Title: 2-Groups in quantum gravity

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Series: Quantum Gravity

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Abstract: Quantum groups are the proper tool to describe quantum gravity in three dimensions. Several arguments suggest that 2-groups should be used to formulate four dimensional quantum gravity. I will review these motivations and will discuss in particular how 2-groups can be used to extend the definition of a phase space associated to a triangulation or to modify the notion of group field theory to generate topological models. I will also highlight how the kappa Poincaré deformation arises in the 2-group context.

2-Groups in quantum gravity

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Motivations

Phase space of a triangulation

2-Group field theory

Outlook

Quantum gravity: a constrained topological model

Gravity in four dimensions can be formulated as a constrained BF (topological) field theory

$$\mathcal{S} = \int_{\mathcal{M}} \langle B \wedge F \rangle + \Lambda \int_{\mathcal{M}} \langle B \wedge B \rangle \,, \quad B = *(e \wedge e)\,.$$

SL(2,C) Chern–Simons theory, a non-planar graph operator, and 4D quantum gravity with a cosmological constant: Semiclassical geometry (2015).

Haggard, Han, Kaminski, Riello

$\Longrightarrow$ **State sum** models used to describe quantum geometries in four dimensions (*spin foam models*).

How to describe the states of a four dimensional curved quantum geometry?

Three dimensional quantum gravity

- Three dimensional Euclidean gravity with cosmological constant

$$\mathcal{S} = \int_{\mathcal{M}} \langle e \wedge F \rangle + \Lambda \int_{\mathcal{M}} \langle A \wedge e \wedge e \rangle \,, \quad \begin{cases} \textit{connection} & A : \mathfrak{su}\text{-valued 1-form} \\ \textit{triad} & e : \mathfrak{an}_\Lambda\text{-valued 1-form} \end{cases}$$

- **Discretization**: Poisson Lie group as decoration of curved cellular decompositions

edges of a triangulation	← triad →	*links* of the dual complex
links of the dual complex	← connection →	*edges* of a triangulation

- Quantum groups describe quantum states of discrete (homogeneously) curved geometries.

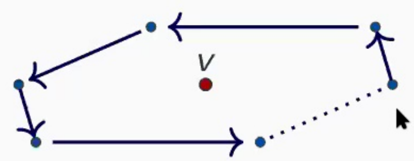
On the origin of the quantum group symmetry in 3d quantum gravity (2020).

Dupuis, Freidel, Girelli, Osumanu, Rennert

Three dimensional quantum gravity

Aspects of three dimensional quantum gravity based on *(quantum) groups*:

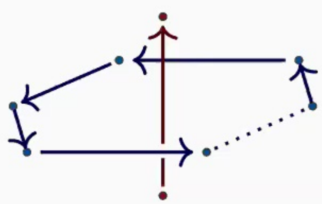
- **metric** degrees of freedom;
- quantum states of **curved discrete geometries**;
- topological features: curvature around a **vertex**.

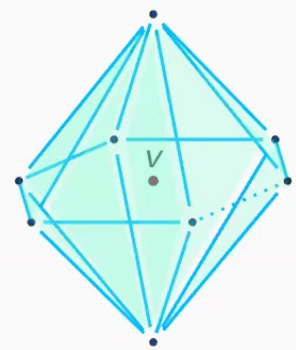


Four dimensional quantum gravity

Based on the features of three dimensional quantum gravity, what do we expect in four dimensions?

- **metric** degrees of freedom: degenerate geometries;
- quantum states of **curved discrete geometries** (encoding the cosmological constant);
- topological features: curvature around an **edge** (*1-holonomy*) and curvature around a **vertex** (*2-holonomy*).



