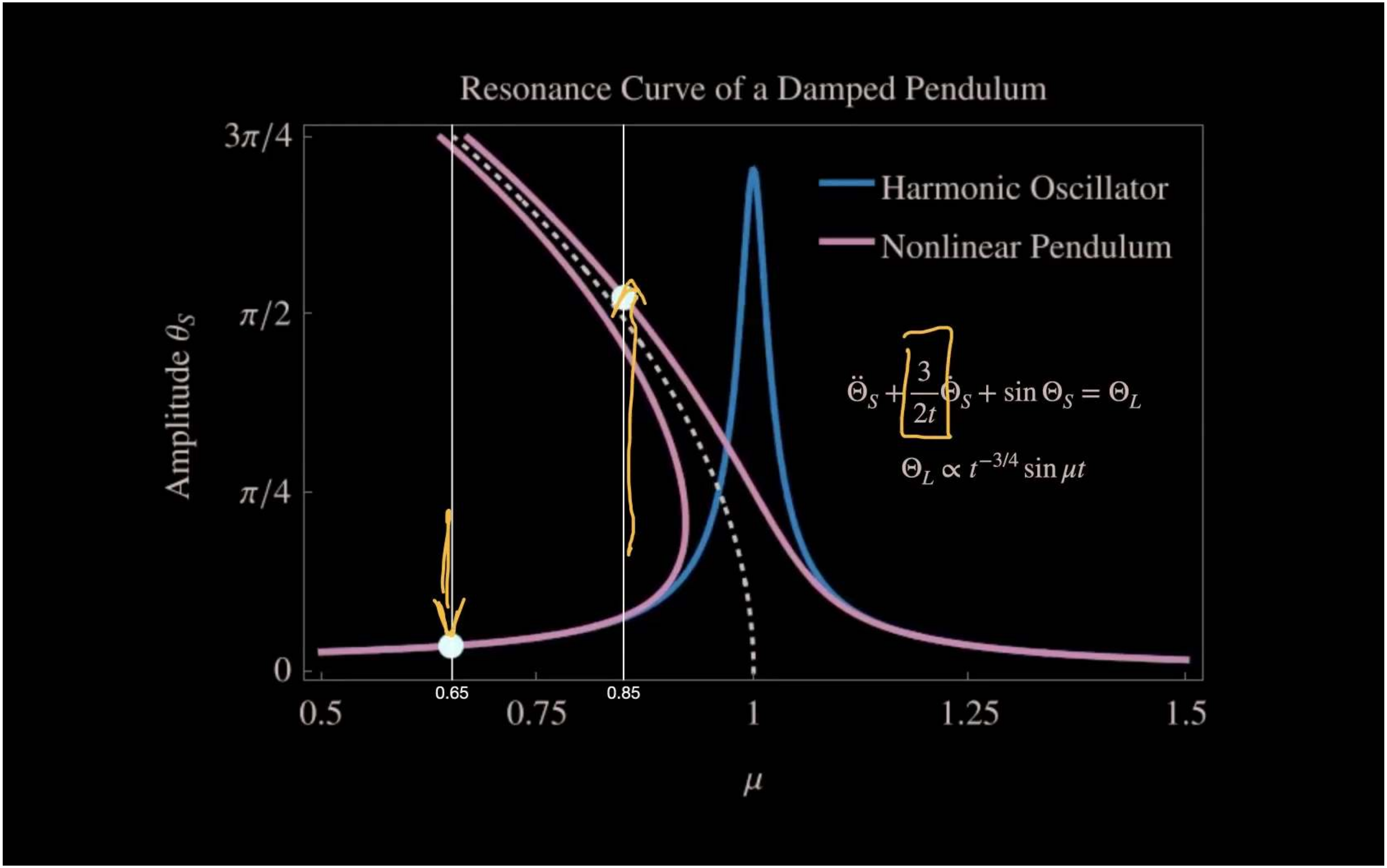
0.65
0.85
0.5
0.75
1
1.25
1.5
μ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$\pi/2$
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0.65
0.85
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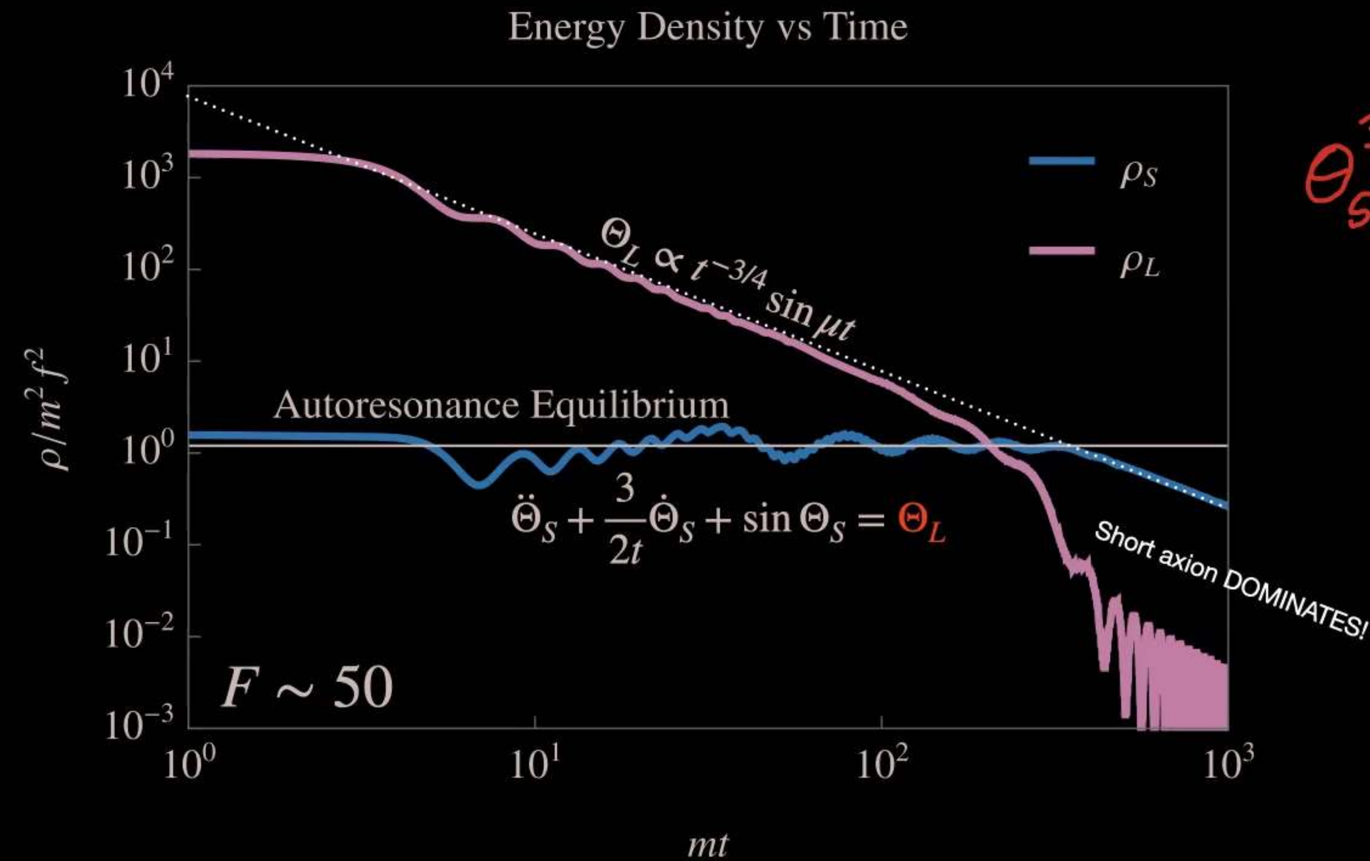
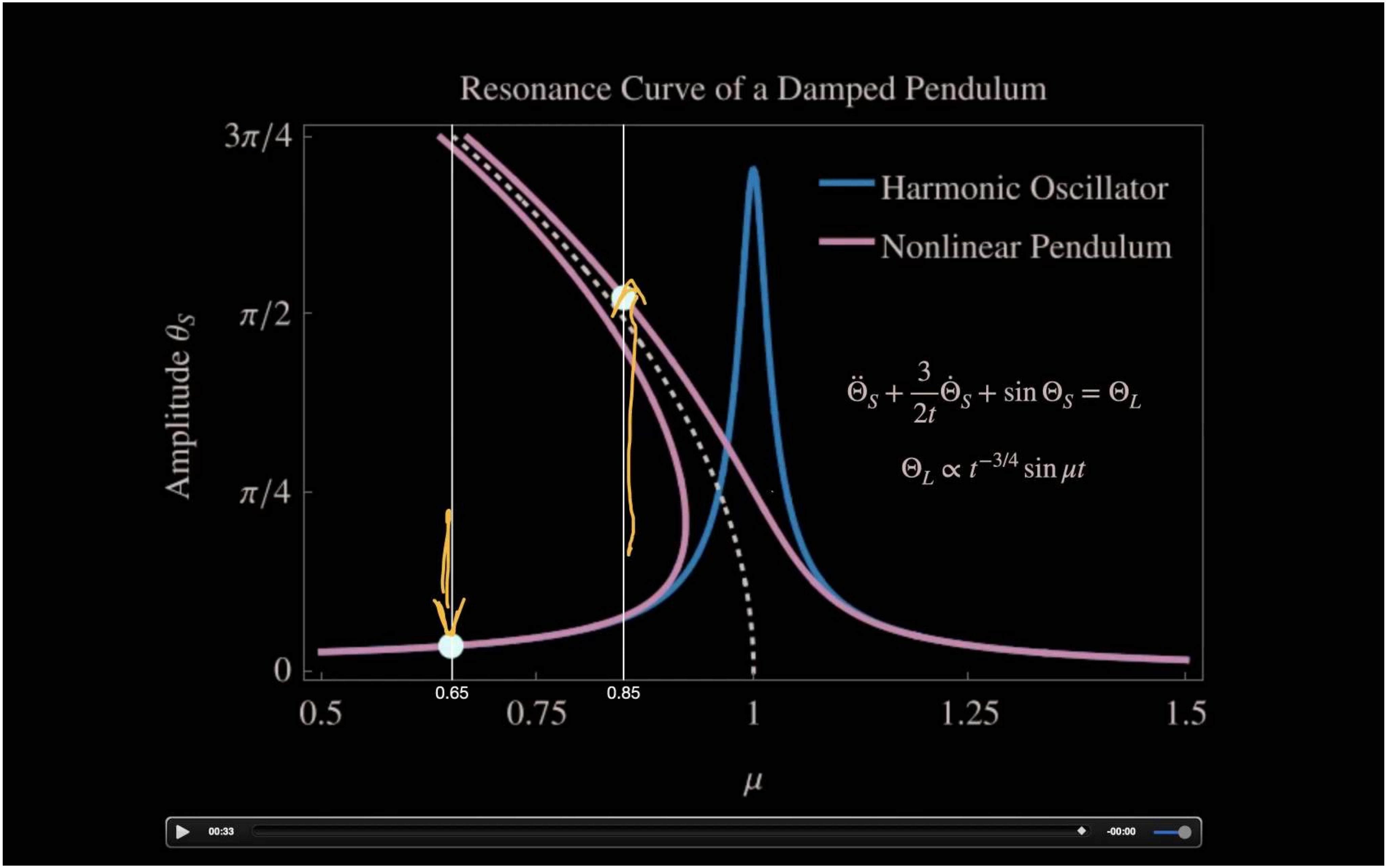
0.75
1
1.25
1.5
μ

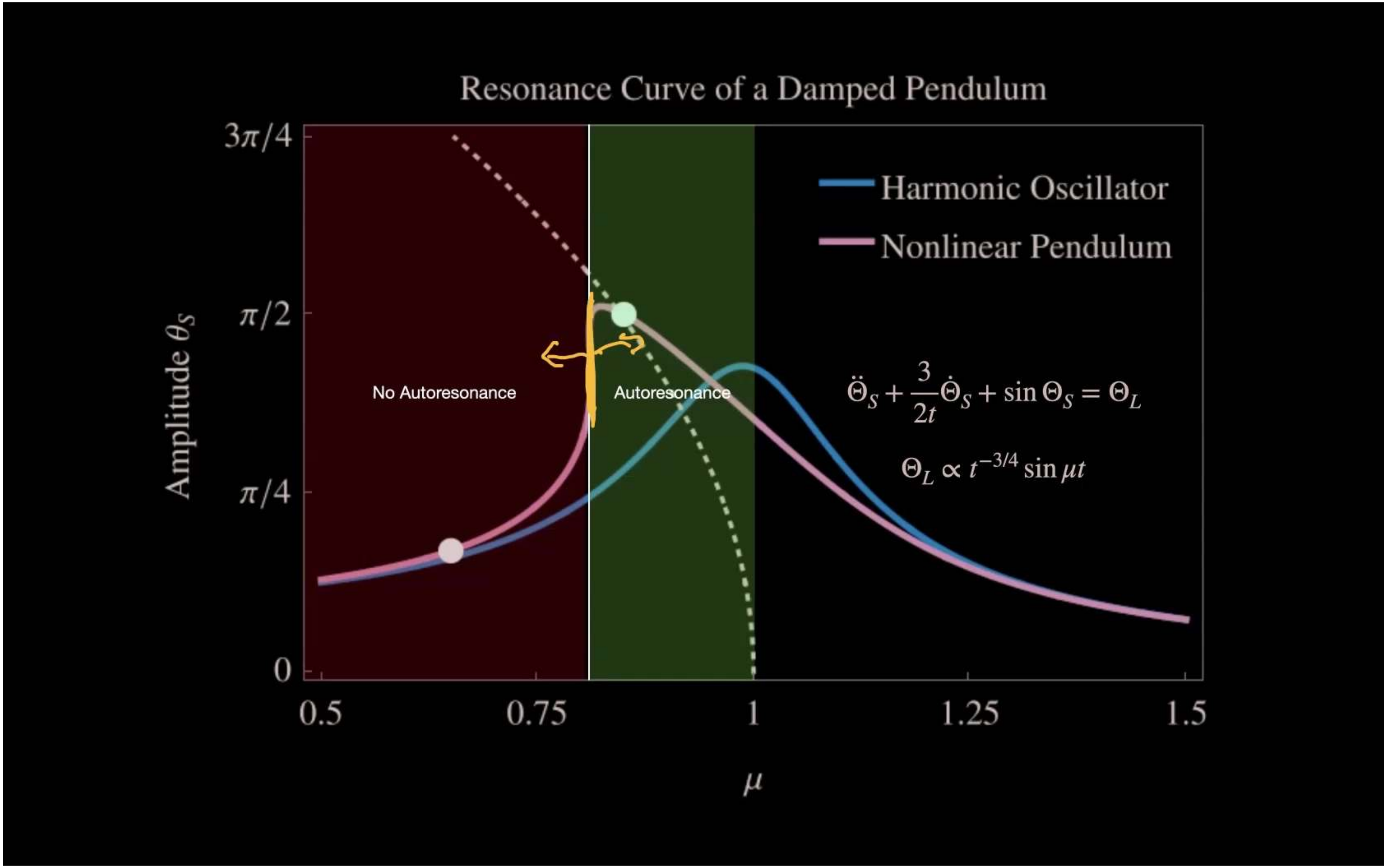
$$V(\theta_S, \theta_L) = m^2 f^2 \left[\left(1 - \cos\left(\theta_S + \theta_L\right) \right) + \mu^2 F^2 \left(1 - \cos\theta_L\right) \right]$$



