Title: The anomaly of the duality symmetry in type IIB string theory

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Series: Mathematical Physics

Date: November 19, 2021 - 1:15 PM

URL: https://pirsa.org/21110024

Abstract: Type IIB string theory has a duality symmetry given by the pin+ cover of GL(2, Z). In joint work with Markus Dierigl, Jonathan J. Heckman, and Miguel Montero, we show that this symmetry is anomalous, and describe how to cancel the anomaly, up to a few calculations we were unable to determine, by adding a Chern-Simons term. I will talk about the setup of the problem in terms of computing the partition function of an invertible topological field theory; a sketch of how the computation goes in terms of bordism and the Adams spectral sequence; and how we cancel the anomaly.

The anomaly of the duality symmetry in type IIB string theory
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November 19, 2021

arXiv:2107.14227

Outline

1. Anomalies and invertible field theories
2. The IIB duality group, and setting up the computational question
3. The bordism computation
4. The anomaly is nonzero!
5. Canceling the anomaly

Anomalies and invertible field theories

- If you ask four people what anomalies in QFT are, you will get five different answers
- We take the perspective that "a QFT is anomalous" means that it is a boundary theory to an *invertible* field theory in one dimension higher

Anomalies and invertible field theories

- If you ask four people what anomalies in QFT are, you will get five different answers
- We take the perspective that "a QFT is anomalous" means that it is a boundary theory to an *invertible* field theory in one dimension higher
  - à la Freed-Teleman "Relative quantum field theory"

Anomalies and invertible field theories

- Thinking of field theories in the Atiyah-Segal formalism, a field theory is a functor

$$Z\colon \mathsf{Bord}_n \longrightarrow C$$

  where Bord_n is a bordism category (possibly geometric!) and C is some target symmetric monoidal category
- An *invertible field theory* (IFT) is a functor whose image is contained in the subcategory of invertible objects, morphisms, . . . , of C
  - On objects, invertibility means under $\otimes$
  - On morphisms, invertibility means under composition
- Means, for example: partition functions are nonzero, state spaces are one-dimensional, . . .
- Intuition: almost, but not quite, trivial

Invertible field theories are classified

- Freed-Hopkins show that unitary invertible field theories are classified by the Anderson duals of bordism spectra
  - The specific type H of bordism is the one that's needed to define the data of the theory (e.g. spin bordism if you have fermions)
  - Freed-Hopkins' result is a theorem for invertible *topological* field theories, and a conjecture for non-topological ones
- There is a short exact sequence

$$0 \to \mathrm{Tors}(\mathrm{Hom}(\Omega_n^H, \mathbb{C}^\times)) \to \{\text{unitary IFTs}\} \to \mathrm{Hom}(\Omega_{n+1}^H, \mathbb{Z}) \to 0$$

The rightmost group is a group of characteristic classes; it gives the anomaly polynomial of an anomaly IFT

A couple notes about IIB specifically

- Sometimes people only think mapping tori in the bordism category are physically relevant, but in supergravity theories, the topology of spacetime can change, so we consider the entire bordism category
- The anomaly polynomial of IIB string theory vanishes, which is a famous calculation of Alvarez-Gaumé and Witten
- Upshot: we only need to look at the invertible TFTs, which amounts to understanding $\mathrm{Hom}(\Omega_{11}^{H}, \mathbb{C}^\times)$
- So the first step is to determine H

The duality group of type IIB string theory

- Conventionally thought of as $\mathrm{SL}_2(\mathbb{Z})$ but Hsieh-Tachikawa-Yonekura noticed it's a little different
- S doesn't quite satisfy $S^4 = 1$: instead, $S^4 = (-1)^F$
- So the symmetry group is $\mathrm{Mp}_2(\mathbb{Z})$, the double cover of $\mathrm{SL}_2(\mathbb{Z})$, and $-1 = (-1)^F$
- You can also include worldsheet orientation reversal, enlarging the group to the pin$^+$ cover of $\mathrm{GL}_2(\mathbb{Z})$, which is denoted $\mathrm{GL}_2^+(\mathbb{Z})$