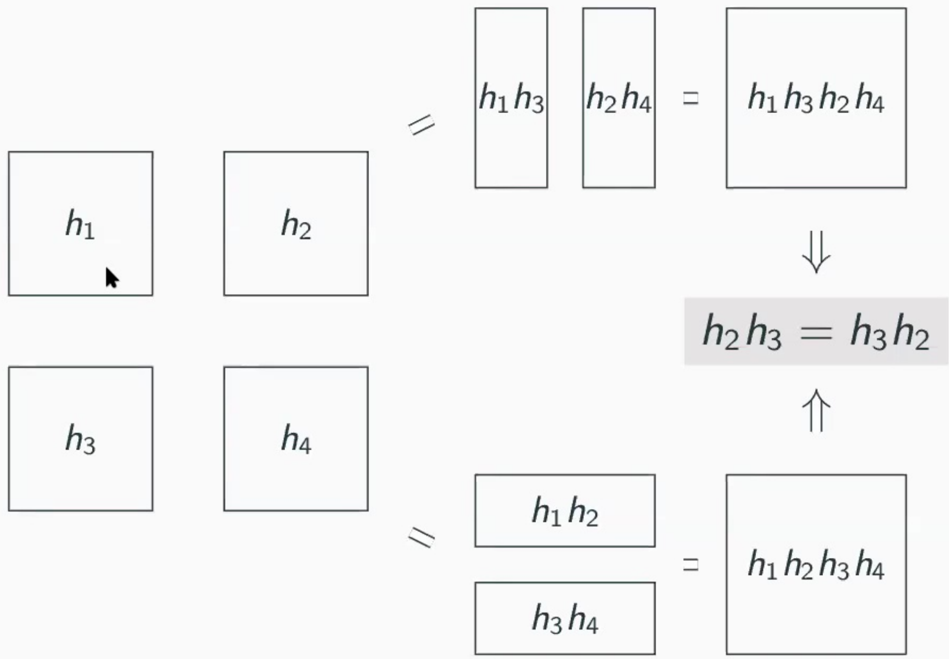


A more sensitive Lorentzian state sum (2013). *Crane, Yetter*

The Eckman-Hilton argument

Eckman-Hilton argument

By decorating *only* the $2d$ surfaces of a cellular decomposition, the group of such decorations **must** be abelian.



Discretization of four dimensional BF theory

The fields of four dimensional BF theory based on the gauge group G are

- *connection:* A $\mathfrak{g}$-valued 1-form;
- *bi-vector:* B $\mathfrak{g}^*$-valued 2-form.

One can discretize the four dimensional BF theory in two ways:

Bi-vector: faces of a triangulation.
Connection: links of a dual complex.

- ✗ metric degrees of freedom;
- ✗ curved geometry;
- ? curvature around a vertex;
- ✓ curvature around an edge.

Connection: edges of a triangulation.
Bi-vector: wedges of a dual complex.

- ✓ metric degrees of freedom;
- ? curved geometry;
- ✓ curvature around a vertex;
- ✗ curvature around an edge.

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- ✓ metric degrees of freedom;
- ? curved geometry;
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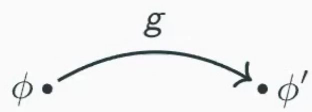
- ✗ curvature around an edge.

Need decorations on links, faces *and* on wedges, edges.

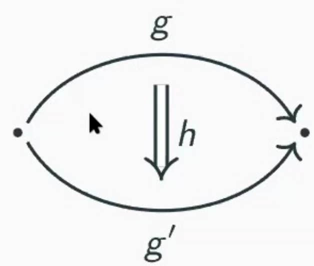
Groups and 2-groups

Groups G are used to describe gauge symmetries: group elements (*holonomies*) define transformations between points of a target space $\mathcal{M}$:



$$\phi' = \pi(g)\phi$$

2-Groups $\mathcal{G}$ introduces an extra layer: group elements g (*1-holonomies*) define transformations between points of a target space $\mathcal{M}$, and group elements h (*2-holonomies*) define transformations between 1-holonomies:



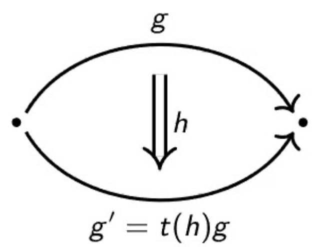
$$\begin{cases} \phi' = \pi(g)\phi \\ g' = t(h)g \end{cases}$$