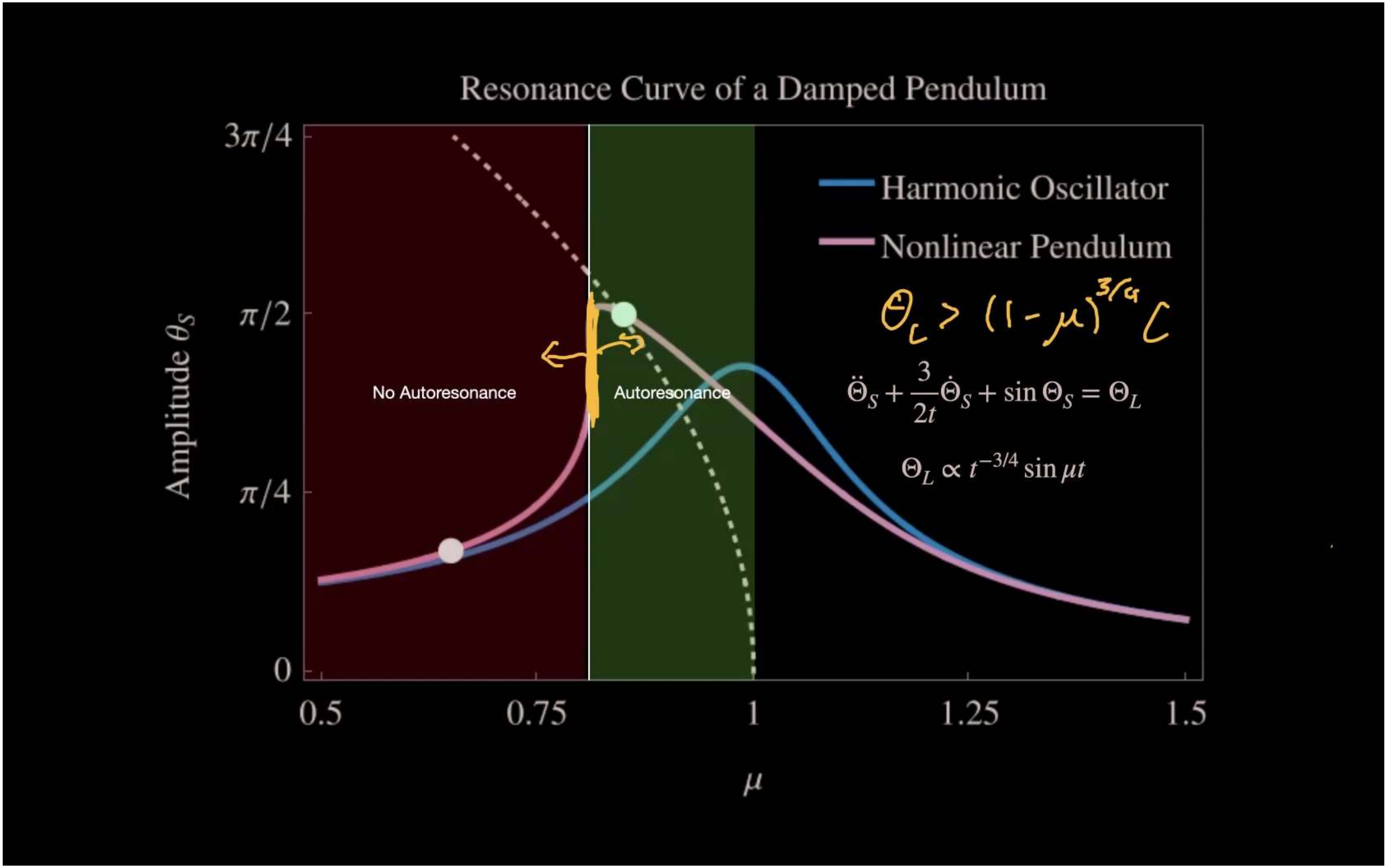
$\theta_S^3 > \theta_L \theta_S^2$

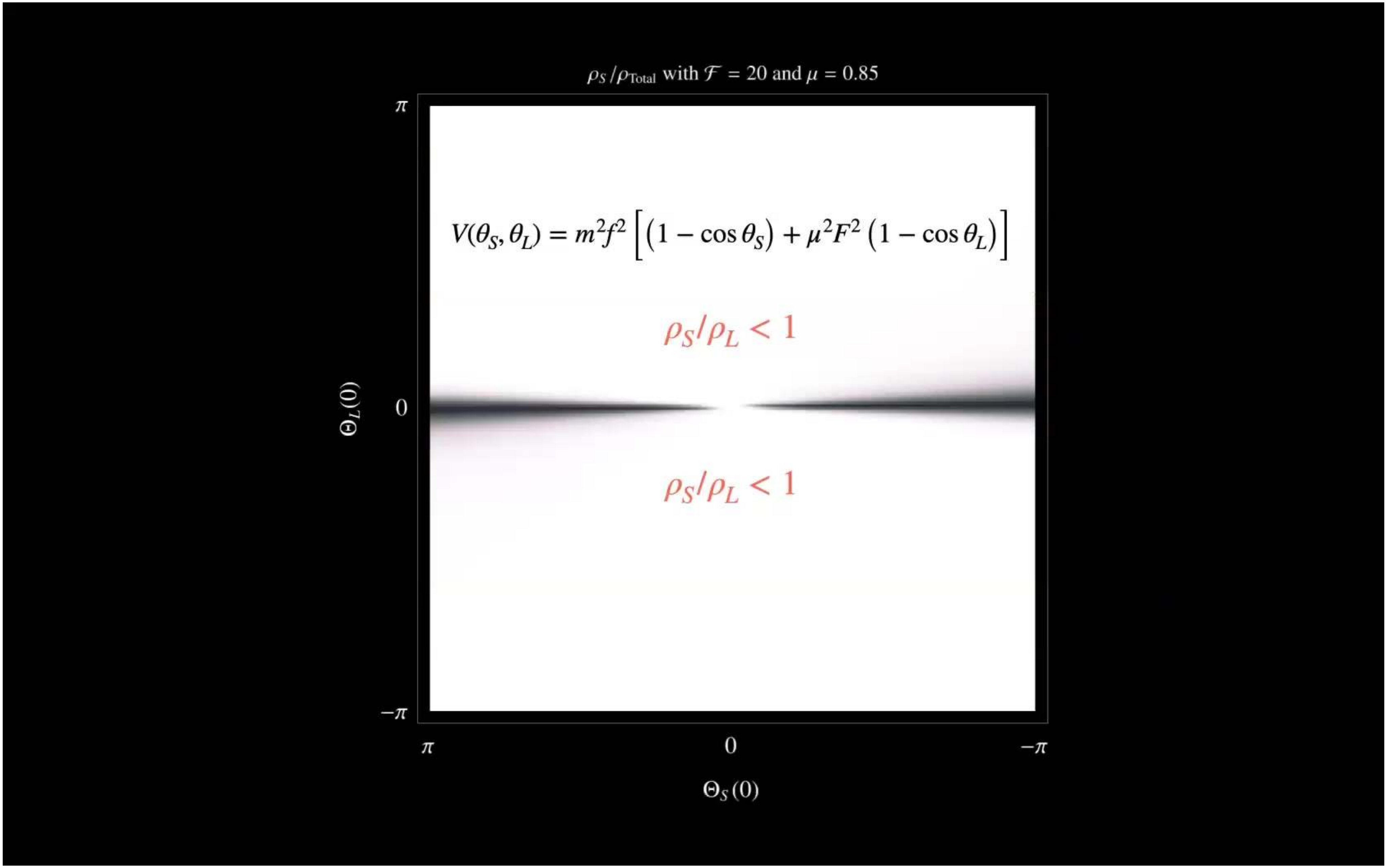


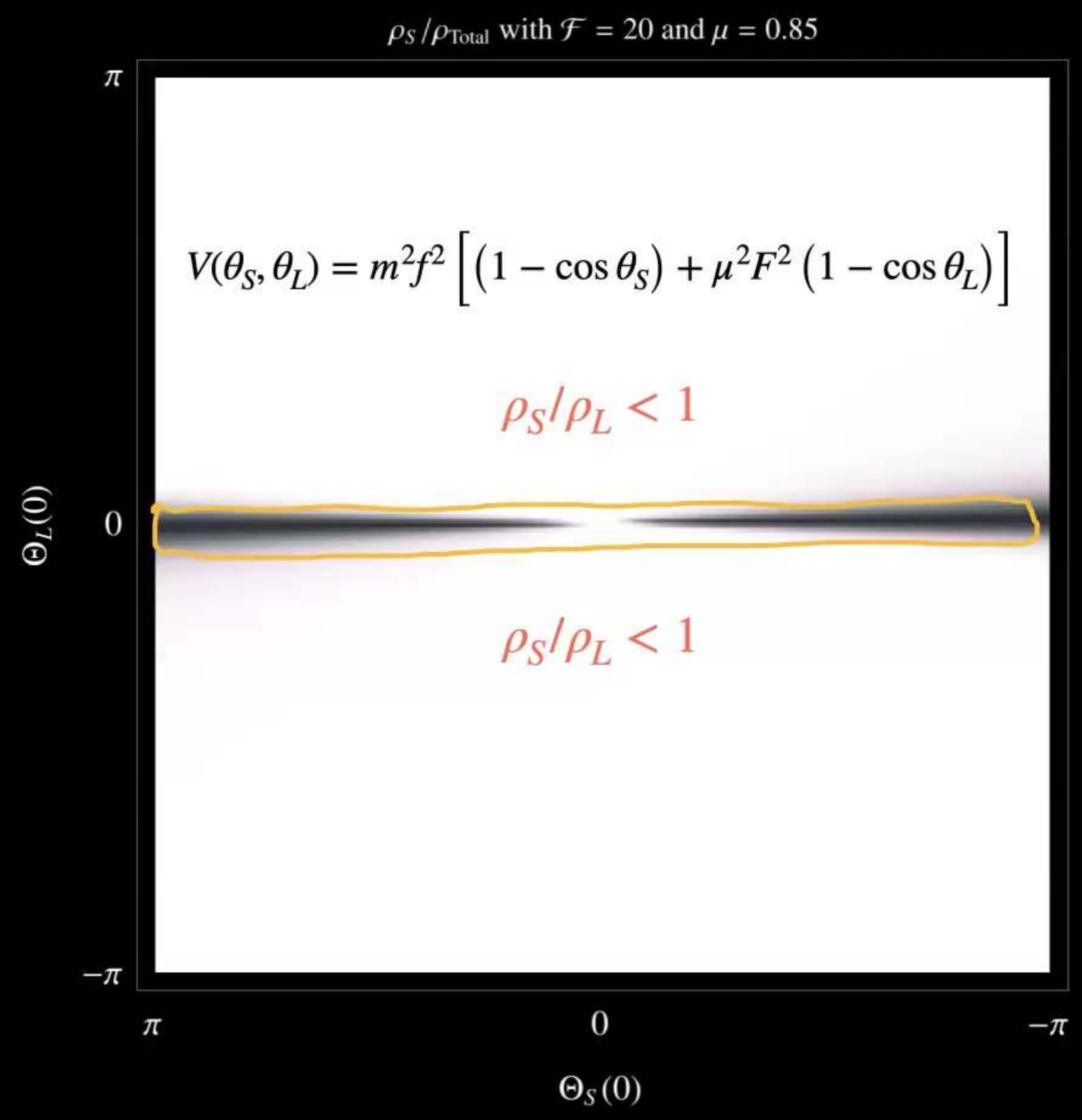
Resonance Curve of a Damped Pendulum
Harmonic Oscillator
Nonlinear Pendulum
$\ddot{\Theta}_S + \frac{3}{2t}\dot{\Theta}_S + \sin\Theta_S = \Theta_L$
$\Theta_L \propto t^{-3/4} \sin \mu t$
Amplitude θ_S
$3\pi/4$
$\pi/2$
$\pi/4$
0
0.5
0.65
0.75
0.85
1
1.25
1.5
μ
00:33
-00:00



Resonance Curve of a Damped Pendulum
Harmonic Oscillator
Nonlinear Pendulum
No Autoresonance
Autoresonance
$\ddot{\Theta}_S + \frac{3}{2t}\dot{\Theta}_S + \sin\Theta_S = \Theta_L$
$\Theta_L \propto t^{-3/4} \sin\mu t$
Amplitude θ_S
$3\pi/4$
$\pi/2$
$\pi/4$
0
0.5
0.75
1
1.25
1.5
μ


