Title: A convergent inflation hierarchy for quantum causal structures

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Abstract: Abstract: TBD

A convergent inflation hierarchy for quantum causal structures

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Outline

Introduction

- Correlation vs. Causation

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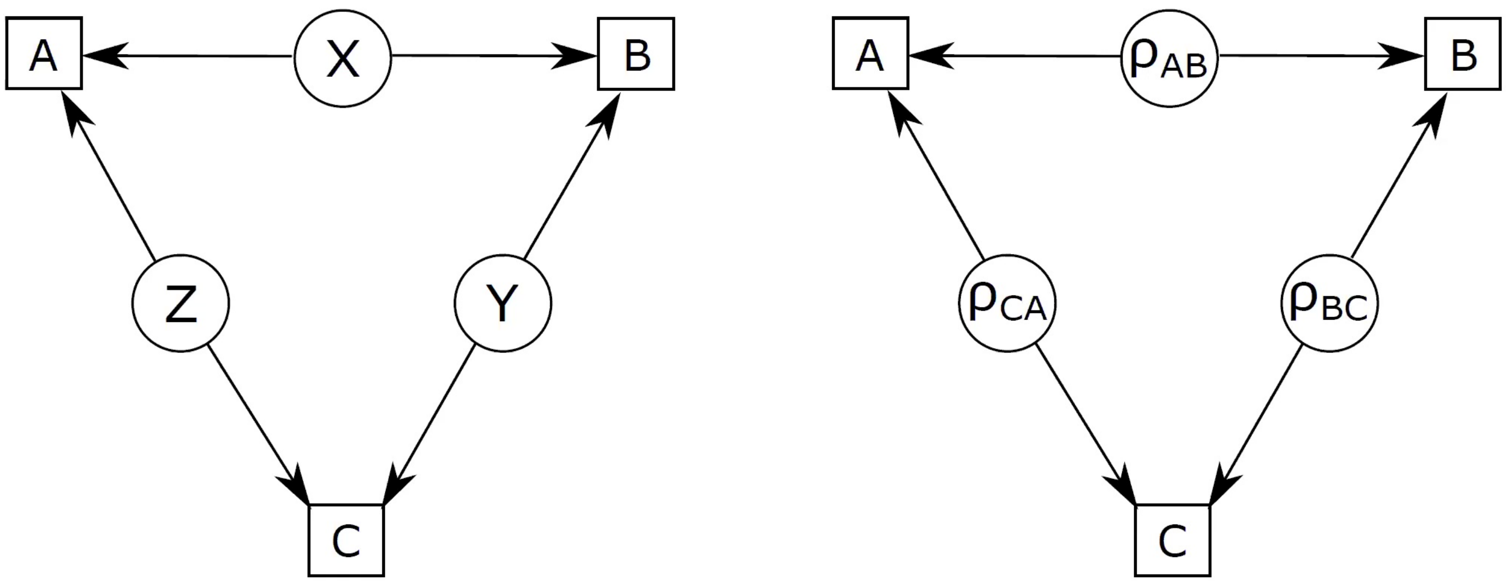
- Can a *causal structure* be deduced just from the observed statistics?
- Important in medicine, economics, physics

