Title: The Markov gap for geometric reflected entropy

Speakers: Jonathan Sorce

Series: Perimeter Institute Quantum Discussions

Date: November 17, 2021 - 3:30 PM

URL: https://pirsa.org/21110037

Abstract: This talk concerns the "Markov gap," a tripartite-entanglement measure with a simple geometric dual in holographic quantum gravity. I will prove a new inequality constraining the Markov gap of classical states in quantum gravity, and interpret this inequality as a lesson about multipartite entanglement in holography. I will also speculate about signatures of the inequality in non-holographic field theories, and conjecture a new universal entanglement feature of two-dimensional CFTs.

Zoom Link: https://pitp.zoom.us/j/98327637522?pwd=TUJOQ0d1aU5Gc0RLTlJLd3B3Ty9LUT09

The Markov gap for geometric reflected entropy

based on arXiv:2107.00009 with Patrick Hayden and Onkar Parrikar

Jon Sorce

Perimeter Institute Seminar

November 16, 2021

Outline

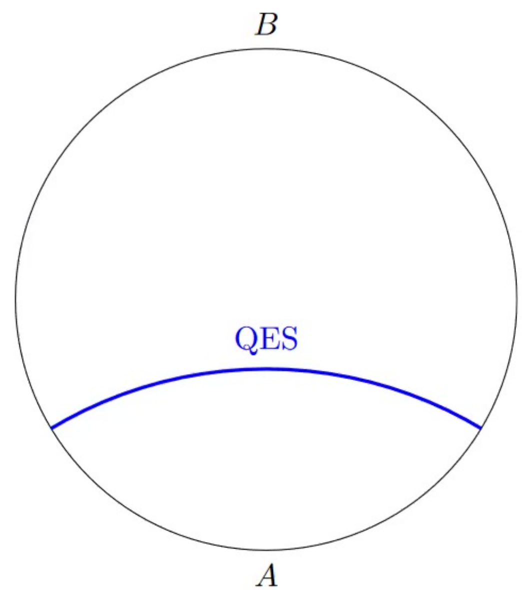
Back to basics: holographic entanglement entropy

Quantum extremal surface formula:

[Ryu-Takayanagi, Hubeny-Rangamani-Takayanagi,

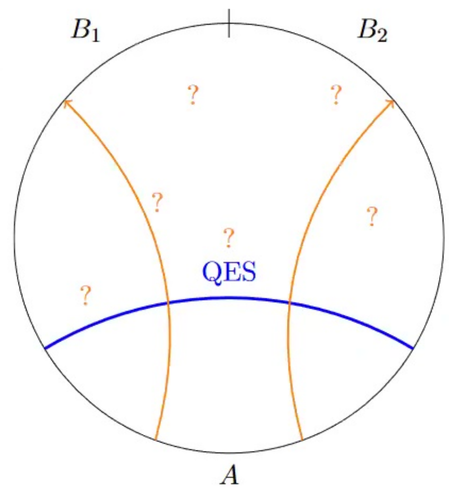
Faulkner-Lewkowycz-Maldacena, Engelhardt-Wall, ...]

$$S_{\mathrm{vN}}(A)_\rho = \frac{\mathrm{area(QES)}}{4G_N} + S_{\mathrm{bulk}}.$$



Ryu-Takayanagi is not enough!

- The von Neumann entropy only depends on a bipartition of the boundary state, so it can't tell us anything about multipartite entanglement.



- If we want to understand the rich structure of holographic entanglement, we need more detailed entanglement measures.

Back to basics: holographic entanglement entropy

Quantum extremal surface formula:

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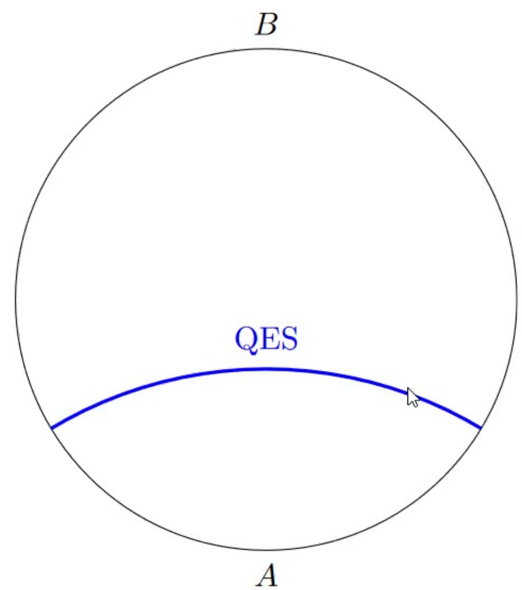
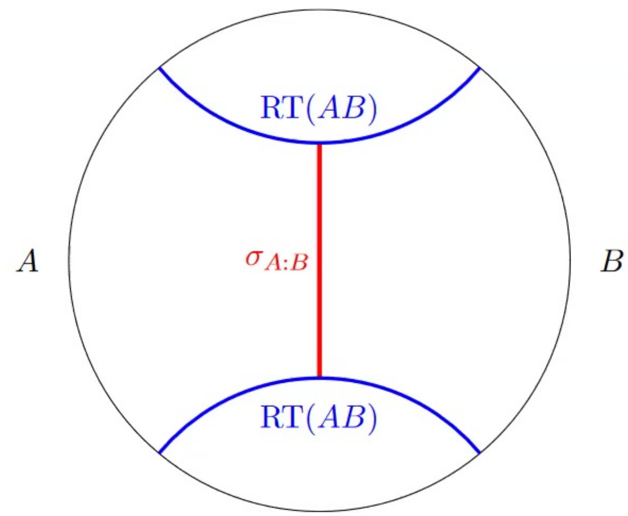


The entanglement wedge cross-section

- Start simple: tripartitions.
- Natural geometric object associated with a tripartition: the **entanglement wedge cross-section** $\sigma_{A:B}$ [Nguyen-Devakul-Halbasch-Zaletel-Swingle, Umemoto-Takayanagi]



Canonical purifications

Defining the canonical purification

Given a density matrix ρ, write

$$\rho = \sum_j p_j |j\rangle\langle j| .$$

Define $|\psi_\rho\rangle \equiv \sum_j \sqrt{p_j} |j\rangle |j\rangle$.

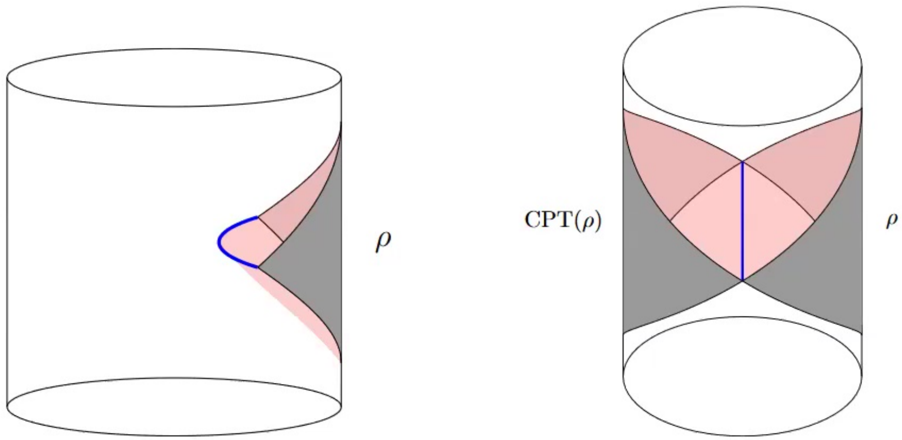
Example

If $\rho = \frac{1}{Z} \sum_j e^{-\beta E_j} |E_j\rangle\langle E_j|$ is thermal, then

$$|\psi_\rho\rangle = \frac{1}{\sqrt{Z}} \sum_j e^{-\beta E_j/2} |E_j\rangle |E_j\rangle \equiv |\mathrm{TFD}\rangle .$$