Resonance Curve of a Damped Pendulum
Amplitude θ_S
$3\pi/4$
$\pi/2$
$\pi/4$
0
0.5
0.75
1
1.25
1.5
μ
Harmonic Oscillator
Nonlinear Pendulum
No Autoresonance
Autoresonance
$\Theta_L > (1-\mu)^{3/4} C$
$\ddot{\Theta}_S + \frac{3}{2t}\dot{\Theta}_S + \sin\Theta_S = \Theta_L$
$\Theta_L \propto t^{-3/4} \sin \mu t$


$\rho_S/\rho_{\rm Total}$ with $\mathcal{F} = 20$ and $\mu = 0.85$
$V(\theta_S, \theta_L) = m^2 f^2 \left[\left(1 - \cos\theta_S\right) + \mu^2 F^2 \left(1 - \cos\theta_L\right)\right]$
$\rho_S/\rho_L < 1$
$\rho_S/\rho_L < 1$
π
0
$-\pi$
π
0
$-\pi$
$\Theta_L(0)$
$\Theta_S(0)$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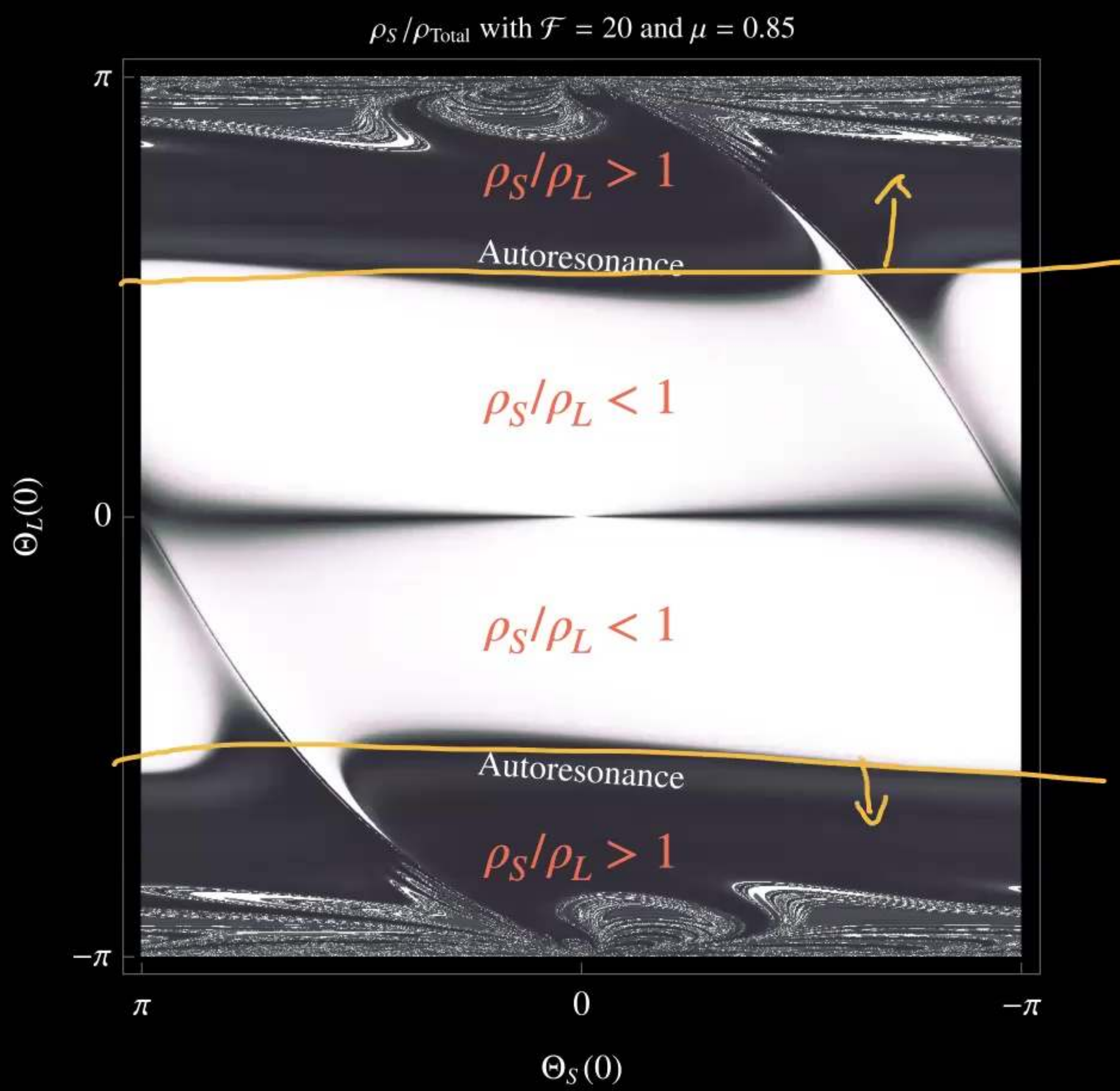
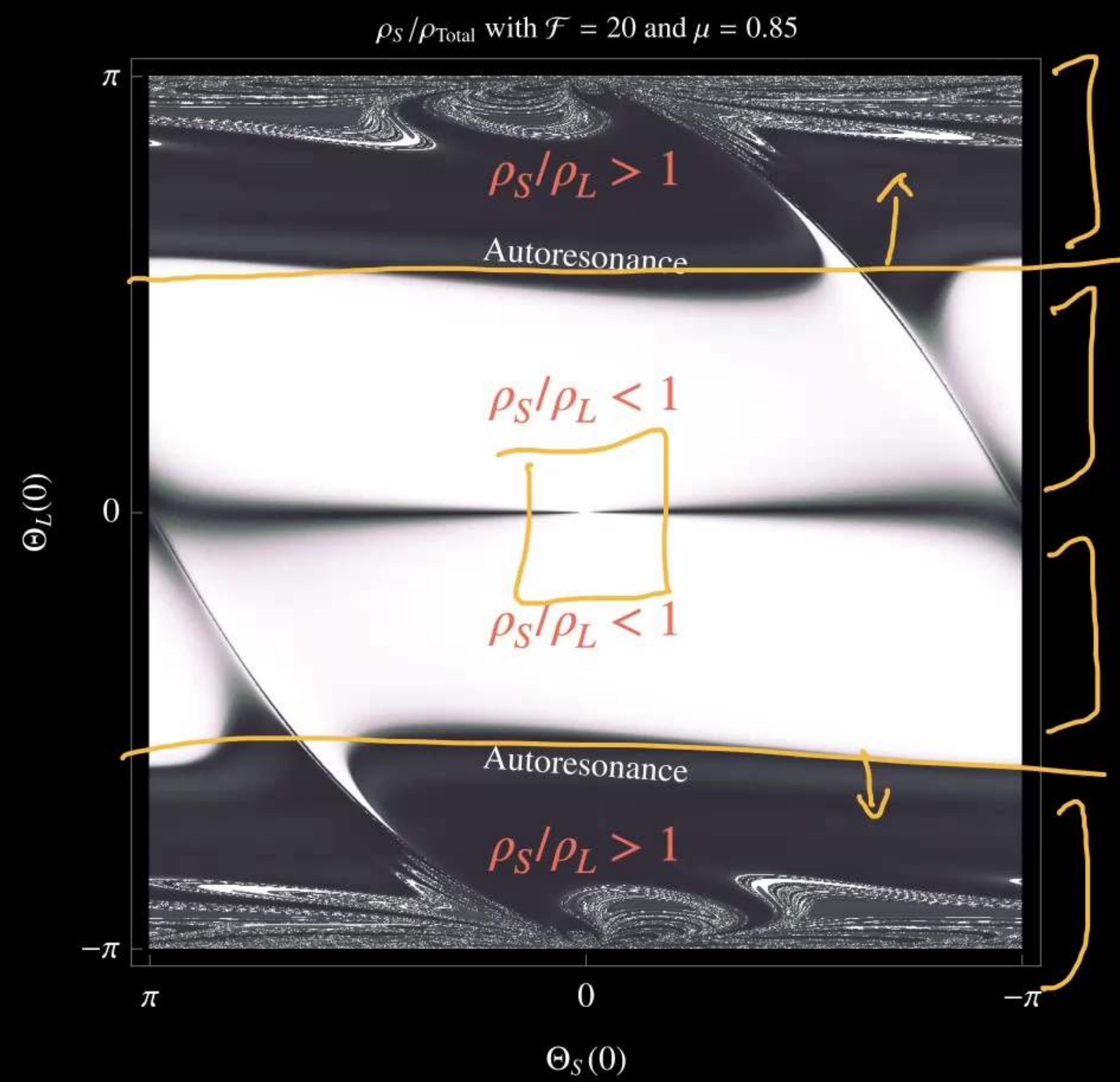
$V(\theta_S, \theta_L) = m^2 f^2 \left[\left(1 - \cos\theta_S\right) + \mu^2 F^2 \left(1 - \cos\theta_L\right)\right]$
$\rho_S/\rho_L < 1$
$\rho_S/\rho_L < 1$
π
0
$-\pi$
π
0
$-\pi$
$\Theta_L(0)$
$\Theta_S(0)$



$\rho_S / \rho_{\text{Total}}$ with $\mathcal{F}$ = 20 and μ = 0.85
$\rho_S/\rho_L > 1$
Autoresonance
$\rho_S/\rho_L < 1$
$\rho_S/\rho_L < 1$
Autoresonance
$\rho_S/\rho_L > 1$
π
0
$-\pi$
$\Theta_L(0)$
π
0
$-\pi$
$\Theta_S(0)$


$\rho_S/\rho_{\text{Total}}$ with $\mathcal{F} = 20$ and $\mu = 0.85$
π
0
$-\pi$
$\Theta_L(0)$
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Autoresonance
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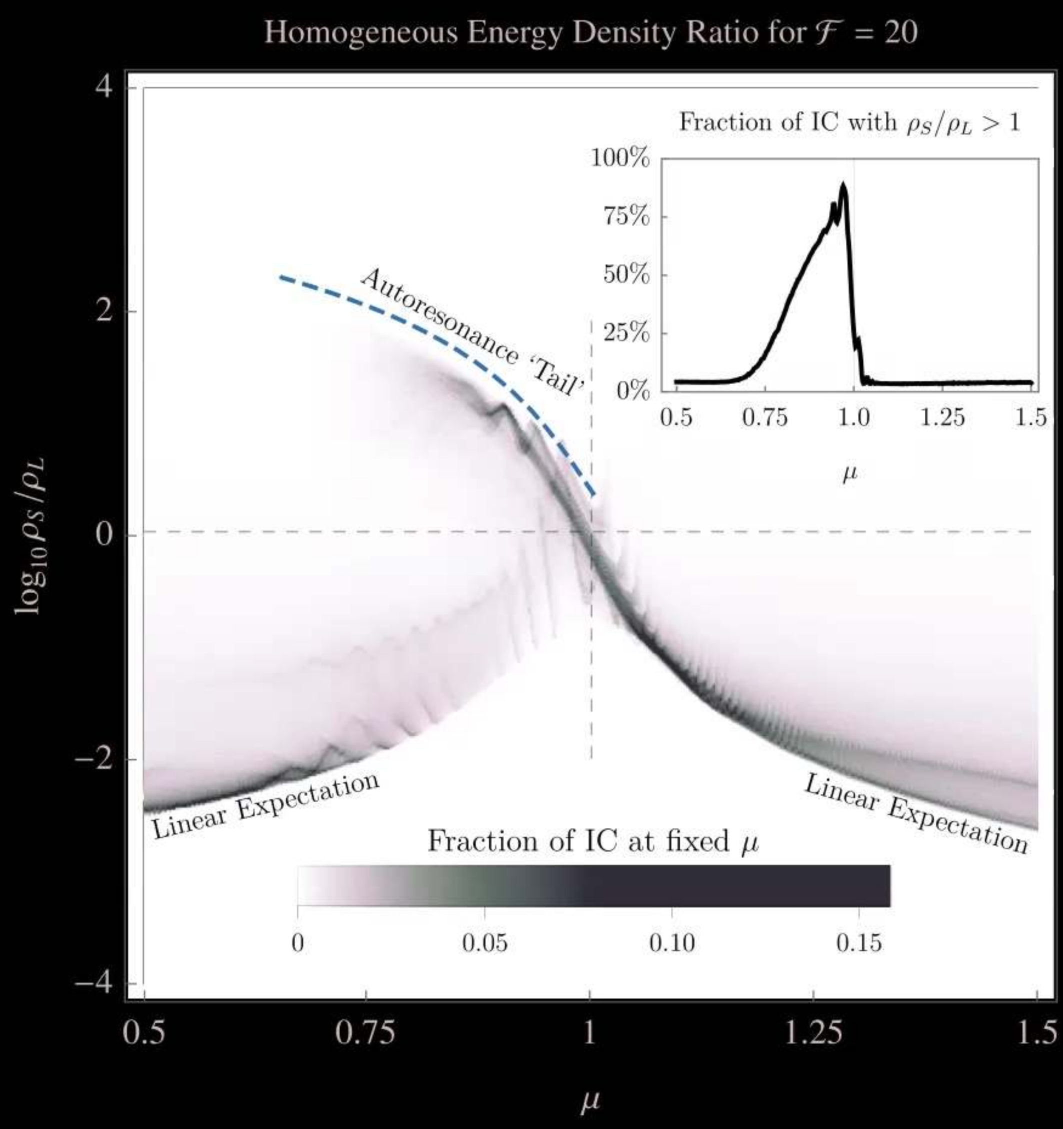
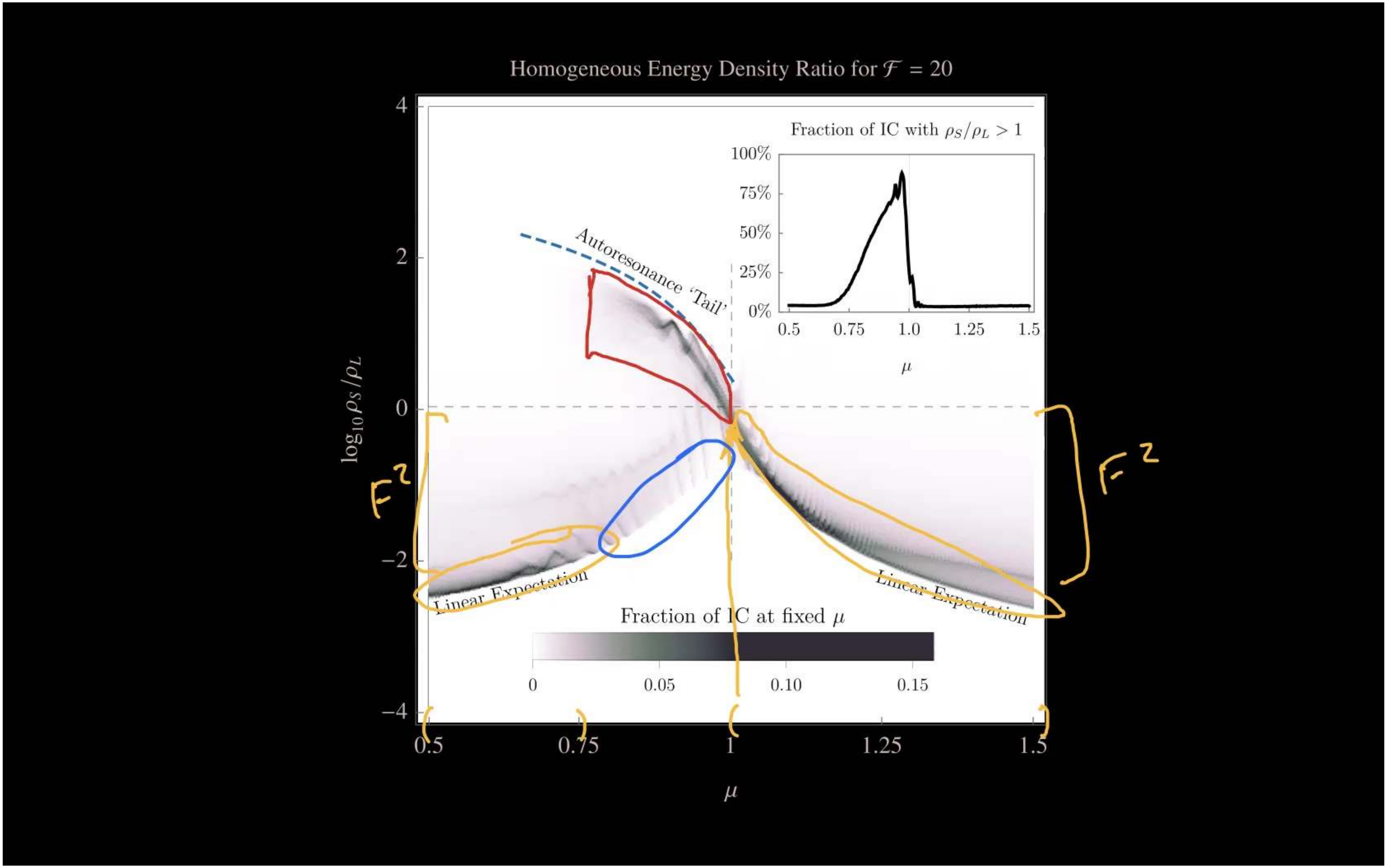
$\rho_S/\rho_L < 1$
Autoresonance
$\rho_S/\rho_L > 1$
π
0
$-\pi$
$\Theta_S(0)$