Problem description: causal compatibility

Nobel prize in economics 2021

Classical vs. Quantum causal structures



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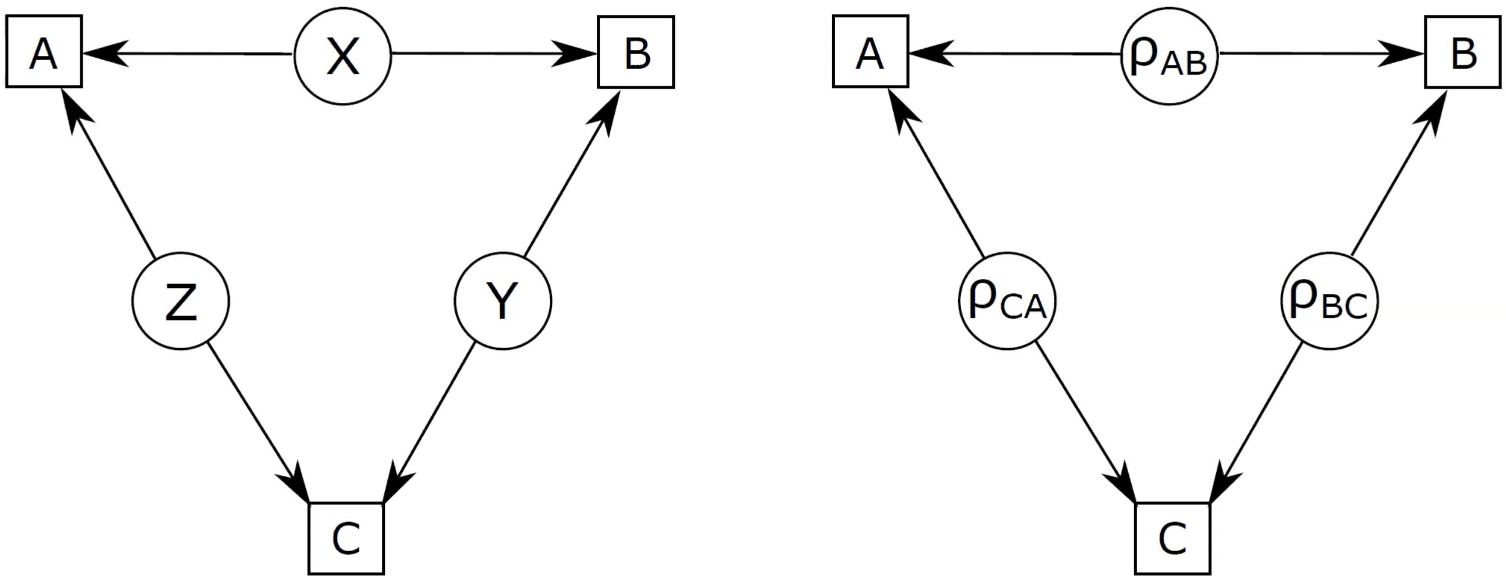

$$P_{\Delta}(a,b,c) = \sum_{x,y,z} P(a|xz)P(b|xy)P(c|yz)P(x)P(y)P(z)$$

Problem description: causal compatibility

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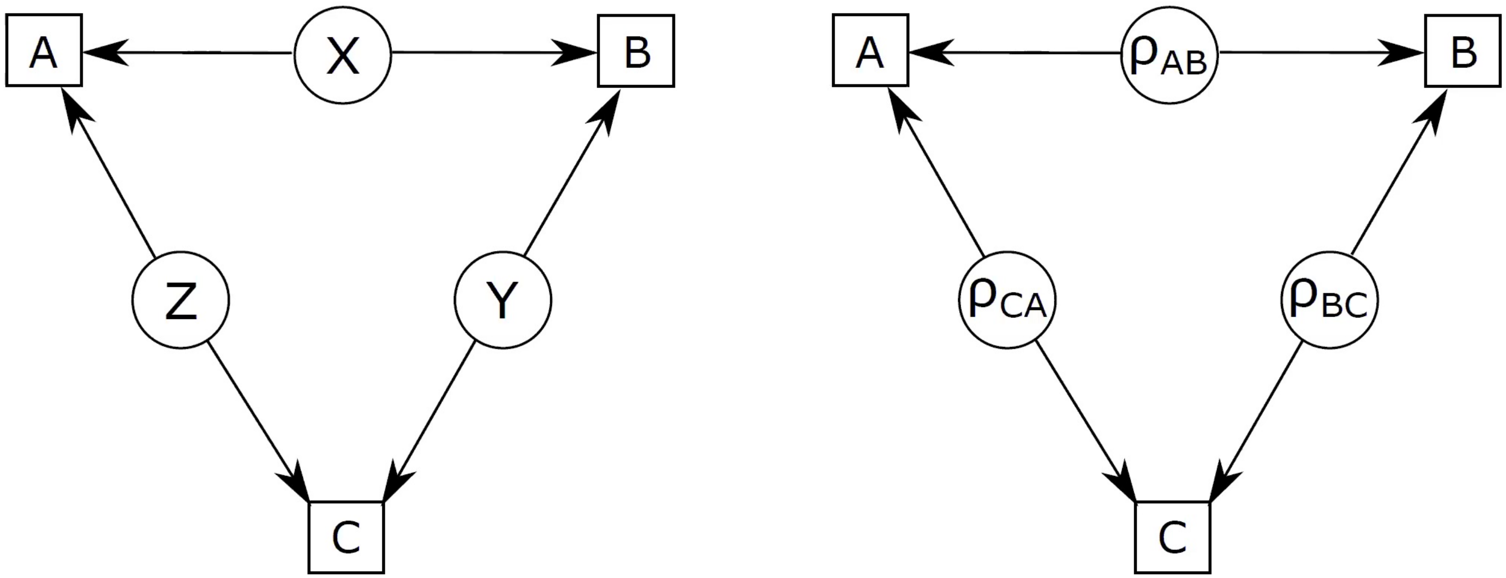