Translating this symmetry into a structure on spacetime

- We want to couple to gauge fields for this symmetry; the presence of $(-1)^F$ in the symmetry group makes this trickier
- Model example: suppose the symmetry were Spin_2 with $-1 = (-1)^F$
- Then, we can work with manifolds with a spin structure and a principal Spin_2-bundle, but we could also work with manifolds which aren't quite spin, and don't quite have a Spin_2-bundle, but in ways that cancel out
- So our transition functions aren't valued in $\mathrm{Spin}_n \times \mathrm{Spin}_2$, but rather in $\mathrm{Spin}_n \times_{\{\pm 1\}} \mathrm{Spin}_2 = \mathrm{Spin}^c_n$
- So we can study the theory on spin^c manifolds

Spin-$\mathrm{Mp}_2(\mathbb{Z})$ structures and spin-$\mathrm{GL}_2^+(\mathbb{Z})$ structures

- A manifold with "spin-$\mathrm{Mp}_2(\mathbb{Z})$ structure," i.e. transition functions for TM valued in $\mathrm{Spin}_n \times_{\{\pm 1\}} \mathrm{Mp}_2(\mathbb{Z})$, doesn't quite have a spin structure or a principal $\mathrm{Mp}_2(\mathbb{Z})$-bundle, but the obstructions cancel each other out
- A spin-$\mathrm{Mp}_2(\mathbb{Z})$ manifold does have an associated $\mathrm{Mp}_2(\mathbb{Z})/\{\pm 1\} = \mathrm{SL}_2(\mathbb{Z})$-bundle, akin to the determinant bundle of a spin^c structure (a $\mathrm{Spin}_2/\{\pm 1\} = \mathrm{SO}_2$-bundle)
- Similarly, a spin-$\mathrm{GL}_2^+(\mathbb{Z})$ structure is a lift of the transition functions to $\mathrm{Spin}_n \times_{\{\pm 1\}} \mathrm{GL}_2^+(\mathbb{Z})$
- Spin-$\mathrm{GL}_2^+(\mathbb{Z})$ manifolds are oriented but not spin, and not necessarily spin^c. Have an associated $\mathrm{GL}_2(\mathbb{Z})$-bundle

The bordism question

- The anomaly invertible field theory is topological, so is determined by its partition function
- And this partition function is a bordism invariant

$$\Omega_{11}^{\text{Spin-GL}_2^+(\mathbb{Z})} \longrightarrow \mathbb{C}^\times$$

- In principle, we know how to calculate the anomaly on any particular manifold, via a formula told to us by the field content of IIB
- We compute that bordism group and its generators in order to know which test manifolds determine the anomaly theory

The bordism answer

$$\Omega_{11}^{\text{Spin-GL}_2^+(\mathbb{Z})} \cong \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^{\oplus 9} \oplus \mathbb{Z}/27 \oplus \mathbb{Z}/3$$

Generators:

- $\mathbb{Z}/8$ and $\mathbb{Z}/27$: lens spaces (detected by η-invariants)

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- $\mathbb{Z}/8$ and $\mathbb{Z}/27$: lens spaces (detected by η-invariants)
- $\mathbb{Z}/3$ and one $\mathbb{Z}/2$: $\mathbb{HP}^2$ times lens spaces (detected by η-invariants)
- Some more $\mathbb{Z}/2$ summands: $\mathbb{RP}^{11}$, $\mathbb{HP}^2 \times \mathbb{RP}^3$; $X_{10} \times S^1$ (X_{10} a spin 10-manifold with $w_4 w_6 \neq 0$); and “Arcanum XI,” an $\mathbb{RP}^5$-bundle over $\mathbb{RP}^6$

Details about the computations

- Work "one prime at a time" (i.e. allowing maps which merely induce equivalences in p-torsion, even if they are not isomorphisms)
- $\mathrm{GL}_2^+(\mathbb{Z})$ factors as an amalgamated product in a way that makes it simplify considerably if you focus on a specific prime
  - Very similar to the $\mathrm{SL}_2(\mathbb{Z})$ story: $\mathrm{SL}_2(\mathbb{Z}) \cong (\mathbb{Z}/4) *_{\mathbb{Z}/2} \mathbb{Z}/6$, and if you're "working at $p = 2$", you can't tell apart $B\mathrm{SL}_2(\mathbb{Z})$ and $B\mathbb{Z}/4$
- Deduce that the only primes that contribute torsion are $p = 2, 3$