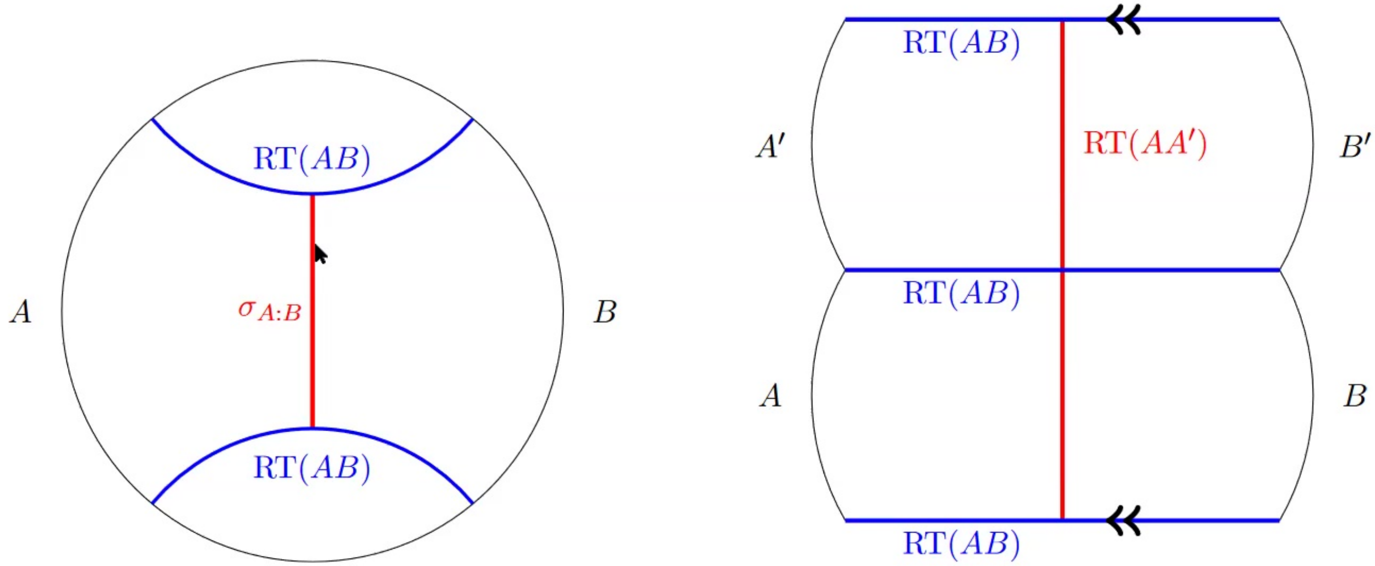
Canonical purifications in gravity

Dutta and Faulkner used the gravitational path integral to argue that for a holographic state, the canonical purification is obtained by sewing two copies of the bulk together along their spatial boundaries. (Think of the black hole $\to$ TFD sewing.)



Back to the entanglement wedge cross-section

Let's look at the canonical purification of our standard bipartite state:



The cross-section entropy is half the entropy of the AA' system.

Reflected entropy

Definition

For a bipartite state ρ_{AB}, the *reflected entropy* is

$$S_R(A:B) \equiv S_{\mathrm{vN}}(AA')_{|\psi_\rho\rangle}.$$

According to Dutta-Faulkner, we have

$$S_R(A:B) = \frac{\mathrm{area}(\sigma_{A:B})}{2G_N} + \text{quantum corrections}$$

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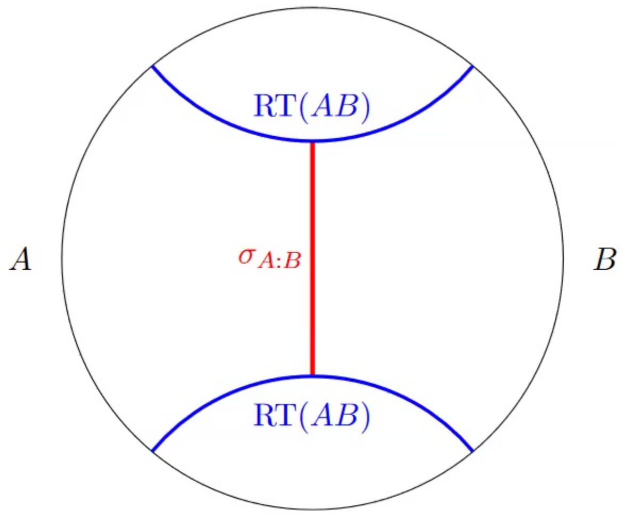