Homogeneous Energy Density Ratio for $\mathcal{F}$ = 20
Fraction of IC with $\rho_S/\rho_L > 1$
100%
75%
50%
25%
0%
0.5
0.75
1.0
1.25
1.5
μ
Autoresonance 'Tail'
Linear Expectation
Linear Expectation
Fraction of IC at fixed μ
0
0.05
0.10
0.15
$\log_{10}\rho_S/\rho_L$
4
2
0
−2
−4
0.5
0.75
1
1.25
1.5
μ



Homogeneous Energy Density Ratio for $\mathcal{F} = 20$
Fraction of IC with $\rho_S/\rho_L > 1$
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1.0
1.25
1.5
μ
Autoresonance 'Tail'
Linear Expectation
Linear Expectation
Fraction of IC at fixed μ
0
0.05
0.10
0.15
4
2
0
−2
−4
$\log_{10} \rho_S/\rho_L$
0.5
0.75
1
1.25
1.5
μ
F^2
F^2

What are the dynamics of autoresonance?

Quasi-equilibrium of a damped driven pendulum

How friendly do axions need to be for autoresonance to be common?

$0.75 \lesssim \mu < 1$ (theoretical range $0.64 < \mu < 1$)

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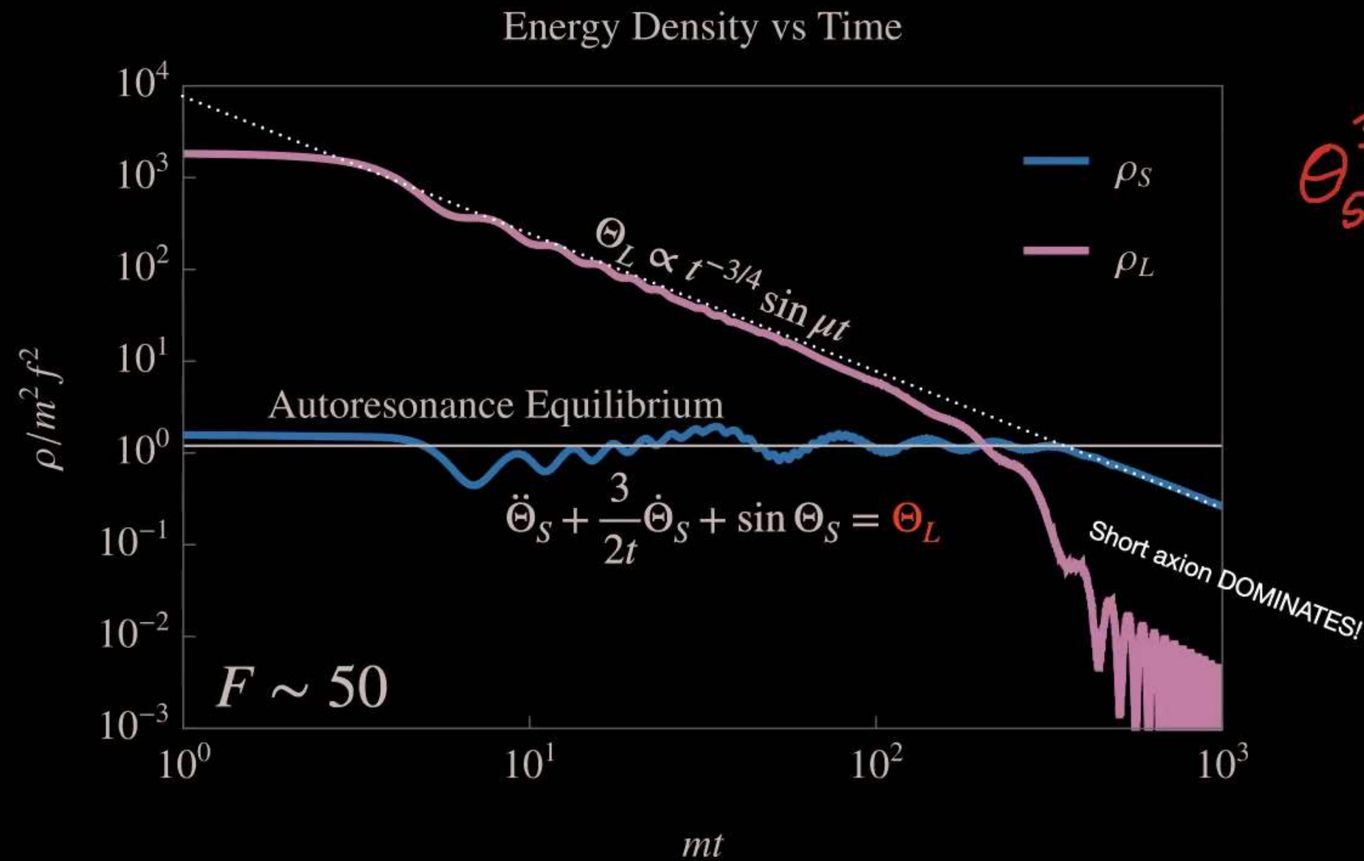
$\theta_L \cos\theta_S$

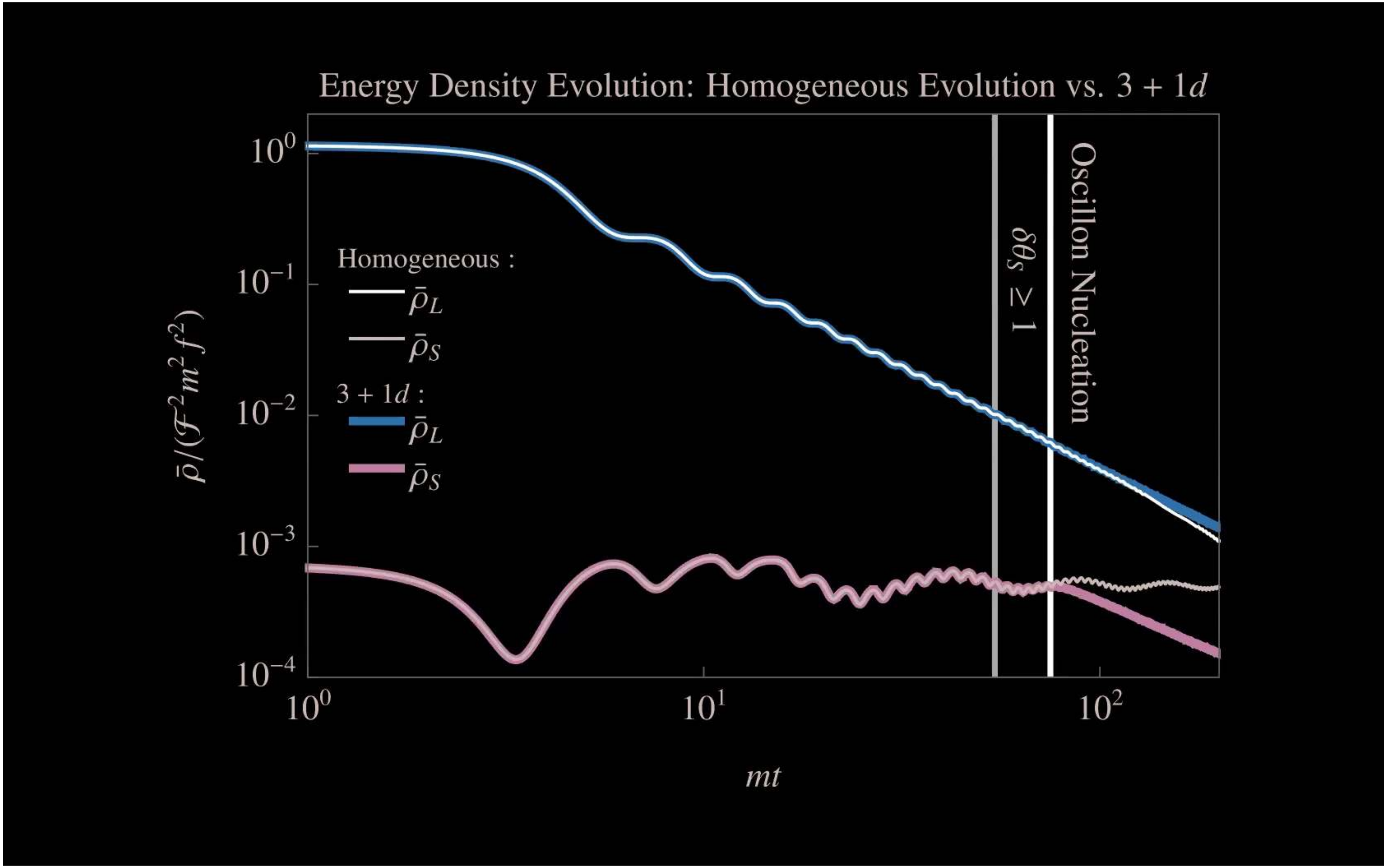
$$V(\theta_S, \theta_L) = m^2 f^2 \left[\left(1 - \cos\left(\theta_S + \theta_L\right)\right) + \mu^2 F^2 \left(1 - \cos\theta_L\right) \right]$$