$$P_{GHZ}(A=a, B=b, C=c) = \begin{cases} \frac{1}{2} & \text{if } a=b=c, \\ 0 & \text{else.} \end{cases}$$

[1] E. Wolfe, R. W. Spekkens, T. Fritz, Journal of Causal Inference 7 (2019)

The quantum compatibility problem

Problem: Approximate quantum causal compatibility

Given $\epsilon \geq 0$, a causal structure and a probability distribution over observable variables P, determine whether there exists a distribution $\tilde{P}$ that can be produced by a quantum description of the causal structure such that $\|\tilde{P} - P\|^2 \leq \epsilon$.

Main result

There is a hierarchy of semidefinite programming relaxations for the approximate quantum causal compatibility problem, which is *complete* in the sense that it can detect any incompatible distribution with measurement operators of a given Schmidt rank r.
Every compatible P can be arbitrarily well approximated by a finite Schmidt rank model.

How to tackle the problem

It's difficult!

- For a long time unclear what to do
- Is it even decidable?
- Inflation technique[1]
- Complete hierarchy of LPs in the classical case[2]
- Quantum inflation technique[3]

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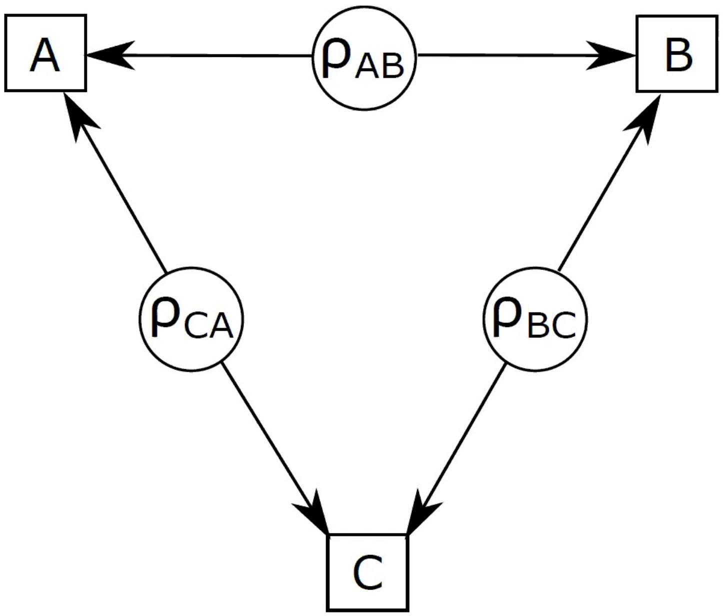
1 E. Wolfe, R.W. Spekkens, T. Fritz, Journal of Causal Inference 7 (2019) [1]
2 M. Navascués, E. Wolfe, Journal of Causal Inference 8, 70 (2020)[2]
3 E. Wolfe, *et al.*, Physical Review X 11, 021043 (2021)[3]