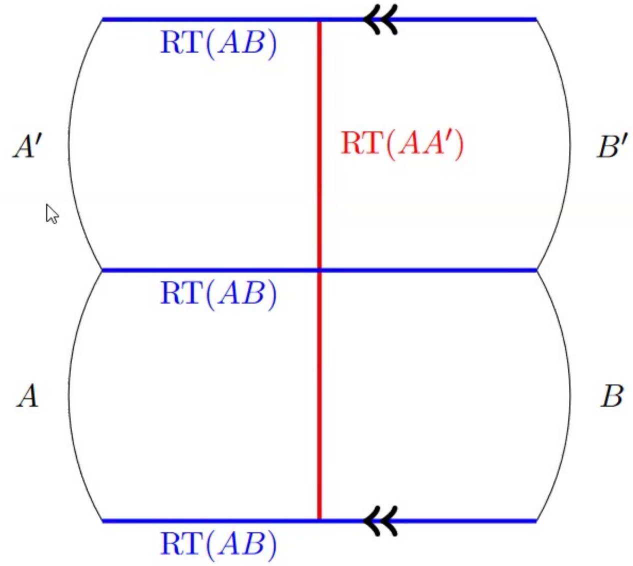

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Some fundamental properties

- For the quantum info experts in the audience:[1]

$$S_R(A:B)_\rho - I(A:B)_\rho = I(A:B'|B)_{|\psi_\rho\rangle}.$$

- This implies

$$S_R(A:B) - I(A:B) \geq 0.$$

  We will call $S_R - I$ the *Markov gap.*

- If a holographic state has $O(1/G_N)$ Markov gap, then it must have $O(1/G_N)$ multipartite entanglement. [Akers-Rath]

[1] $I(A:B) = S(A) + S(B) - S(AB)$.

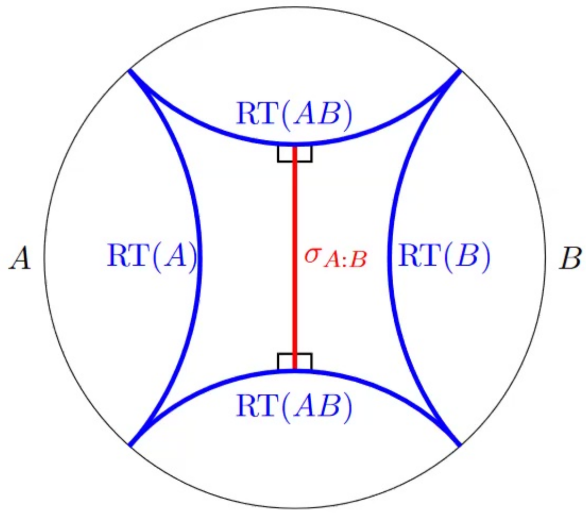
Main technical result

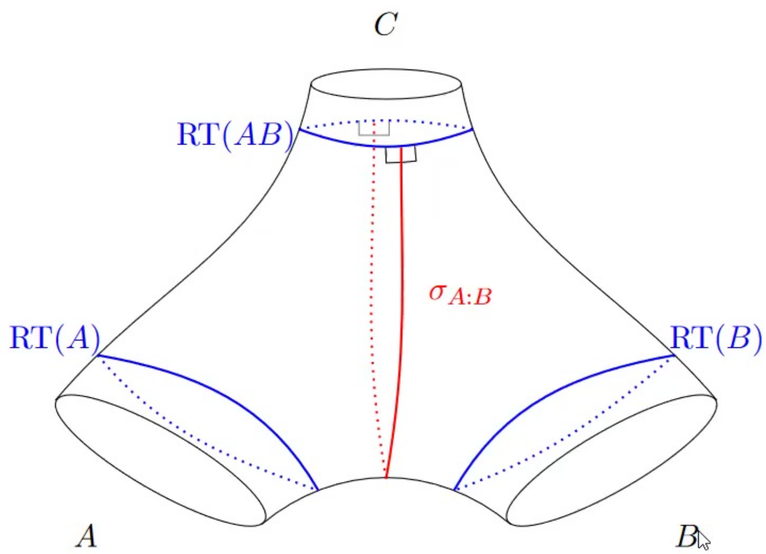
For (time-symmetric) AdS_3 states with no matter, we have proven the following inequality:

$$S_R(A:B) - I(A:B) \geq \frac{\log(2)}{2G_N} |\partial \sigma_{A:B}|.$$

The quantity $|\partial \sigma_{A:B}|$ is the *cardinality of the cross-section boundary*, i.e., the number of cross-section endpoints.