An invitation to higher gauge theories (2015). *Baez, Huerta*

More on 2-groups

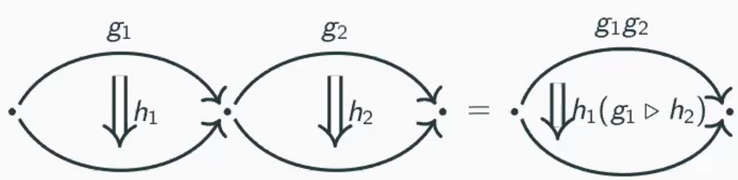
Definition

A 2-group (also known as **crossed module** of groups) $\mathcal{G} = (\mathrm{G}, \mathrm{H}, \triangleright, t)$ is:

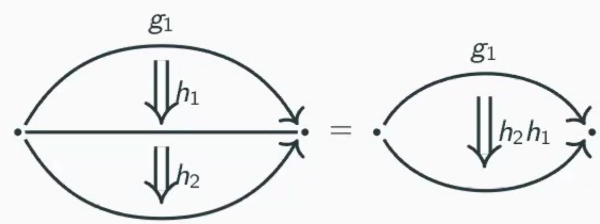


- a group G of 1-holonomies;
- a group H of 2-holonomies;
- a group homomorphism $t : \mathrm{H} \to \mathrm{G}$;
- a group action $\triangleright : \mathrm{G} \times \mathrm{H} \to \mathrm{H}$;

Horizontal composition



Vertical composition



There is a compatibility relation between *horizontal* and *vertical* compositions which implies compatibility conditions between the *action* and the *t map*.

Quantum gravity with 2-groups

A model of four dimensional quantum gravity based on **2-groups** has access to one and two dimensional decorations in both the triangulation and the dual complex: decorations on links and wedges, edges and faces:

- ✓ metric degrees of freedom;
- ✓ quantum states of a curved discrete geometry;
- ✓ curvature around a vertex (*2-holonomy*);
- ✓ curvature around an edge (*1-holonomy*).

Motivations

Phase space of a triangulation

2-Group field theory

Outlook

Skeletal 2-groups

As a starting point we deal with **skeletal 2-groups**.

Definition

Skeletal 2-groups are a class of 2-groups with a *trivial* t *map*. This implies that the group H of 2-holonomies is abelian.

The skeletal 2-group, regarded as a crossed module, is nearly equivalent to a *semi-direct product of groups*:

$$\mathcal{G} = \mathrm{G} \ltimes \mathrm{H}\,.$$

Examples of skeletal 2-groups are the *Euclidean* and the *Poincaré* 2-groups:

- Lorentz group of 1-holonomies $\mathrm{G} = \mathrm{SO}(3,1)$;
- group of translations of 2-holonomies $\mathrm{H} = \mathbb{R}^4$;
- trivial t map: $t(h) = 1$, for all $h \in \mathrm{H}$;
- natural action of the Lorentz group on 4-vectors;

Phase space of a skeletal 2-group

- If a Lie group G is the space of configurations, its co-tangent bundle is interpreted as the associated phase phase

$$T^*\mathrm{G} \cong \mathrm{G} \ltimes \mathbb{R}^d \quad \Rightarrow \quad \text{flat momentum space}$$

- If a Poisson Lie group G is the space of configurations, the phase space is its *Heisenberg double*, which encodes a curved momentum space.

Heisenberg double $\mathcal{H}$

The Heisenberg double of a group G is the group based on the double cross product $\mathrm{G} \bowtie \mathrm{G}^*$ equipped with a *symplectic* Poisson structure.

We define the phase space of a triangulation decorated by a skeletal 2-group $\mathcal{G}$ as the Heisenberg double of the semi-direct product $\mathrm{G} \ltimes \mathrm{H}$.