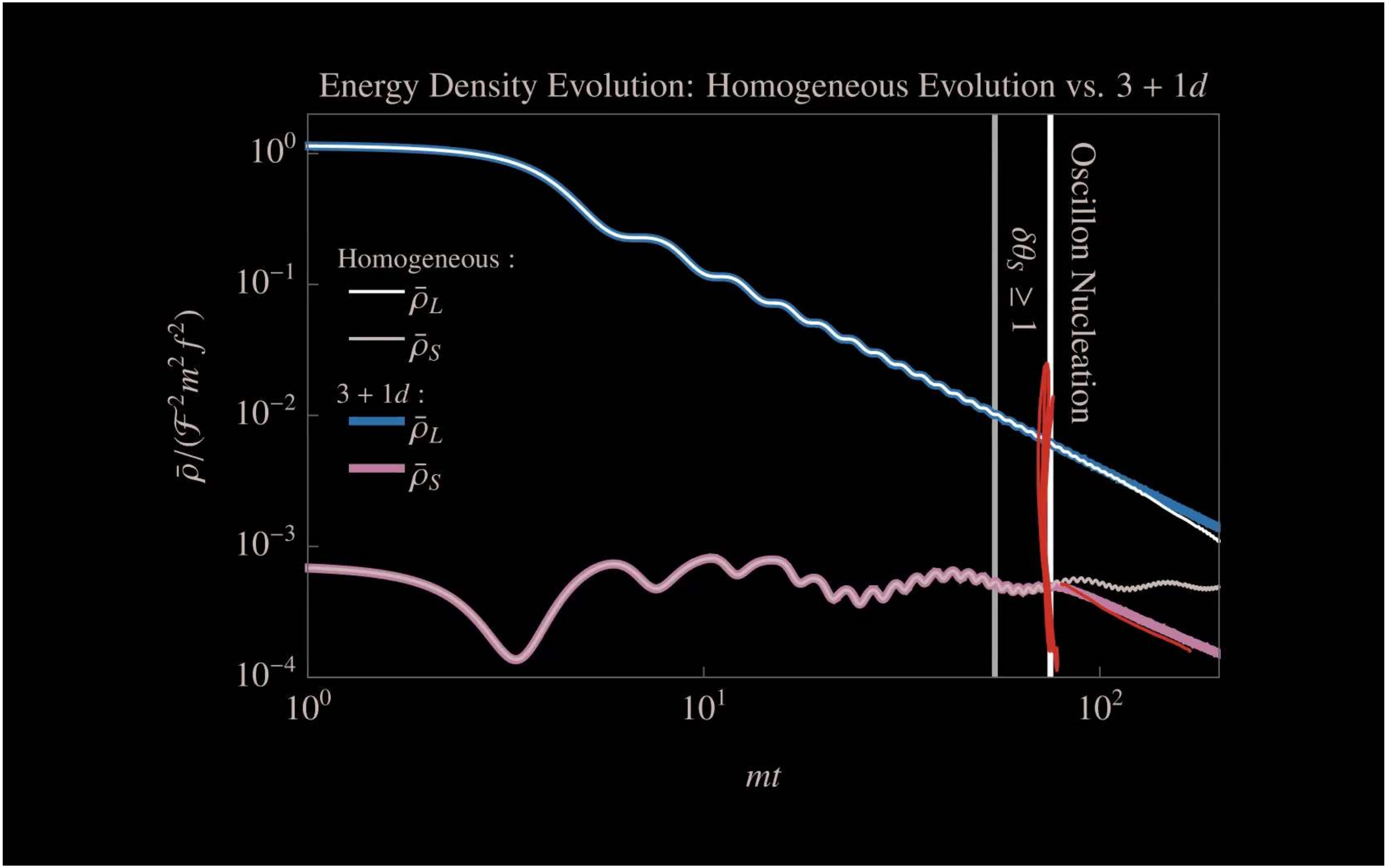


$\theta_S^3 > \theta_L \theta_S^2$

Inhomogeneous Modes

- During autoresonance, the short axion is held at large amplitudes where attractive self-interactions are strong
  - Horizon-size modes get produced when the axion starts oscillating
- If these perturbations grow nonlinear they can quench autoresonance
  - $F \gtrsim 20$ (slight μ dependence)

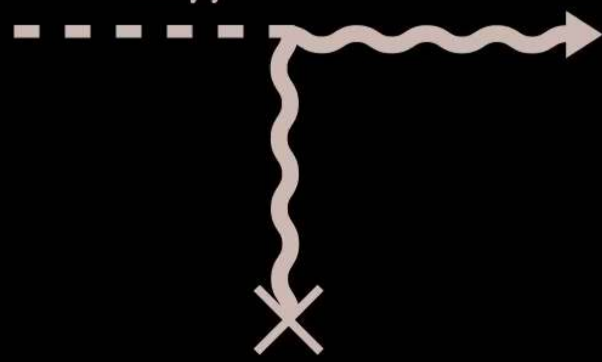

Energy Density Evolution: Homogeneous Evolution vs. 3 + 1d
Homogeneous :
$\bar{\rho}_L$
$\bar{\rho}_S$
3 + 1d :
$\bar{\rho}_L$
$\bar{\rho}_S$
$\delta\theta_S \geq 1$
Oscillon Nucleation
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$\bar{\rho}/(\mathcal{F}^2 m^2 f^2)$
10^0
10^1
10^2
mt


Energy Density Evolution: Homogeneous Evolution vs. 3 + 1d
$\bar{\rho}/(\mathcal{F}^2 m^2 f^2)$
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
Homogeneous :
$\bar{\rho}_L$
$\bar{\rho}_S$
3 + 1d :
$\bar{\rho}_L$
$\bar{\rho}_S$
$\delta\theta_S \geq 1$
Oscillon Nucleation
10^0
10^1
10^2
mt

Direct Detection

- Haloscopes are sensitive to the combination $g_{a\gamma\gamma}^2 \rho$

$$g_{a\gamma\gamma} \sim \alpha / 4\pi f$$



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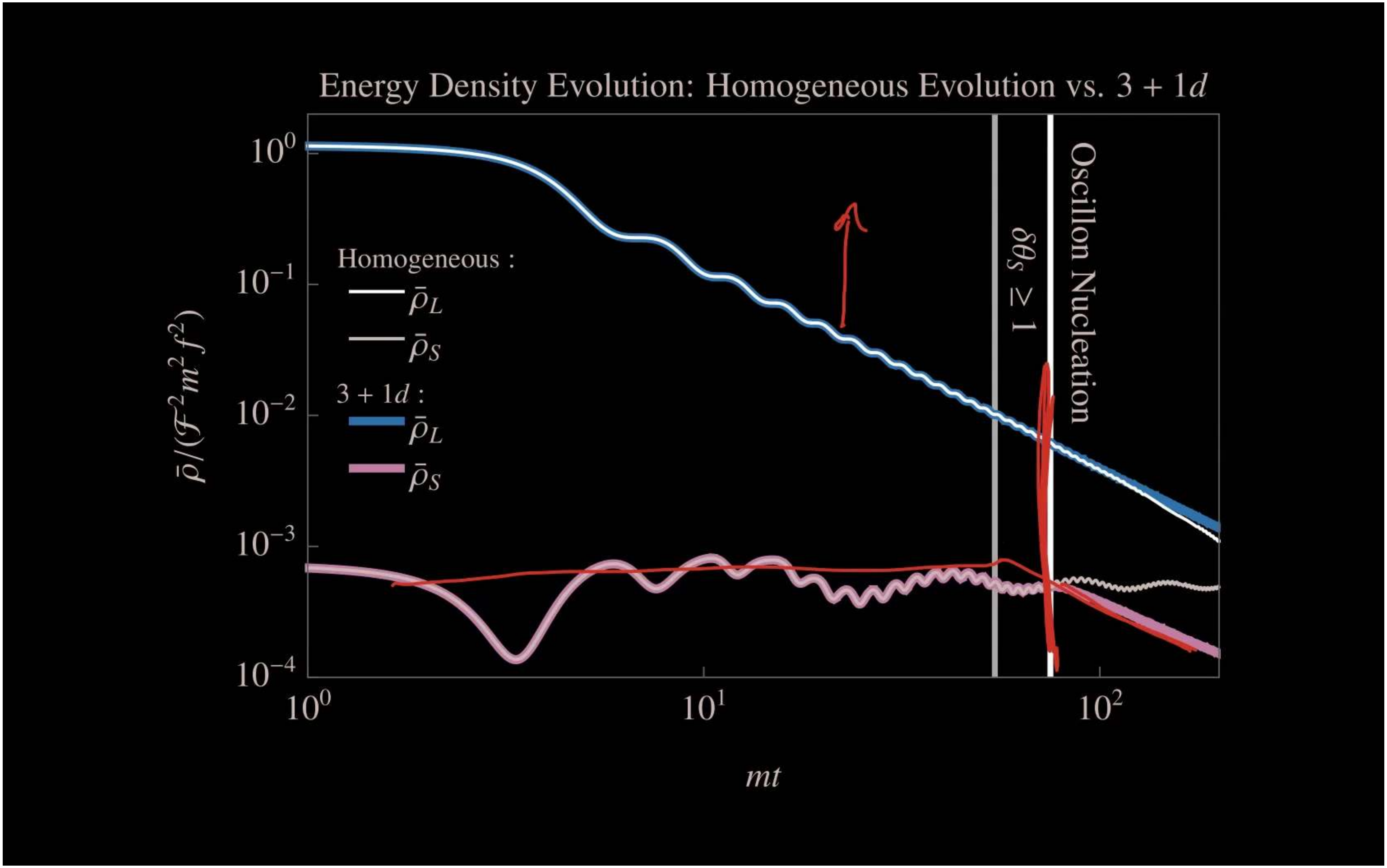


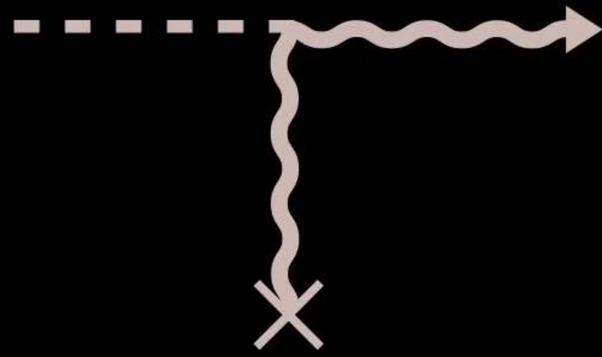
Energy Density Evolution: Homogeneous Evolution vs. 3 + 1d
$\bar{\rho}/(\mathcal{F}^2 m^2 f^2)$
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
Homogeneous :
$\bar{\rho}_L$
$\bar{\rho}_S$
3 + 1d :
$\bar{\rho}_L$
$\bar{\rho}_S$
$\delta\theta_S \geq 1$
Oscillon Nucleation
10^0
10^1
10^2
mt


Signatures: Direct Detection

Direct Detection

- Haloscopes are sensitive to the combination $g_{a\gamma\gamma}^2 \rho$

$$g_{a\gamma\gamma} \sim \alpha / 4\pi f$$



Direct detection prospects: Lonely Axion

- Consider a lonely axion, living in a cosine potential

$$V(\phi) = m^2 f^2 (1 - \cos \phi / f)$$

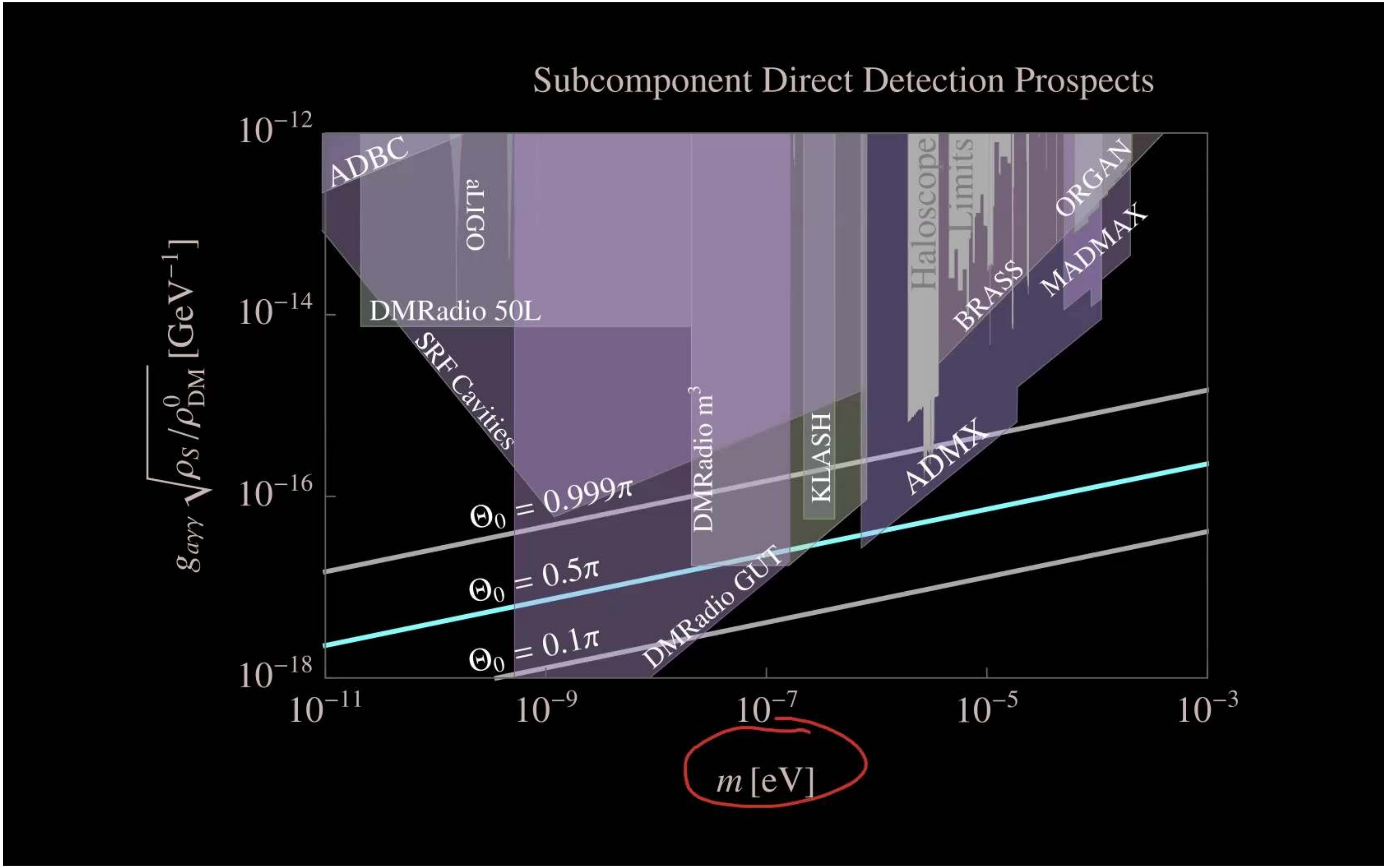
- Relic abundance:

$$\frac{\rho_{\rm Lonely}}{\rho_{\rm crit}} \sim 0.4 \left(\frac{\Theta(0)}{\pi/2}\right)^2 \left(\frac{m}{10^{-17}\text{eV}}\right)^{1/2} \left(\frac{f}{10^{16}\text{GeV}}\right)^2$$

$$g_{a\gamma\gamma} \sim \frac{\alpha}{4\pi f}$$

- Sensitivity to a single axion is independent of f:

$$\left(g_{a\gamma\gamma}^2 \frac{\rho_{\rm Lonely}}{\rho_{DM}^0}\right)^{1/2} \sim 2.3 \times 10^{-17}\text{GeV}^{-1} \left(\frac{\Theta(0)}{\pi/2}\right) \left(\frac{m}{10^{-17}\text{eV}}\right)^{1/4}$$