Two examples





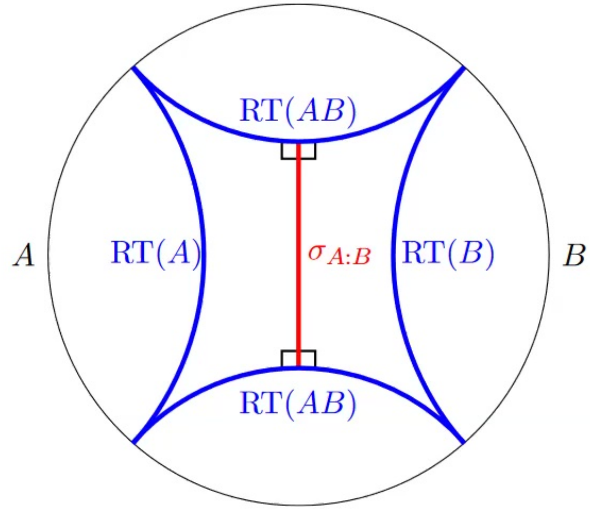
Heuristic interpretation

$$S_R(A:B) - I(A:B) \geq \frac{\log(2)}{2G_N}|\partial\sigma_{A:B}|.$$

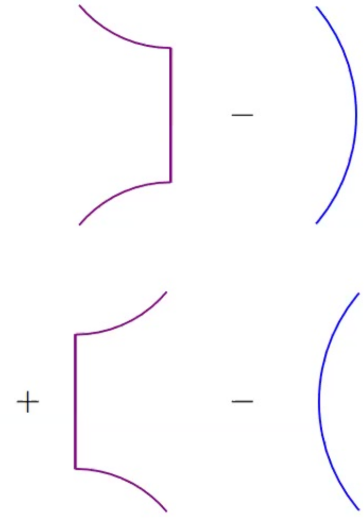
- Because $S_R - I$ is $O(1/G_N)$ only if there is significant tripartite entanglement, this formula suggests that $\partial\sigma_{A:B}$ emerges from irreducible tripartite entanglement.
- Generalizing this inequality to states with matter, or to higher dimensions, would help us better understand universal tripartite entanglement in quantum gravity.

RT and KRT surfaces

Let's look at an example two-party state:



The Markov gap can be expressed visually as



Reframing the question

We want to lower-bound the area difference between KRT and RT in terms of the number of kinks in KRT.

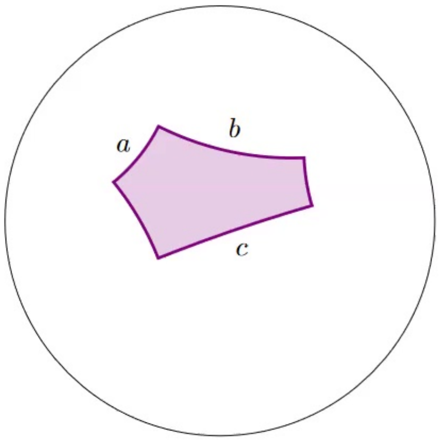
Question

On a hyperbolic 2-manifold, if KRT is a geodesic with right-angled kinks and RT the minimal homologous geodesic, can we show

$$\text{area}(\text{KRT}) - \text{area}(\text{RT}) \geq \log(2) \times |\text{kinks}|?$$

A simple example using right-angled pentagons

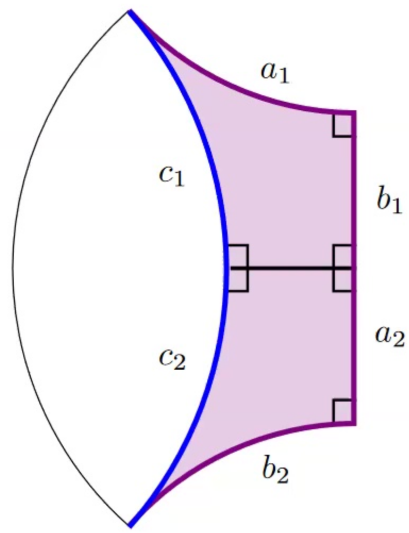
In hyperbolic space, the right-angled pentagon