Causal compatibility



Given $P(A, B, C)$, are there:

- POVMs $\{E_a\}$, $\{F_b\}$, $\{G_c\}$
- $\rho = \rho_{AB} \otimes \rho_{BC} \otimes \rho_{CA}$

Independence

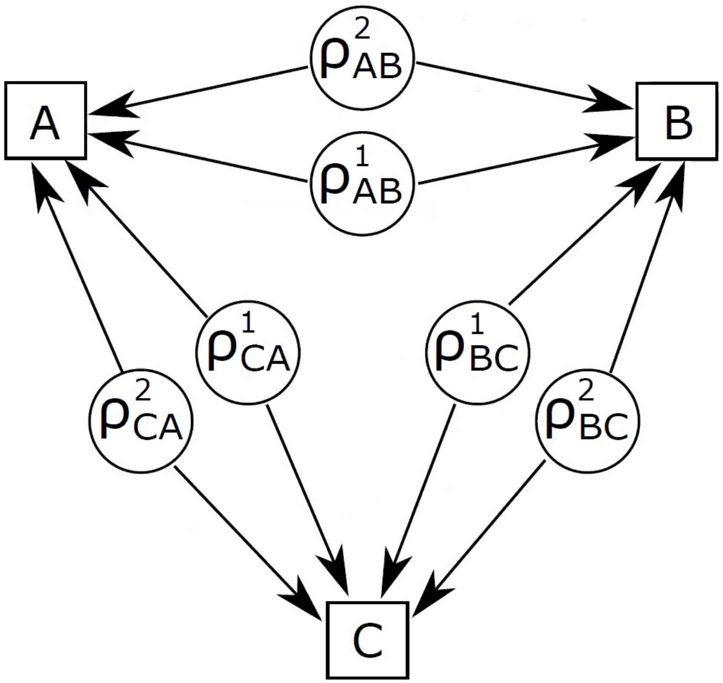
Problem: To describe independence we need product states!

- Difficult set to handle
- Not a convex set

Idea: Relaxation via symmetry [4]

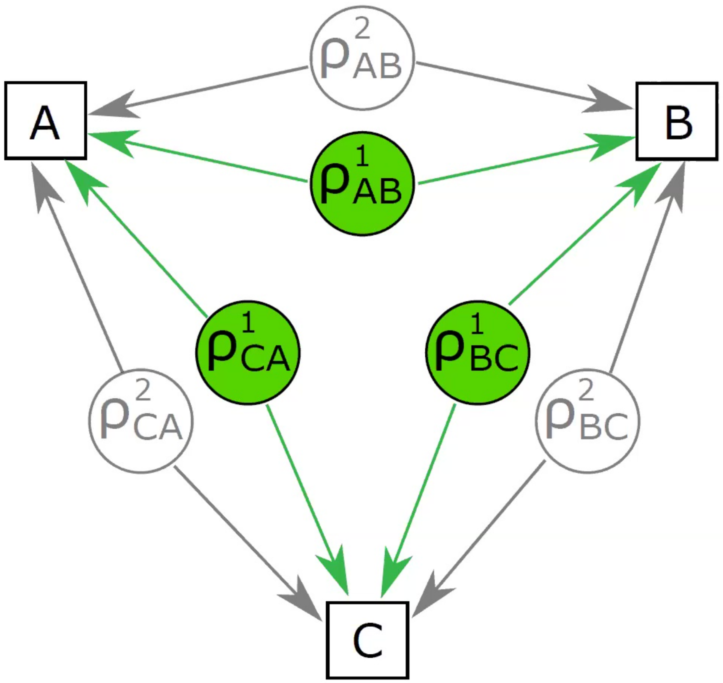
[4] E. Wolfe, R.W. Spekkens, T. Fritz, Journal of Causal Inference 7 (2019) [1]

Inflation



If P is compatible, there exists an *inflated* causal structure with:

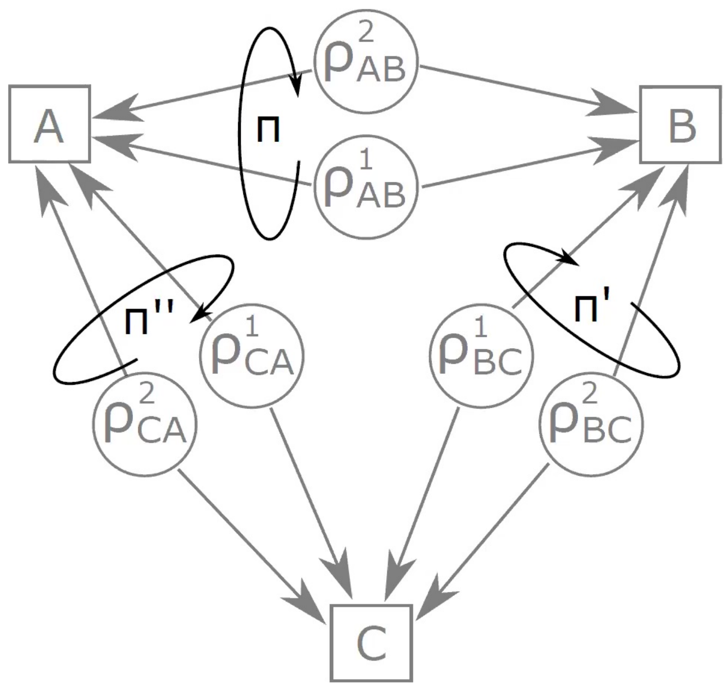
Inflation



If P is compatible, there exists an *inflated* causal structure with:

- n copies of the latent systems
- n^2 copies of the POVMs: $\{E_a^{ij}\}, \{F_b^{ij}\}, \{G_c^{ij}\}$
- Permutation symmetry: e.g. $\rho(E_a^{11} F_b^{11} G_c^{11}) = \rho(E_a^{12} F_b^{21} G_c^{11})$

Inflation



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Can one decide this?

Non-commutative polynomial optimization (NPO) [5]

Hierarchy of semidefinite programs (SDPs)

For the triangle scenario:
Input: $P(A, B, C)$ and n
Output: Operators $\{\hat{E}_a^{ij}\}$, $\{\hat{F}_b^{ij}\}$, $\{\hat{G}_c^{ij}\}$ and a state ρ on their algebra such that

- ρ has permutation invariance over n levels of inflation
- $P(a, b, c) = \rho(E_a^{11} F_b^{11} G_c^{11})$

[5] S. Pironio, M. Navascués, A. Acín, SIAM Journal on Optimization 20, 2157 (2010) [4]

To do's

1. Converging SDP hierarchy for quantum problems

To do's

1. Converging SDP hierarchy for quantum problems ✓ NPO
2. **Deduce independence from symmetry ⇒ Fitting de Finetti theorem**
3. Make sure the quantum model of NPO is such that the observables of Alice, Bob and Charlie factorize in the first place!

(Quantum) de Finetti Theorems

Permutation invariant states

$$\rho^{\otimes k}, \qquad \sum_i p_i \rho_i^{\otimes k}$$

$$|\psi^n\rangle = \frac{1}{\sqrt{2}}\Big(|0\ldots0\rangle + |1\ldots1\rangle \Big)$$

Subsystems will be equally entangled with each other and therefore "not very entangled". For example:

$$\Big|\psi^k\Big\rangle\Big\langle\psi^k\Big| = \mathrm{Tr}_{k+1,\ldots,n}(|\psi^n\rangle\langle\psi^n|) = \mathbb{I}_{2^k}/2^k$$

Many versions exist, but none fit our purpose

Hilbert space vs Commuting observable algebras

1. Hilbert space tensor product

2. Commuting observable algebras

[6] Ji *et al.*, $\mathrm{MIP}^* = \mathrm{RE}$, arxiv:2001.04383 (2020) [5]