Spin-$GL_2^+(\mathbb{Z})$ bordism at $p = 3$

- Using the amalgamation, we learn that there is a map $\Omega_*^{\mathrm{Spin}}(BD_6) \to \Omega_*^{\mathrm{Spin}\text{-}GL_2^+(\mathbb{Z})}$ which is an isomorphism on 3-torsion
- Can compute $\Omega_*^{\mathrm{Spin}}(BD_6)$ with the Atiyah-Hirzebruch spectral sequence like normal
  - Working 3-locally, spin bordism is equivalent to a sum of simpler pieces, namely copies of *Brown-Peterson homology*, and we actually use the AHSS for this
- Alternatively, you can combine the AHSS for spin bordism with the Leray-Serre spectral sequence for the fibration

$$B\mathbb{Z}/3 \to BD_6 \to B\mathbb{Z}/2$$

to approach this problem with the
"Leray-Serre-Atiyah-Hirzebruch spectral sequence"

$$E^2_{p,q} = H_p(B\mathbb{Z}/2; \Omega_q^{\mathrm{Spin}}(B\mathbb{Z}/3)) \Longrightarrow \Omega_q^{\mathrm{Spin}}(BD_6)$$

Spin-$GL_2^+(\mathbb{Z})$ bordism at $p = 2$

- At $p = 2$, the amalgamation simplifies the computation to bordism for the group $\mathrm{Spin} \times_{\{\pm 1\}} D_{16}$
- The plan: express this as the spin bordism of something, then throw the Adams spectral sequence at it
- This is less crazy than it might sound: the Adams SS for spin bordism at $p = 2$ is dramatically simpler than the general case, and its structure deals with many differentials and extension problems better than Atiyah-Hirzebruch in this range

Expressing spin-D_{16} bordism as twisted spin bordism

- If $V \to X$ is a vector bundle, define an (X, V)-twisted spin structure on a vector bundle $E \to M$ to be a map $f\colon M \to X$ and a spin structure on $E \oplus f^*(V)$
- For example, pin^- structures are equivalent to $(B\mathrm{O}_1, \mathrm{taut})$-twisted spin structures
- Showing G-structures are equivalent to twisted spin structures tends to feel group-theoretic rather than homotopical, which can be nice

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- Showing G-structures are equivalent to twisted spin structures tends to feel group-theoretic rather than homotopical, which can be nice
- And the Pontrjagin-Thom theorem tells us that the (X,V)-twisted spin bordism groups are isomorphic to the spin bordism groups of the *Thom spectrum* X^V

Computing spin-D_{16} bordism: the Adams spectral sequence

- Let V be the standard (two-dimensional, real) representation of D_8
- Then spin-D_{16} structures are the same thing as $(BD_8, V \oplus 3\mathrm{Det}(V))$-twisted spin structures

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- Then spin-D_{16} structures are the same thing as $(BD_8, V \oplus 3\mathrm{Det}(V))$-twisted spin structures
- So we need to understand $\Omega_*^{\mathrm{Spin}}((BD_8)^{V \oplus 3\mathrm{Det}(V) - 5})$

Computing spin-D_{16} bordism: the Adams spectral sequence

- Anderson-Brown-Peterson showed that at $p = 2$, Ω_*^{Spin} splits into simpler homology theories:

$$\Omega_*^{\mathrm{Spin}}(X) \cong ko_*(X) \oplus ko_{*-10}(X) \oplus (ko_J)_{*-10}(X) \oplus \cdots$$

For computations up to dimension 11, what's above suffices

  - ko denotes *connective real K-theory*
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  For computations up to dimension 11, what's above suffices
  - ko denotes *connective real K-theory*
  - ko_J is a variant of ko; coincides with mod 2 homology in cases of interest today
- The Adams spectral sequence for ko-theory is much simpler than the general case:

$$E_2^{s,t} = \mathrm{Ext}_{\mathscr{A}(1)}(H^*(X;\mathbb{Z}/2), \mathbb{Z}/2) \Longrightarrow ko_*(X)$$

  (usually there's $\mathscr{A}$ in place of $\mathscr{A}(1)$, but $\mathscr{A}(1)$ is much smaller!)