Subcomponent Direct Detection Prospects
ADBC
aLIGO
DMRadio 50L
SRF Cavities
DMRadio m^3
KLASH
Haloscope Limits
BRASS
ORGAN
MADMAX
ADMX
DMRadio GUT
$\Theta_0 = 0.999\pi$
$\Theta_0 = 0.5\pi$
$\Theta_0 = 0.1\pi$
$g_{a\gamma\gamma}\sqrt{\rho_S/\rho^0_{DM}}$ [GeV^{-1}]
m [eV]
10^{-12}
10^{-14}
10^{-16}
10^{-18}
10^{-11}
10^{-9}
10^{-7}
10^{-5}
10^{-3}

Direct detection prospects: Friendly Axion

- The energy transferred from θ_L to θ_S enhances ρ_S relative to the lonely-axion expectation:

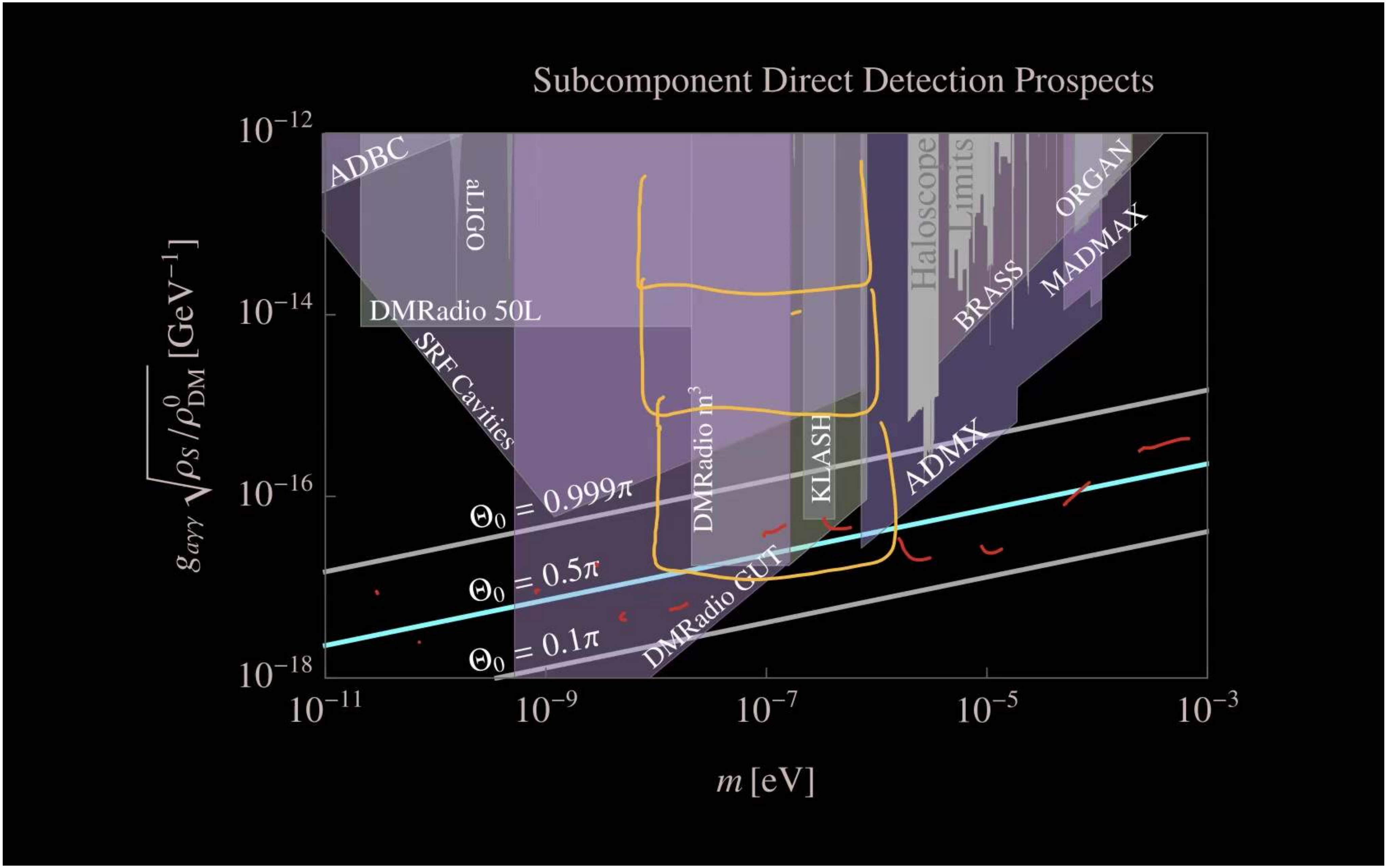
$$\rho_S \approx F^2 \rho_{\text{Lonely}}$$

- θ_S has a smaller f, *and* enhanced energy density:

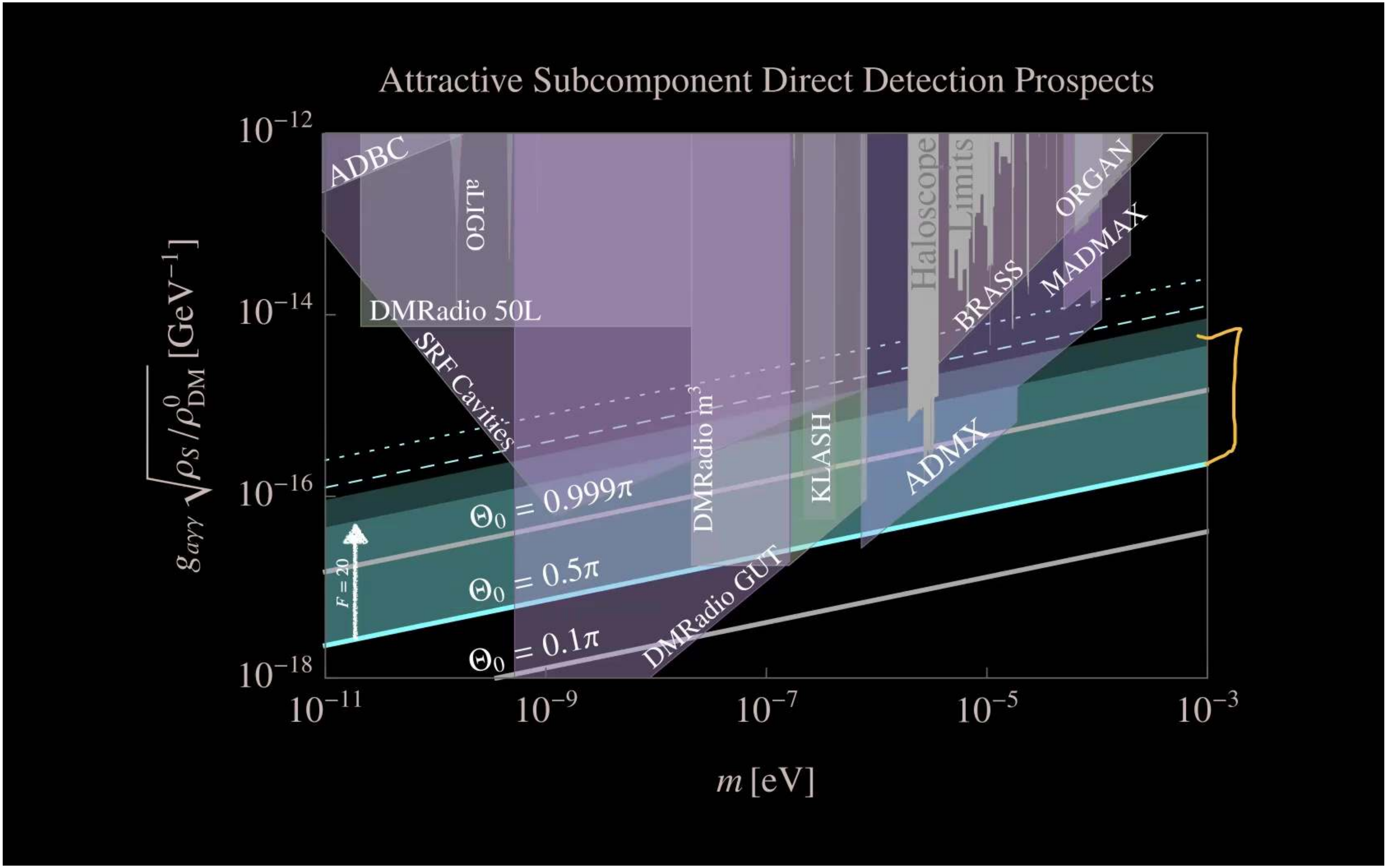
Best of both worlds

- Stronger coupling *and* more axions.

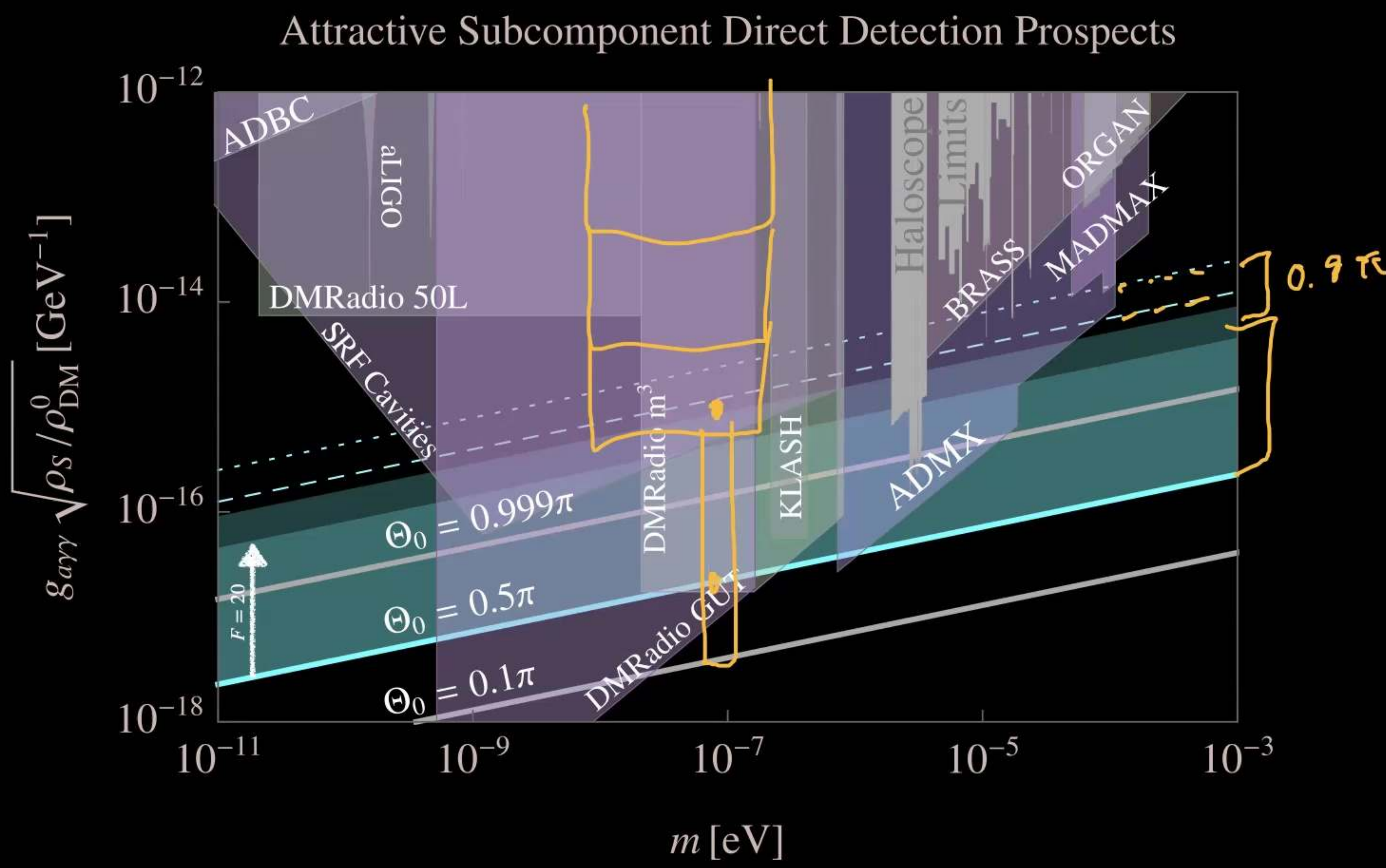
- Does not depend on whether the friendly pair is the DM



Subcomponent Direct Detection Prospects
$g_{a\gamma\gamma}\sqrt{\rho_S/\rho^0_{DM}}$ [GeV^{-1}]
10^{-12}
10^{-14}
10^{-16}
10^{-18}
10^{-11}
10^{-9}
10^{-7}
10^{-5}
10^{-3}
m [eV]
ADBC
aLIGO
DMRadio 50L
SRF Cavities
DMRadio m^3
KLASH
Haloscope Limits
BRASS
ORGAN
MADMAX
ADMX
DMRadio CUT
$\Theta_0 = 0.999\pi$
$\Theta_0 = 0.5\pi$
$\Theta_0 = 0.1\pi$