satisfies

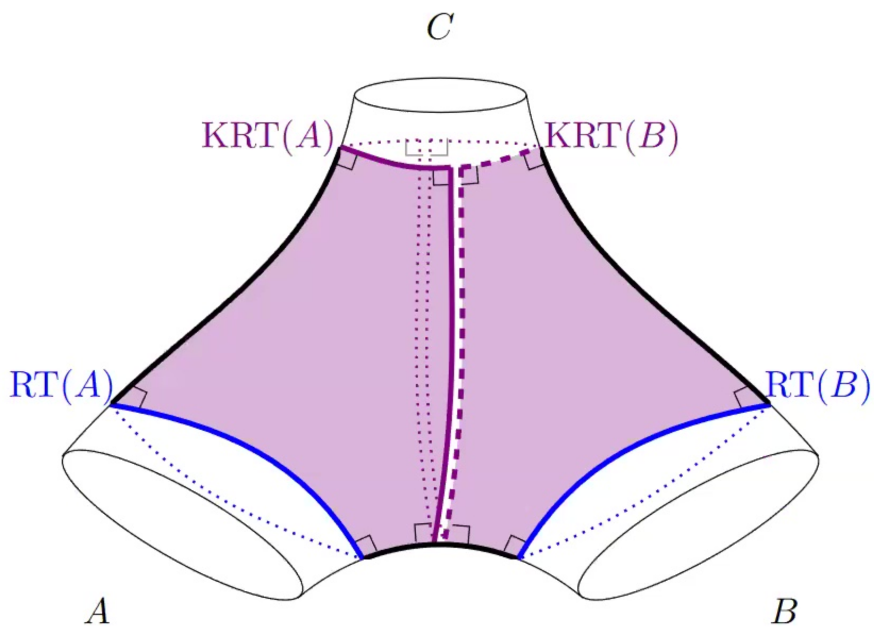
$$a + b - c \geq \log(2).$$

Can use these to tile between KRT and RT:



One more example

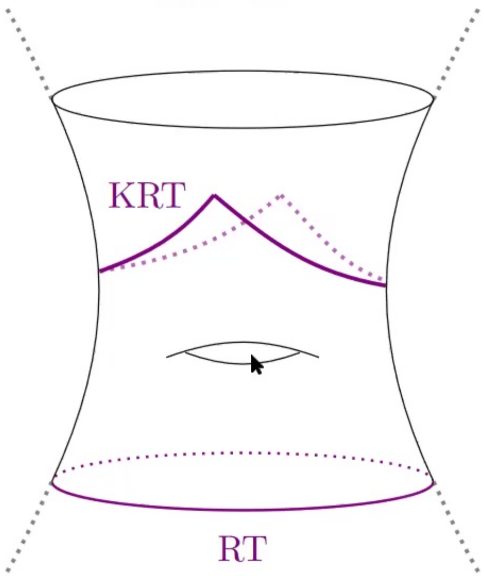
Generally, our strategy will be to tile the region between RT and KRT with pentagons:



Obstacles

Two main issues must be addressed:

1. These tilings can't exist for higher-genus manifolds:



2. Even in the absence of topological obstructions, how do we know these tilings exist?

Strategy

Using covering space theory, we

1. Show that KRT is *homotopic* to a smooth geodesic γ.
2. Show that the *homotopy* region can be tiled with pentagons.
3. Use

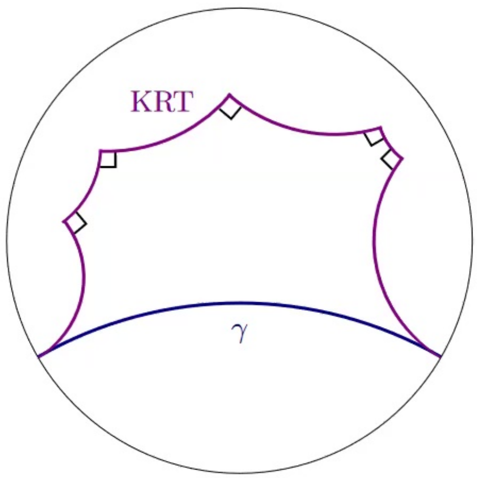
$$\gamma \sim_{\text{homology}} \text{KRT} \sim_{\text{homology}} \text{RT}$$

to show

$$\text{area}(\text{RT}) \leq \text{area}(\gamma) \leq \text{area}(\text{KRT}) - \log(2) \times |\text{kinks}|.$$

The Poincaré disk

Let KRT be a kinked geodesic on the Poincaré disk with well-defined boundary endpoints:



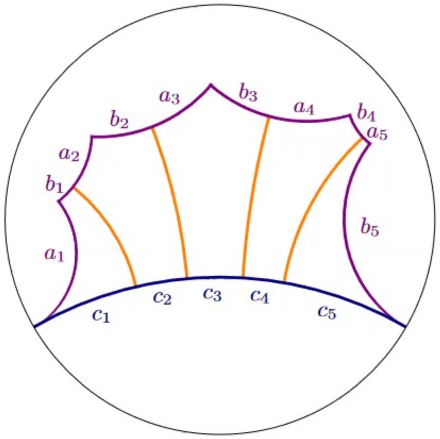
There is a unique geodesic γ between those two endpoints. KRT is homotopic to γ.

The Poincaré disk

Fact

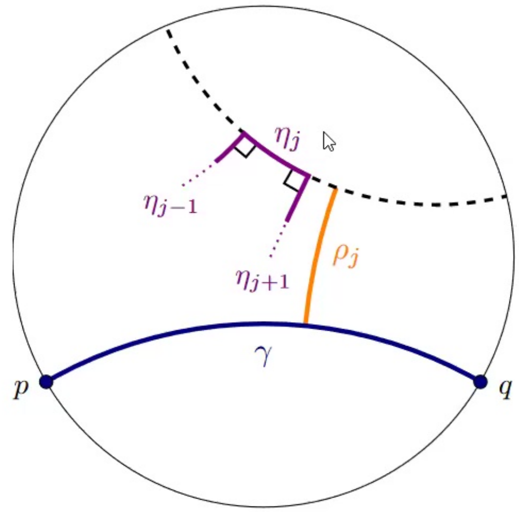
For any two geodesics on the Poincaré disk, there is a *unique* geodesic intersecting them both at right angles.

For each segment of KRT, draw the unique such geodesic connecting it to γ:



Existence?

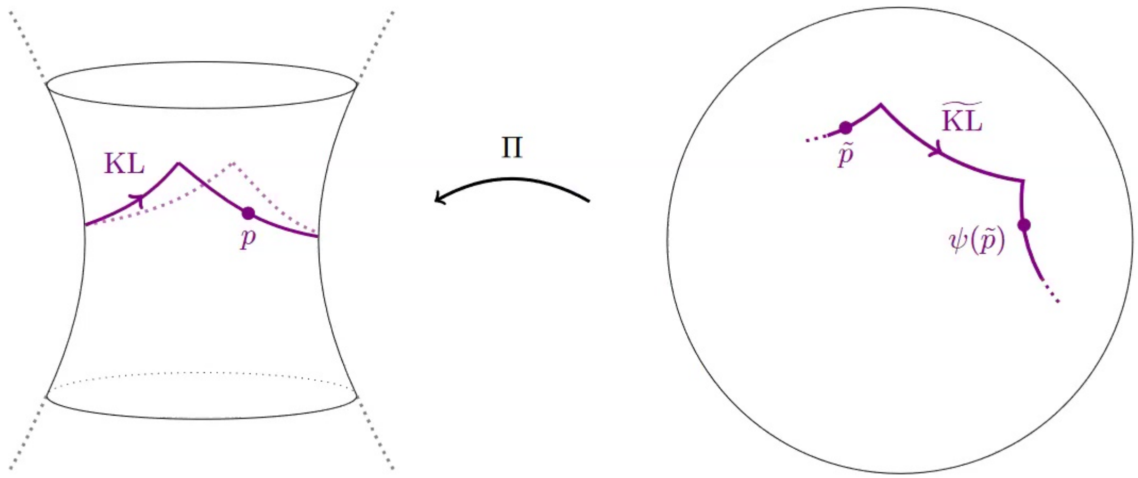
There's an important caveat — the unique geodesic connecting a segment to γ might not lie on the segment itself:



In the paper, we show that this cannot happen unless KRT self-intersects.

Sketching the extension

To prove the general theorem, we use the fact that every hyperbolic 2-manifold is universally covered by the Poincaré disk:



By tracking lengths between the manifold and the universal cover carefully, we complete the proof.

What did we learn?

$$S_R(A:B) - I(A:B) \geq \frac{2\log(2)|\partial\sigma_{A:B}|}{4G_N}.$$

In the proof we used, every point in $\partial\sigma_{A:B}$ was treated on equal footing. **To each endpoint, we associated two right-angled hyperbolic pentagons.** To each pentagon, we associate a minimal area difference of $2\log(2)$.