Hilbert space vs Commuting observable algebras

1. Hilbert space tensor product
   - Associate to each subsystem a **Hilbert space** $\mathcal{H}_i$
   - The joint Hilbert space is e.g. $\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$
   - Operators from $\mathcal{A}_i = L(\mathcal{H}_i)$ are embedded into $\mathcal{A}_{12} = L(\mathcal{H}_{12})$ by padding with identities
2. Commuting observable algebras
   - Associate to each subsystem an **observable algebra** $\mathcal{A}_i$
   - The joint system is an algebra $\mathcal{A}_{12}$, which has $\mathcal{A}_1$, $\mathcal{A}_2$ as commuting subalgebras

These descriptions are not the same![6]
We need a de Finetti theorem for the second model.

[6] Ji *et al.*, MIP* = RE, arxiv:2001.04383 (2020) [5]

Max tensor product Quantum de Finetti Theorem

Theorem (Extension of [6] to the max tensor product)

Let ρ be a symmetric state on an infinite maximal tensor product

$$\mathcal{D}^{\infty} = \lim_{n\to\infty} \mathcal{D}^{\otimes^n_{\max}}.$$

Then there exists a unique probability measure $\mathrm{d}\mu$ over states on $\mathcal{D}$ such that for all $x \in \mathcal{D}^{\infty}$,

$$\rho(x) = \int \Pi_{\sigma}(x)\,\mathrm{d}\mu(\sigma),$$

where Π_{σ} is the infinite symmetric product state on $\mathcal{D}^{\infty}$ associated to the state σ on $\mathcal{D}$.

[6] G. Raggio, R. Werner, Helv. Phys. Acta 62 (1989)

To do's

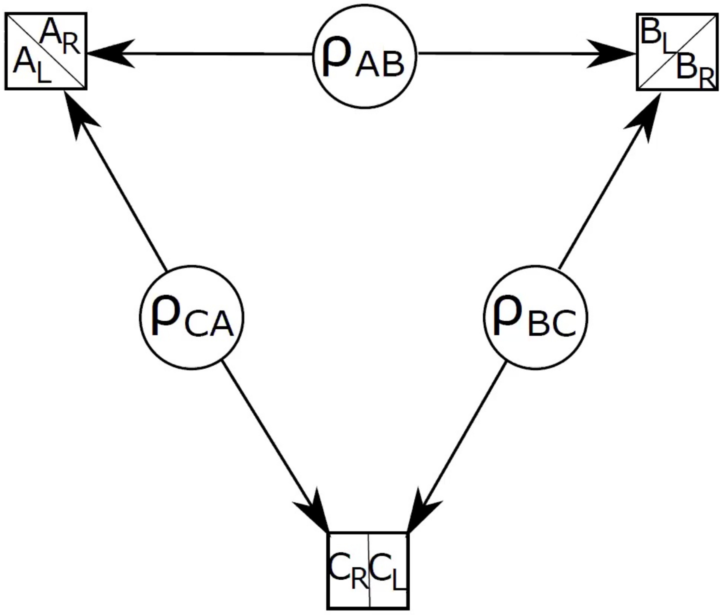
1. Converging SDP hierarchy for quantum problems ✓ NPO
2. Deduce independence from symmetry ⇒ Fitting de Finetti theorem ✓
3. **Make sure the quantum model of NPO is such that the observables of Alice, Bob and Charlie factorize in the first place!**

Local subsystems

From the original inflation NPO of Wolfe *et al.* one gets

- POVMs $\{E_a^{ij}\}$, $\{F_b^{ij}\}$, $\{G_c^{ij}\}$
- Permutation invariant ρ

We want for:



$$A_L \in \mathcal{A}_L, \quad B_L \in \mathcal{B}_L, \quad C_L \in \mathcal{C}_L,$$
$$A_R \in \mathcal{A}_R, \quad B_R \in \mathcal{B}_R, \quad C_R \in \mathcal{C}_R,$$

$$\rho(A_L A_R B_L B_R C_L C_R) = \rho(C_R A_L)\rho(A_R B_L)\rho(B_R C_L)$$

Independence and NPO

$$\rho(A_L A_R B_L B_R C_L C_R) = \rho(C_R A_L)\rho(A_R B_L)\rho(B_R C_L),$$

The independence property is defined in terms of observables that are not being measured!
Local operators do not obviously come out of the original inflation NPO