Attractive Subcomponent Direct Detection Prospects
ADBC
aLIGO
DMRadio 50L
SRF Cavities
DMRadio m^3
KLASH
Haloscope Limits
BRASS
ORGAN
MADMAX
ADMX
DMRadio GUT
$\Theta_0 = 0.999\pi$
$\Theta_0 = 0.5\pi$
$\Theta_0 = 0.1\pi$
$F = 20$
$g_{a\gamma\gamma}\sqrt{\rho_S/\rho^0_{DM}}$ [GeV^{-1}]
m [eV]
10^{-12}
10^{-14}
10^{-16}
10^{-18}
10^{-11}
10^{-9}
10^{-7}
10^{-5}
10^{-3}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0.9 π
10^{-12}
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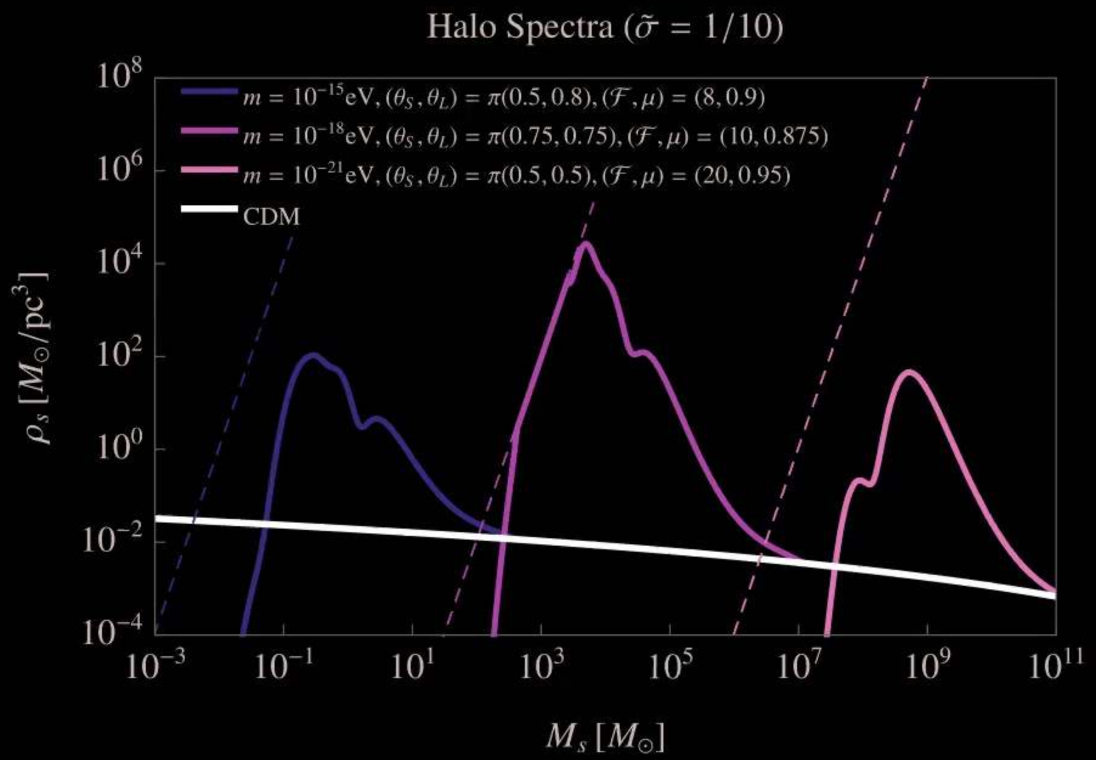
10^{-3}
$g_{a\gamma\gamma}\sqrt{\rho_S/\rho^0_{DM}}$ [GeV^{-1}]
m [eV]

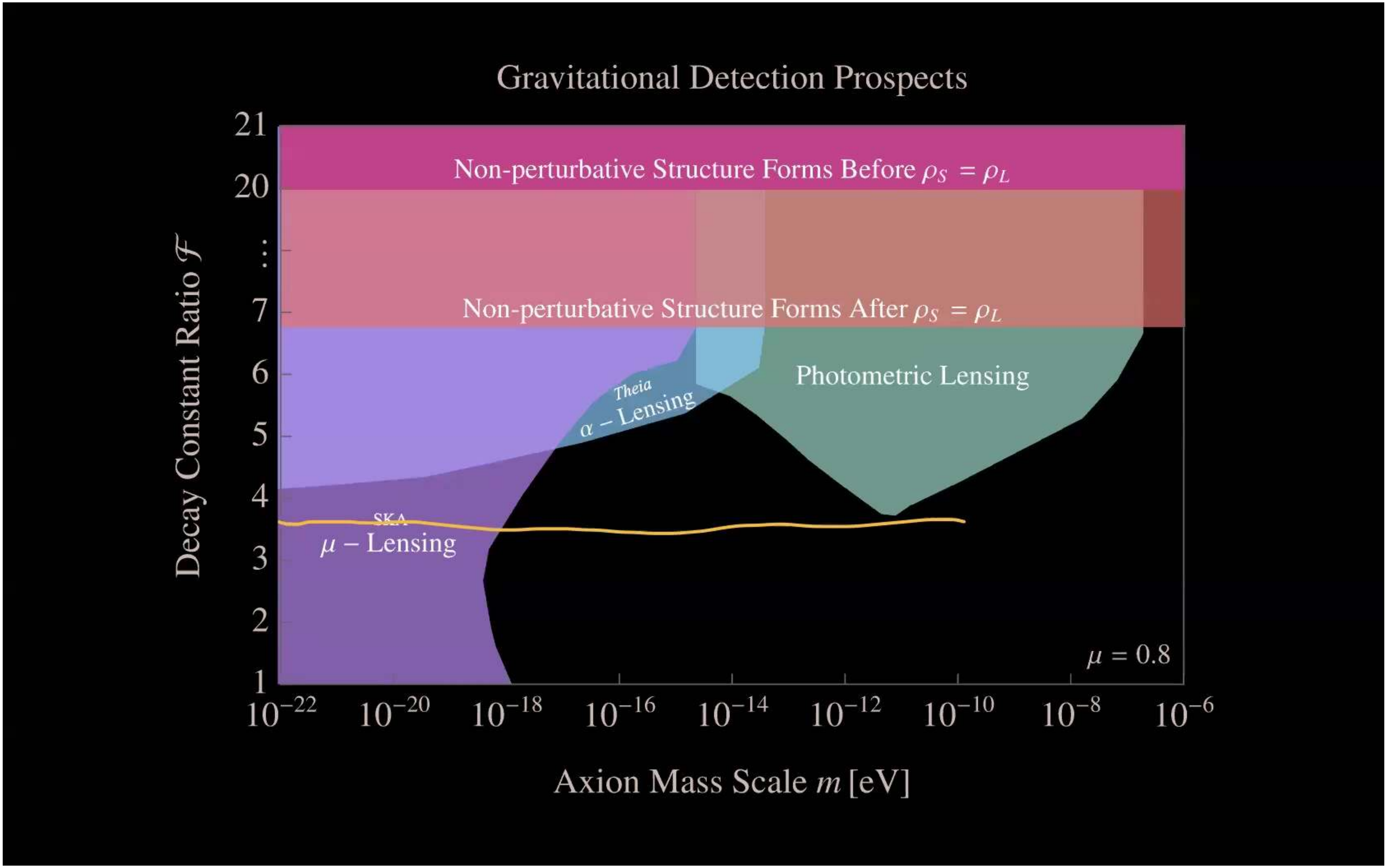
Gravitational detection prospects

- If the friendly pair is the DM, density perturbations form axion mini halos:

$$M \sim 1.2 \times 10^4 M_{\odot} \left(\frac{10^{-19}\text{eV}}{m} \right)^{3/2}$$

- Gravitational signatures vanish if autoresonance is quenched by perturbation growth




Gravitational Detection Prospects
Non-perturbative Structure Forms Before $\rho_S = \rho_L$
Non-perturbative Structure Forms After $\rho_S = \rho_L$
Photometric Lensing
Theia
α – Lensing
SKA
μ – Lensing
$\mu = 0.8$
Decay Constant Ratio $\mathcal{F}$
Axion Mass Scale m [eV]
21
20
7
6
5
4
3
2
1
10^{-22}
10^{-20}
10^{-18}
10^{-16}
10^{-14}
10^{-12}
10^{-10}
10^{-8}
10^{-6}

Summary

- Friendly axions are more visible than lonely axions

- Discovering a highly-visible axion should prompt a search for more weakly coupled axions at nearby masses

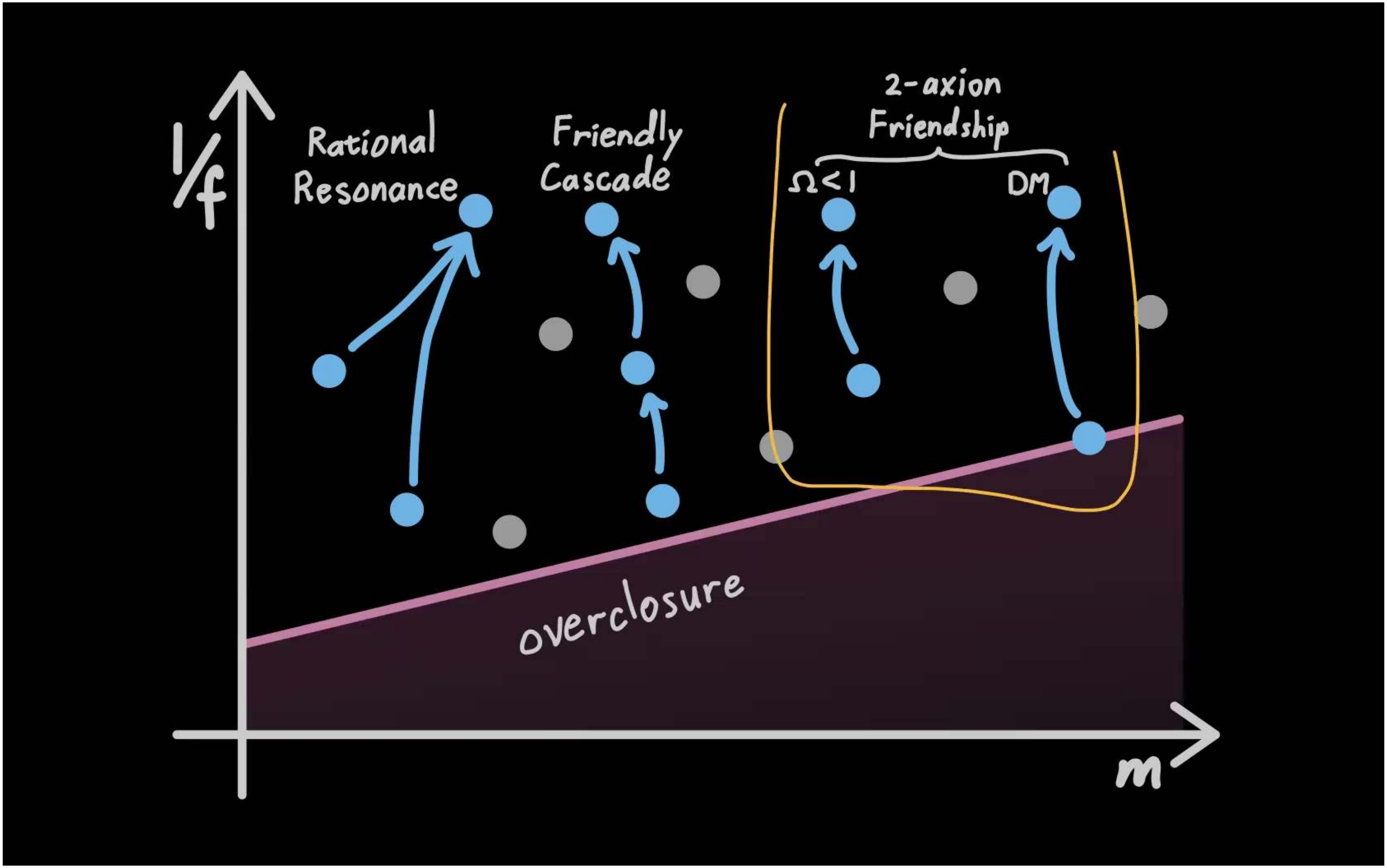
- Discovery of a friendly pair would be evidence that we live in a dense axiverse