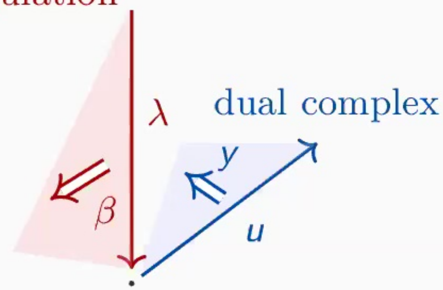
Symplectic reduction

* The Heisenberg double defines the fundamental *building block* of phase space: the **atomic phase space**

$$\mathcal{P} = (\mathrm{G} \ltimes \mathrm{H}) \bowtie (\mathrm{G}^* \rtimes \mathrm{H}^*)$$

configuration: (link $u \in \mathrm{G}$, wedge $y \in \mathrm{H}$)

momentum: (edge $\lambda \in \mathrm{H}^*$, face $\beta \in \mathrm{G}^*$)

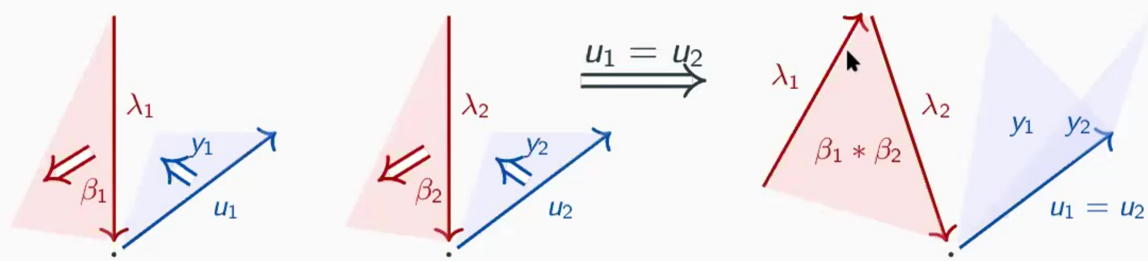


* We **fuse** atomic phase spaces through **symplectic reduction**:

1. impose a *geometric constraint* to fuse the (atomic) phase spaces. This condition, called **momentum map**, is enforced as a constraint on some group elements;
2. as a consequence, the variables dual to the ones that have been constrained are *glued*;
3. such fusion of (atomic) phase spaces is still a phase space.

Symplectic reduction: examples

Example 1: consider two atomic phase spaces and impose the identification of the link variables $u \in \mathrm{G}$.



- The *geometric constraint* is the identification of the links;
- it is imposed through the *momentum map* $u_1 u_2^{-1} \in \mathrm{G}$;
- dually, the face decorations are *glued* in a total one: $\beta_1 * \beta_2$;
- If $\mathcal{P}_1, \mathcal{P}_2$ are the initial phase spaces, the resulting fused phase space is

$$\mathcal{P} = (\mathcal{P}_1 \times \mathcal{P}_2)//\mathrm{G}^* \, .$$

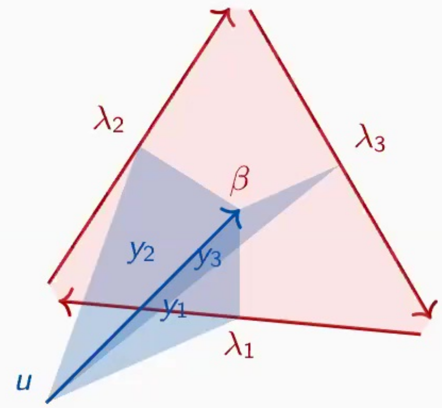
Symplectic reduction: examples

Example 2: triangle phase space. To construct the phase space of a triangle, consider three atomic phase spaces $\mathcal{P}_i$ and impose three *geometric constraints*: identification of the three links and the closure of the triangle.