The gameplan

1. Compute the $\mathscr{A}(1)$-module structure on $H^*((BD_{16})^{V+\mathrm{Det}(V)-3};\mathbb{Z}/2)$ (routine, can do by hand or by computer)
2. Compute Ext of this $\mathscr{A}(1)$-module (i.e. look up Ext of its summands)
3. Compute differentials: not as easy, but falls to standard tools (e.g. Margolis' theorem, May-Milgram theorem): the Adams SS has a lot of structure that constrains differentials

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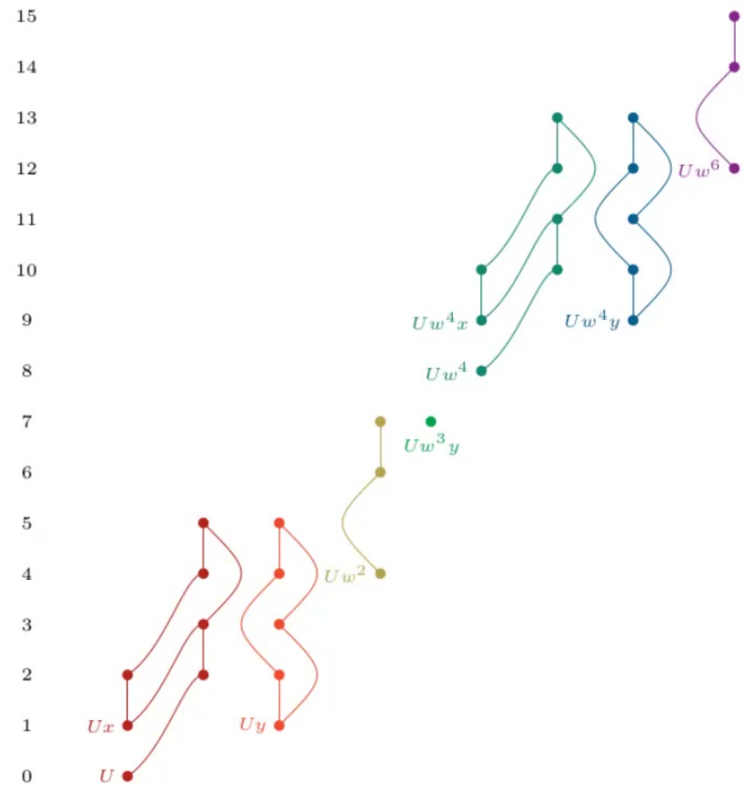
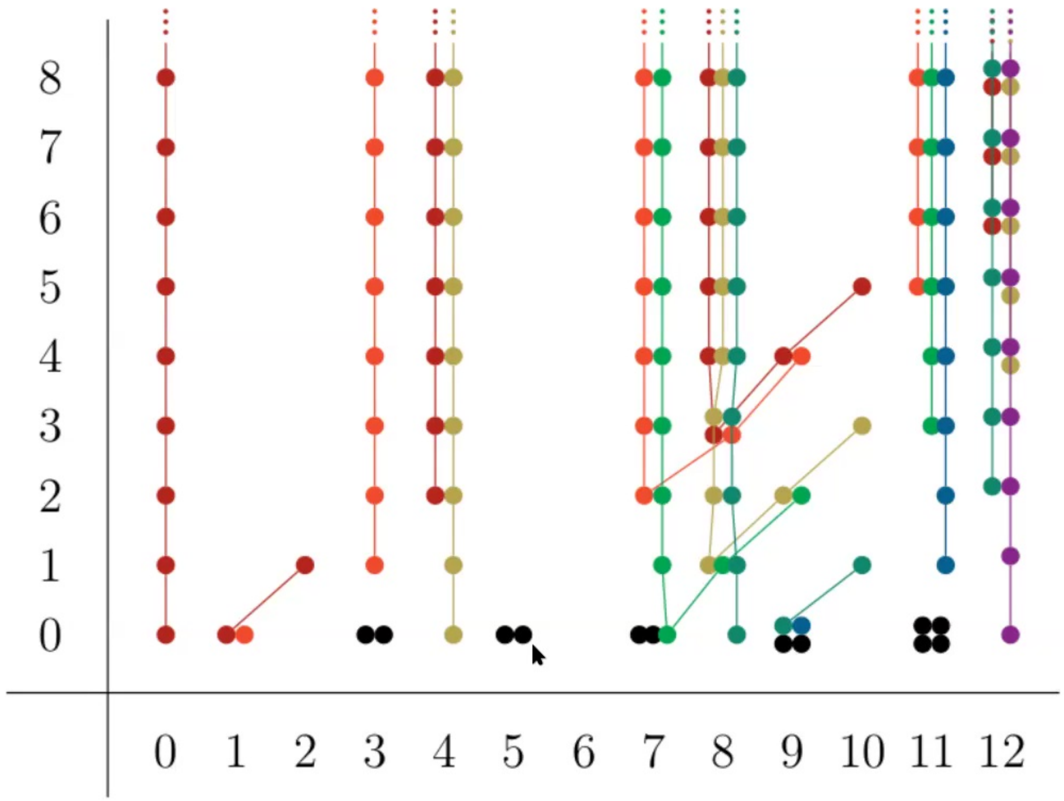
4. Address extension problems: mostly easy but a few tricky problems