Future Directions

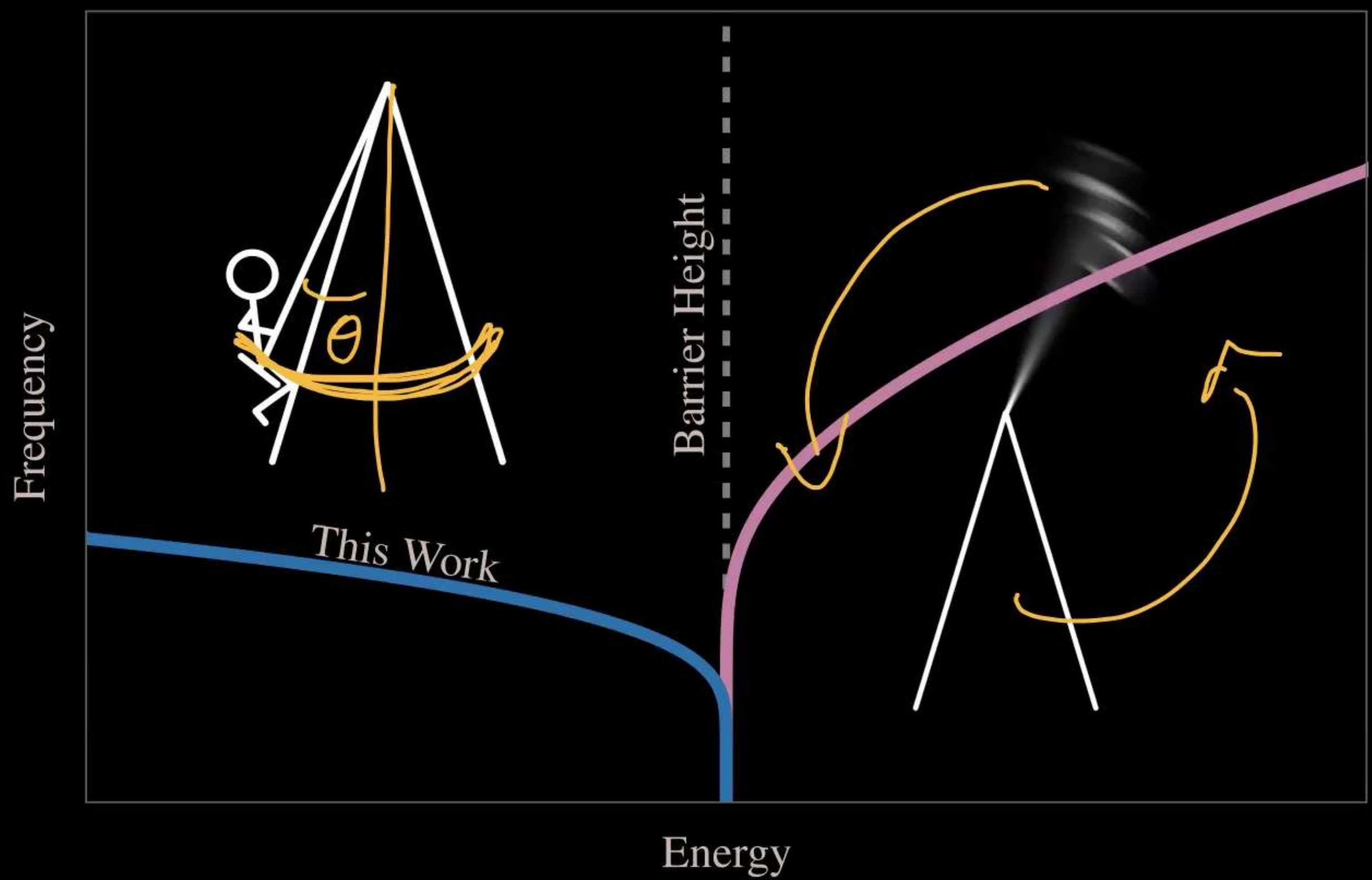
- Dynamics of axions in realistic string compactifications [ongoing work with Viraf Mehta, Tudor Giurgica-Tiron, Olivier Simon, Jed Thompson]

- Simulations of matter power spectrum resulting from nonlinear structures during autoresonance

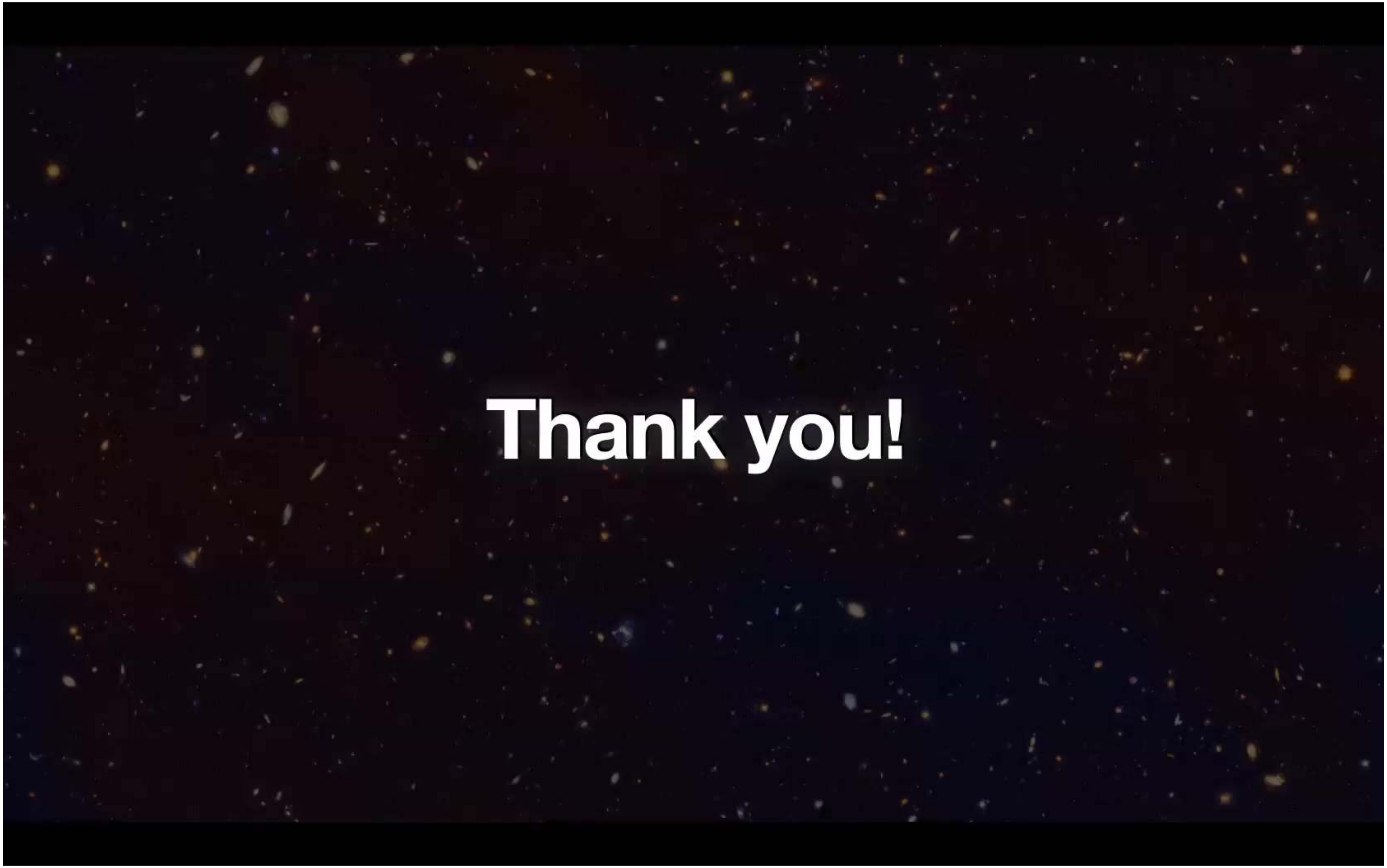
- Simulations of sustained nonlinear structures in θ_S (enhanced oscillon longevity)



1/f
Rational Resonance
Friendly Cascade
2-axion Friendship
Ω<1
DM
overclosure
m



Frequency
Barrier Height
This Work
Energy


Thank you!

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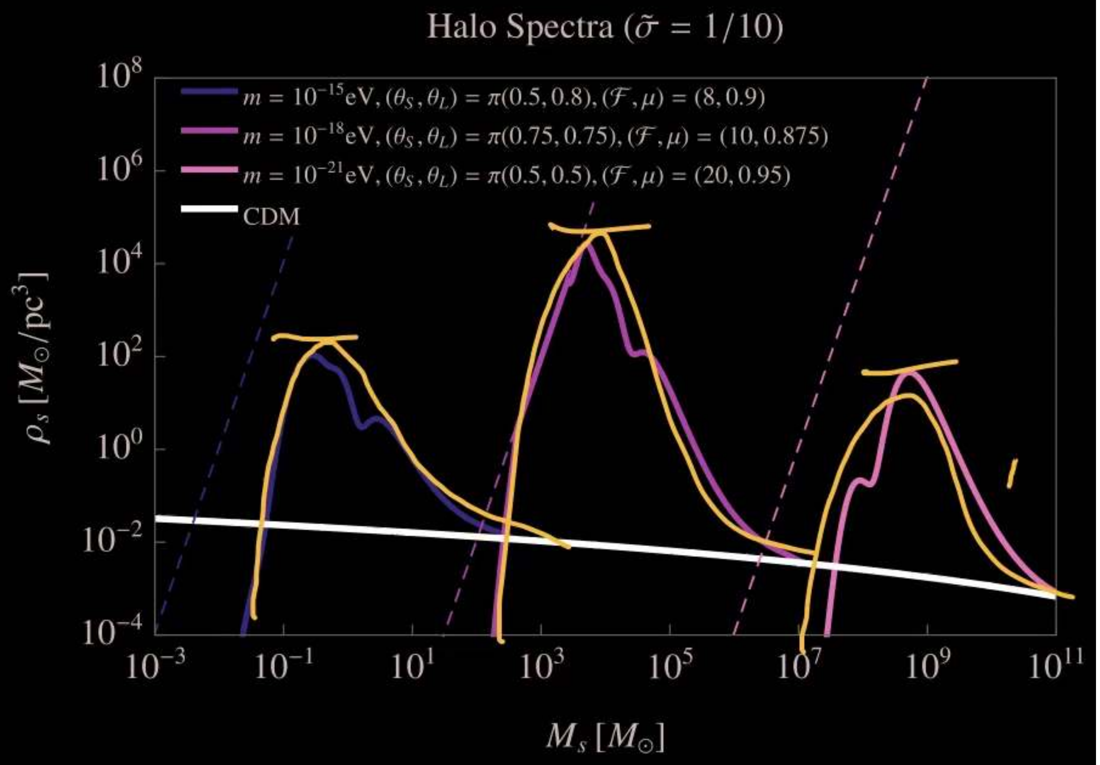

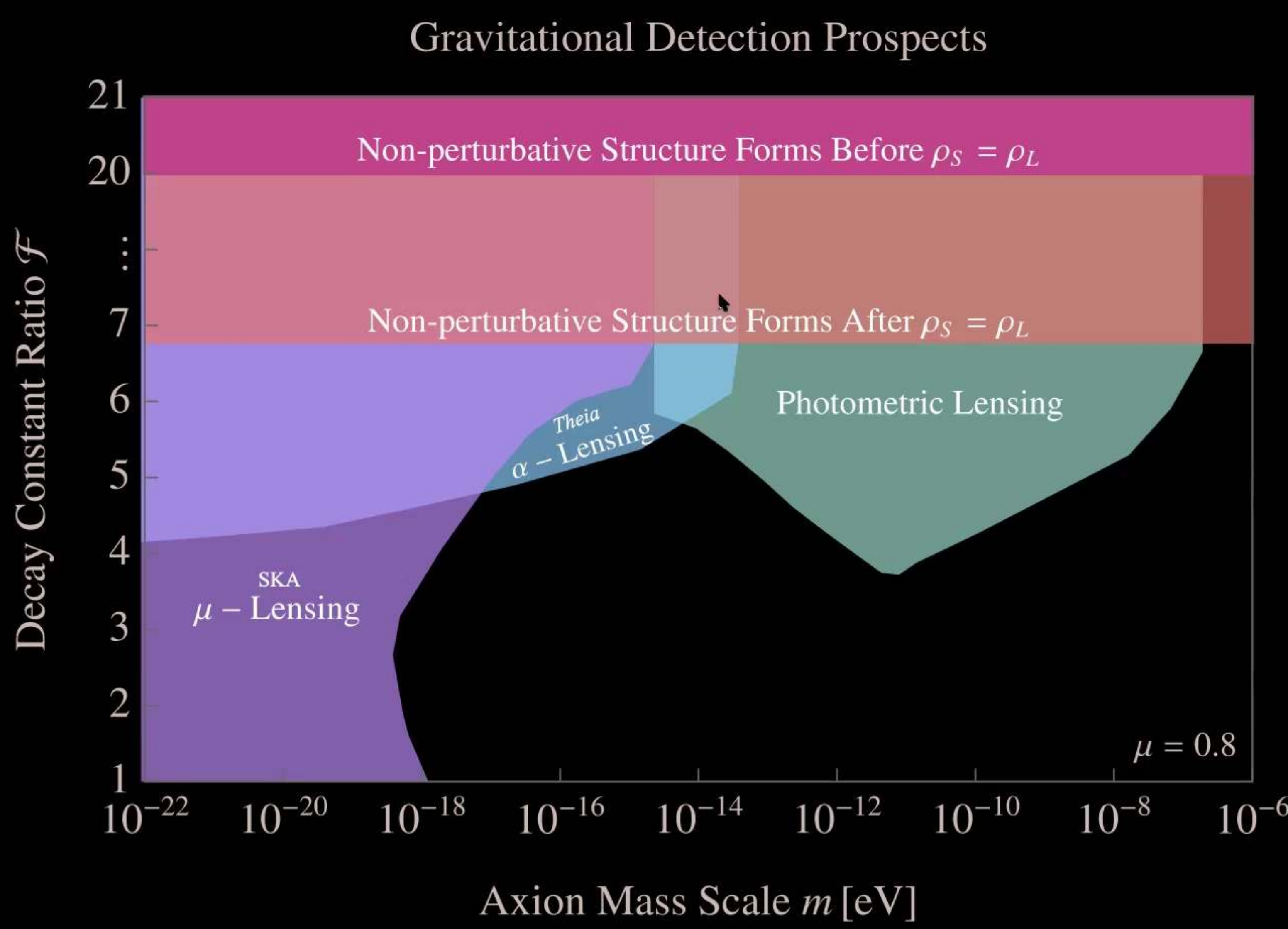
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21
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2
1
10^{-22} 10^{-20} 10^{-18} 10^{-16} 10^{-14} 10^{-12} 10^{-10} 10^{-8} 10^{-6}
Axion Mass Scale m [eV]

Statistics

- What we need: Λ_1^2/Λ_2^2 lies in some $\mathcal{O}(0.25)$ interval
- $\Lambda_i^4 = M_{\rm UV}^4 e^{-S_i} \implies c \lesssim |S_1 - S_2| \lesssim c + {\rm d}S$, with ${\rm d}S \sim 0.5$
- If S_i are uniformly distributed over $[S_{\rm min}, S_{\rm max}]$:
    - Average number of friendly S-pairs: $\langle \# \,|\, S_i - S_j - c \,|\, < {\rm d}S \rangle \sim N^2 \times {\rm d}S/S_{\rm max}$ for $c \ll S_{\rm max}$, when $0 < S_{\rm min} \ll S_{\rm max}$
- Expect at least one coincidence:
    - $S_{\rm max} \lesssim N^2 {\rm d}S$
    - 100 axions: $S_{\rm max} < 5 \times 10^3$
    - 491 axions: $S_{\rm max} < 1.2 \times 10^5$