Momentum maps

$$\left.\begin{array}{l} u_1 = u_2 \\ u_2 = u_3 \end{array}\right\} \Rightarrow \underbrace{\beta = \beta_1 * \beta_2 * \beta_3}_{\textbf{total face decoration}}$$

$$\lambda_1 \lambda_2 \lambda_3 = 1$$



Denote $\mathcal{C}_t = (\mathrm{G}^* \times \mathrm{G}^* \times \mathrm{H})$ the set of groups respect to which the symplectic reduction is taken. The triangle phase space is

$$\mathcal{P}_t = (\mathcal{P}_1 \times \mathcal{P}_2 \times \mathcal{P}_3)//\mathcal{C}_t \,.$$

Outcome

- Phase space of a triangulation with **edge** and (flat) **face** decorations.
  - *triangle* phase space: symplectic reduction respect to $\mathcal{C}_t$;
  - *tetrahedron* phase space: symplectic reduction respect to $\mathcal{C}_\tau$;
  - phase space of a *full link*: symplectic reduction respect to $\mathcal{C}_l$.

  The phase space of a triangulation $\mathcal{T}$ with dual complex $\mathcal{T}^*$ is

$$\mathcal{P} = \Big(\bigtimes_i \mathcal{P}_i \Big) // \Big(\bigtimes_{t\in\mathcal{T}} \mathcal{C}_t \bigtimes_{\tau\in\mathcal{T}} \mathcal{C}_\tau \bigtimes_{l\in\mathcal{T}^*} \mathcal{C}_l \Big) .$$

- Applications:
  - *Poincaré 2-group*: our construction recovers the G-networks
    - *configuration:* $\mathrm{SO}(3,1)$ on links $\mathbb{R}^4$ on wedges
    - *momentum:* $\mathbb{R}^4$ on edges $\mathbb{R}^6$ on faces

Quantum geometry for higher gauge theory (2019).
Asante, Dittrich, Girelli, Riello, Tsimiklis

  - *κ-Poincaré 2-group*: a generalization of G-netowrks is obtained
    - *configuration:* $\mathrm{SO}(3,1)$ on links $\mathbb{R}^4$ on wedges
    - *momentum:* $\mathrm{AN}(3)$ on edges $\mathbb{R}^6$ on faces

Polyhedron phase space using 2-groups: κ-Poincaré as a Poisson 2-group (2021).
Girelli, L., Tsimiklis

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∗ *configuration:*	$\mathrm{SO}(3,1)$ on links	$\mathbb{R}^4$ on wedges
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- **Insights** on the proper geometric conditions needed for gluing cellular decompositions with decorations on edges and faces.

Polyhedron phase space using 2-groups: κ-Poincaré as a Poisson 2-group (2021).

Girelli, L., Tsimiklis

Motivations

Phase space of a triangulation

2-Group field theory

Outlook

Kinematics: field and Fourier transform

- The **field** in a *3+1d 2-group field theory* is a function on the *four links* and the *six wedges* that compose the 2-graph dual to a tetrahedron:

$$\phi(u_1, \ldots, u_4; y_1, \ldots, y_6) \in F(\mathrm{G}^{\times 4} \times \mathrm{H}^{\times 6}) .$$

It can be written as a function on twelve copies of a 2-group $\mathcal{G}$, $\phi(\{(y_{i;a}, u_{i;a})\}) \in F(\mathcal{G}^{\times 12})$ with the proper geometric constraints.

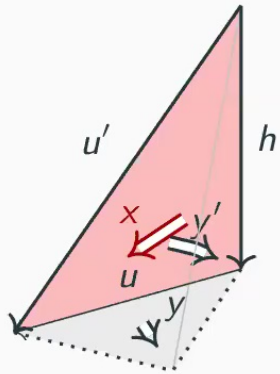
- The **Fourier transform** is an invertible map between dual 2-groups $\mathcal{F} : \mathcal{G} \to \mathcal{G}^*$ defined as

$$\mathcal{F}[f](\lambda, \beta) = \int \mathrm{d}u\mathrm{d}y\, E\big((y, u)\,,\, (\lambda, \beta)\big)\, f\,(y, u)\,.$$

The kernel of this Fourier transform is called *2-plane wave*. Its properties encode the proper geometric conditions to fuse phase spaces.