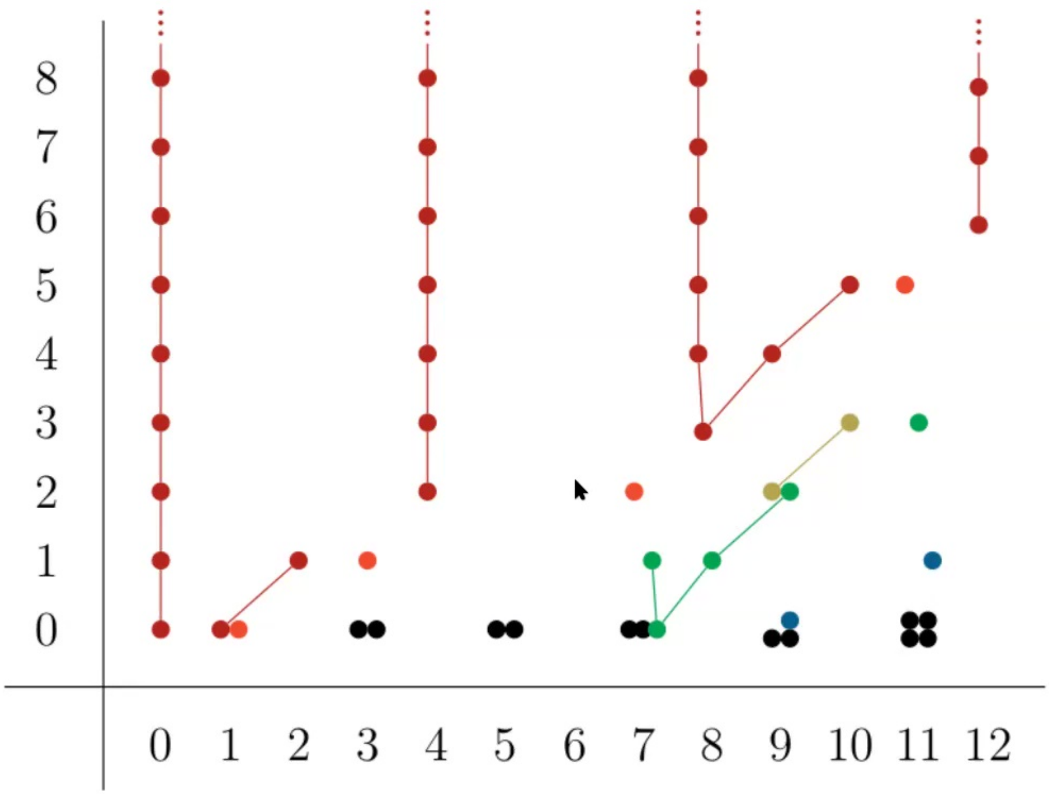
(part of) $H^*(BD_8;\mathbb{Z}/2)$ as an $\mathscr{A}(1)$-module:



The E_2-page of the Adams spectral sequence



The E_∞-page of the Adams spectral sequence



The unreasonable effectiveness of the Adams spectral sequence in physics

- Adams calculations over $\mathscr{A}(1)$ are generally routine, and even harder questions (e.g. computing string bordism or *tmf*) can take advantage of the strong structural constraints in low dimensions
- There's lots of other bordism questions out there whose answers would be interesting to physicists (e.g. McNamara-Vafa cobordism conjecture for various symmetry types coming out of string theory)

The anomaly of the self-dual field

- The anomalies of self-dual fields have been worked out only recently (Monnier; also Hsieh-Tachikawa-Yonekura in this specific case)
- The field strength of the self-dual field is a (differential) cocycle for $H^6(M;L)$, where L is the local system in which reflections in $\mathrm{GL}_2(\mathbb{Z})$ act by -1
- There is a torsion bilinear pairing $\langle -,- \rangle : \mathrm{Tors}(H^6(M;L)) \otimes \mathrm{Tors}(H^6(M;L)) \to \mathbb{R}/\mathbb{Z}$ for M a closed, spin-$\mathrm{GL}_2^+(\mathbb{Z})$ 11-manifold

An important assumption

- Part of the data needed to define the anomaly of the self-dual field is a quadratic refinement of the torsion pairing: a function $q : H^6(M;L) \to \mathbb{R}/\mathbb{Z}$ such that

$$\langle v, w\rangle = q(v+w) - q(v) - q(w) + q(0).$$

- **Here we had to make an assumption:** that there is a canonical quadratic refinement on a closed spin-$\mathrm{GL}_2^+(\mathbb{Z})$ 11-manifold
  - Analogous to how the intersection pairing on a spin 2-manifold has a quadratic refinement

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- **Here we had to make an assumption:** that there is a canonical quadratic refinement on a closed spin-$\mathrm{GL}_2^+(\mathbb{Z})$ 11-manifold
  - Analogous to how the intersection pairing on a spin 2-manifold has a quadratic refinement
  - Justified physically by M/F duality: we think we don't need to supply a quadratic refinement to extend M-theory to pin$^+$ (or m_c) manifolds
  - Maybe the answer is to pass to K-theory — but mixing K-theory and duality in this setting is open

The anomaly of the self-dual field

- Anyways, with that assumption in place, the anomaly of the C-field is (Hsieh-Tachikawa-Yonekura)

$$A(X) = \frac{1}{8}\eta_{\mathrm{Sig}}(X) + \mathrm{Arf}(q) + q(F) \in \mathbb{R}/\mathbb{Z},$$

where Sig denotes the signature Dirac operator, F is the field strength of C, and $\mathrm{Arf}(q)$ is the Arf invariant of the quadratic refinement