The proof technique is aesthetically in line with our guiding principle: that each point in $\partial\sigma_{A:B}$ contributes some irreducible tripartite entanglement. (Here, in the form of two pentagons.)

Generalizations?

1. It would be nice to know if the inequality holds for spacetimes with matter satisfying a suitable energy condition. We'll need a new proof technique to prove it. (Work in progress with Dan Eniceicu.)
2. Will some version of this inequality hold in higher dimensions? Two natural generalizations:

$$S_R(A:B) - I(A:B) \geq C_d \times \text{area}(\partial\sigma_{A:B}),$$
$$S_R(A:B) - I(A:B) \geq C_d \times \text{components}(\partial\sigma_{A:B}).$$

A quantum of tripartite entanglement?

AdS$_3$ gravity has central charge $c = 3/2G_N$. So we may rewrite our inequality as

$$S_R(A:B) - I(A:B) \geq \frac{c}{3}\log(2)|\partial\sigma_{A:B}|.$$

This bound is not satisfied in generic CFTs, but numerics show that *the limiting behavior seems universal*. Where does $c\log(2)/3$ come from in 2D CFT? (Independent work in progress, and work in progress with Yijian Zou.)

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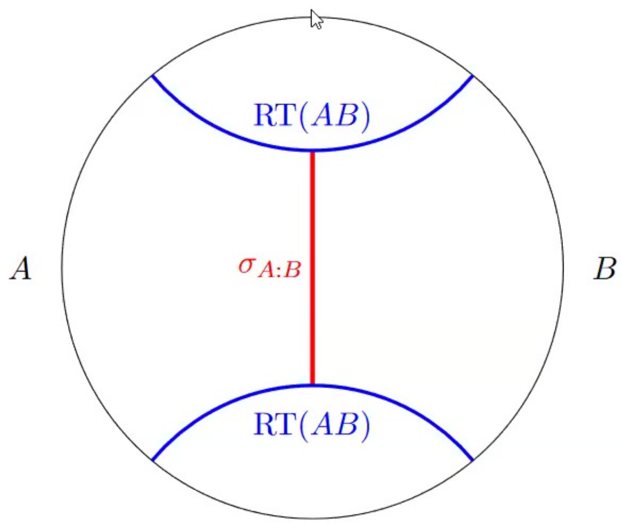

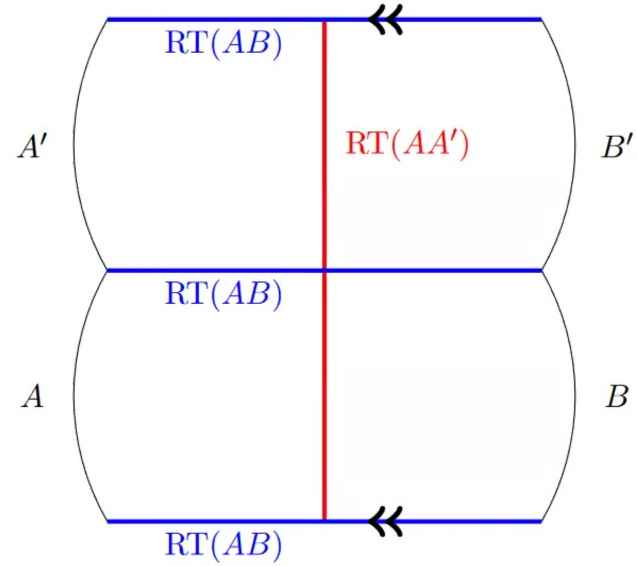

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Markov recovery

- Recall:
$$S_R(A:B)_\rho - I(A:B)_\rho = I(A:B'|B)_{|\psi_\rho\rangle}.$$
- This implies [Fawzi-Renner, ..., Junge-Renner-Sutter-Wilde-Winter]
$$S_R - I \geq -\max_{\mathcal{R}_{A\to AA'}} \log F(\rho_{ABB'}, \mathcal{R}_{B\to BB'}(\rho_{AB})).$$
- Interpret the Markov gap as **an obstruction to producing** $\rho_{ABB'}$ **from** ρ_{AB} **without touching the** A **system.**