2-Gauge transformation and closure constraint

➢ The **2-gauge transformation** on the (y, u) variables:



$$\begin{cases} u' = t(x)hu \\ y' = (h \triangleright y)x^{-1} \end{cases}$$

Here $h \in \mathrm{G}$ and $x \in \mathrm{H}$ are the 2-gauge variables. The field is invariant under the projector P that encodes the 2-gauge transformation.

➢ The Fourier transform of the projected field gives the **closure constraint**

$$\mathcal{F}[(\mathrm{P}\,\phi)] = \hat{\mathcal{C}} \cdot \hat{\phi}\ , \quad \hat{\mathcal{C}}(\{(\lambda_{i;a}, \beta_{i;a})\}) = \hat{\delta}(\beta_1\beta_2\beta_3\beta_4) \prod_{i=1}^{4} \hat{\delta}(\lambda_i t(\beta_i))\,.$$

This encodes the *closure of a tetrahedron.*

Dynamics

- **Action**: in analogy with group field theory, we defined the action of 2-group field theory as a contribution of
  - an *interaction* term: the product of five fields with the combinatorics of a **4-simplex**;
  - a *kinetic* term: the product of two fields. It has the role of gluing 4-simplices by identifying a pair of tetrahedra.

- A model of discrete geometry is topological if it is invariant under the *Pachner moves*. In four dimensions there are three Pachner moves:
  - $P_{1,5}$: takes one 4-simplex in a combination of five;
  - $P_{2,4}$: takes two 4-simplices in a combination of four;
  - $P_{3,3}$: takes three 4-simplices in a combination of three.

  Our definition of 2-group field theory is a **topological model**.

The definition of 2-group field theory is a work in progress...

Dynamics

We expressed the 2-group field theory as a **state sum model**:

- *triangulation*: the state sum is the proper combination of tetrahedra amplitudes

$$\mathcal{Z} = \sum_{\Gamma} \prod \left(\hat{\delta}(\beta_1\beta_2\beta_3\beta_4) \prod_{i=1}^{4} \hat{\delta}(\lambda_i t(\beta_i)) \right),$$

  and coincides with the *Yetter model*;

  TQFT's from homotopy 2 types.J. Knot Theor. Ramifications (1993). *Yetter*

- *dual complex*: the state sum is a combination of polyhedra

$$\mathcal{Z} = \sum_{\Gamma} \prod \left(\delta(x_1 \dots x_N) \prod_{i=1} \delta(t(x_i)h_i) \right),$$

  that encode the *2-curvature around a vertex* and the *1-curvature around the edges*. In the case of the Poincaré 2-group, the model generates the G-networks and is equivalent to the Korepanov-Baratin-Freidel (KBF) model.

  A 2-categorical state sum model (2015). *Baratin, Freidel*

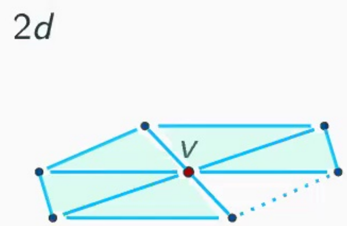
There is much more...

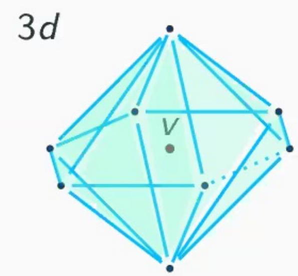
Some aspects of 2-groups to be explored:

- representation theory (Peter-Weyl theorem for 2-groups);
- quantum 2-groups and 2-Hopf algebras;
- beyond 2-groups?

Possible future works:

- the **geometric** version of 2-group field theory;
- higher categories can be used to introduce **matter degrees of freedom**:
  - 2-groups encode topological defects for $2d$ manifolds;
  - 3-groups encode topological defects for $3d$ manifolds.





Higher Gauge Theories Based on 3-groups (2019). *Radenkovic, Vojinovic*

Thanks