Calculation of the anomaly

Factor	Generator	Detector	A(gen.)
$\mathbb{Z}_{27}$	L^{11}_3	$\eta^{\mathrm{D}}_1 - \eta^{\mathrm{D}}_3$	$\frac{1}{3}$
$\mathbb{Z}_3$	$\mathbb{HP}^2 \times L^3_3$	$\eta^{\mathrm{RS}}_1 - \eta^{\mathrm{RS}}_3$	$\frac{1}{3}$
$\mathbb{Z}_8$	Q^{11}_4	$\eta^{\mathrm{D}}_1 - \eta^{\mathrm{D}}_3$	$\frac{k}{4}$
$\mathbb{Z}_2$	$\mathbb{HP}^2 \times L^3_4$	$\tilde{\eta}^{\mathrm{RS}}_1 - 2\tilde{\eta}^{\mathrm{D}}_1 - \tilde{\eta}^{\mathrm{D}}_{-3}$	$\frac{1}{2}$
$\mathbb{Z}_2$	$\mathbb{RP}^{11}$	x^{11}	0
$\mathbb{Z}_2$	$\widetilde{\mathbb{RP}^{11}}$	y^{11}	0
$\mathbb{Z}_2$	$\mathbb{HP}^2 \times \mathbb{RP}^3$	$w_4^2 x^3$	0
$\mathbb{Z}_2$	$\mathbb{HP}^2 \times \widetilde{\mathbb{RP}^3}$	$w_4^2 y^3$	0
$\mathbb{Z}_2$	$X_{10} \times S^1$	$w_4 w_6 x$	0
$\mathbb{Z}_2$	$X_{10} \times \widetilde{S^1}$	$w_4 w_6 y$	0
$\mathbb{Z}_2$	X_{11}	$w_2^4 x^3$	0 or $\frac{1}{2}$
$\mathbb{Z}_2$	$\widetilde{X_{11}}$	$w_2^4 y^3$	0 or $\frac{1}{2}$

Calculation of the anomaly

- We were not able to completely determine the anomaly on all generators
  - Calculating η-invariants on lens space bundles such as Q_4^{11} and X_{11}
  - Studying the intersection pairing on X_{11}

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- We were not able to completely determine the anomaly on all generators
  - Calculating η-invariants on lens space bundles such as Q_4^{11} and X_{11}
  - Studying the intersection pairing on X_{11}
- Working around our assumption about the quadratic refinement: restricted to spin manifolds with trivial duality bundle, the anomaly is a bordism invariant $\Omega_{11}^{\mathrm{Spin}} \to \mathbb{C}^\times$, and $\Omega_{11}^{\mathrm{Spin}} = 0$
  - Most of the generators can be chosen to be spin, and this allows us to calculate the contribution coming from the self-dual field on those generators

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Calculation of the anomaly: some questions still left

- Is there a canonical choice of quadratic refinement for spin-$GL_2^+(\mathbb{Z})$ 11-manifolds? (trivialization of some sort of Wu class)
- Using that, and calculations of η-invariants on the remaining generators, to determine the anomaly on the remaining generators
- Considering additional fields, symmetries, etc.

Canceling the anomaly: $p = 3$

- The U_1-valued term T contains two pieces, one from $\mathbb{Z}/3 \hookrightarrow U_1$ and one from $\mathbb{Z}/4 \hookrightarrow U_1$
- There is an isomorphism $H^*(B\mathrm{GL}_2(\mathbb{Z}); \mathbb{Z}/3) \overset{\cong}{\leftarrow} H^*(BD_6; \mathbb{Z}/3)$, so we have a characteristic class $a \in H^1$ for a spin-$\mathrm{GL} + 2^+(\mathbb{Z})$ manifold
- The $\mathbb{Z}/3$ part of T is

$$\beta(a)^2 + \frac{\lambda}{2}(p_1 \bmod 3),$$

where β is the Bockstein for $0 \to \mathbb{Z}/3 \to \mathbb{Z}/9 \to \mathbb{Z}/3 \to 0$ and λ is a constant

Canceling the anomaly: $p = 3$

- The mod 3 term in T cancels the anomaly on the generators of the 3-torsion in the bordism group
- This relies on a very fortunate fact: the anomaly on the lens space generating the $\mathbb{Z}/27$ in the bordism group is $1/3$ — if it were any of the other 26 possible values, this anomaly cancellation approach wouldn't work!

Canceling the anomaly: $p = 2$

- The part of T coming from $\mathbb{Z}/4 \hookrightarrow \mathrm{U}_1$ is

$$\frac{\lambda'}{2}(p_1 \bmod 4 - P(w_2)) \cup b + \kappa \beta(b)^2 \smile b$$

IIB continued. . .

- Some details to investigate: spin-$GL_2^+(\mathbb{Z})$ structures and the quadratic refinement; anomalies on the remaining generators
- Checking that our anomaly cancellation passes checks coming from M-theory or compactification to lower dimensions
- Plenty of other anomalies out there to calculate! (T- and U-duality symmetries, . . .)