Modified inflation NPO

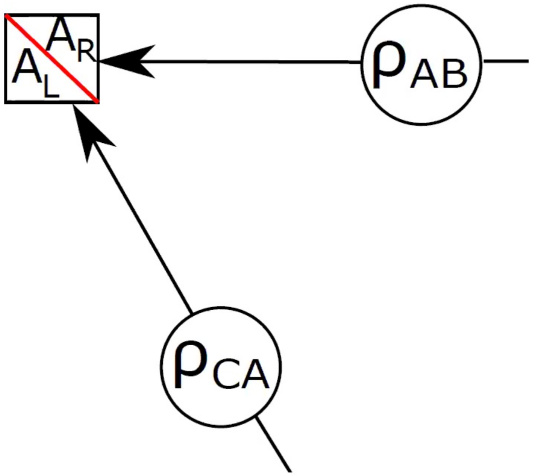
Ask NPO to find generators

- $e^i_L(a,\alpha) \in \mathcal{A}^i_L, \quad \alpha = 1,\dots,r$
- $e^j_R(a,\alpha) \in \mathcal{A}^j_R, \quad \alpha = 1,\dots,r$

such that

$$E^{ij}_a = \sum_{\alpha=1}^{r} e^i_L(a,\alpha) e^j_R(a,\alpha).$$

Always possible for some r if P is compatible, because $\langle \mathcal{A}^i_L \cdot \mathcal{A}^j_R \rangle$ is dense in $\mathcal{A}^{ij}$



To do's

1. Converging SDP hierarchy for quantum problems ✓ NPO
2. Deduce independence from symmetry $\Rightarrow$ Fitting de Finetti theorem ✓
3. **Make sure the quantum model of NPO is such that the observables of Alice, Bob and Charlie factorize in the first place!**

Convergence

- If NPO succeeds for level n, we know there exists a state ρ that is symmetric over all permutations of n copies
- Using the De Finetti theorem, we can then show that for $n \to \infty$ there also exists a *product state* σ, for which

$$P(a, b, c) = \sigma(E_a^{11} F_b^{11} G_c^{11})$$

- We are guaranteed to solve the approximate causal compatibility problem for any ϵ for large enough values of r.

If NPO returns "infeasible" for given r and any n, we can conclude there exists no quantum model of the causal structure with measurement operators of rank r that produces P

Remarks

There is a price to pay:

1. Additional parameter r
2. The convergence for increasing values of r and n is non-monotonous

Summary

We have

- described causal structures in terms of commuting algebras
- proven a quantum de Finetti Theorem
- shown that there exists a converging SDP hierarchy for the causal compatibility problem

Open questions

- Bounds for, or scaling with, r
- Other ways to identify local algebras
- Is the original hierarchy of Wolfe *et al.* convergent?
- Numerical methods

Norm bound C

The generators $e_L(a, \alpha)$, etc. are not automatically bounded. Therefore we add the constraints

$$C^2\mathbb{I} - e_L(a, \alpha)^* e_L(a, \alpha) \geq 0,$$

for some large value of C.

- C is another (inexpensive) parameter of the SDP
- It might be possible to choose $C = 1$

References

[1] E. Wolfe, R. W. Spekkens, T. Fritz, *Journal of Causal Inference* **7** (2019).

[2] M. Navascués, E. Wolfe, *Journal of Causal Inference* **8**, 70 (2020).

[3] E. Wolfe, *et al.*, *Physical Review X* **11**, 021043 (2021).

[4] S. Pironio, M. Navascués, A. Acín, *SIAM Journal on Optimization* **20**, 2157 (2010).

[5] Z. Ji, A. Natarajan, T. Vidick, J. Wright, H. Yuen, *arXiv:2001.04383* (2020).

[6] G. Raggio, R. Werner, *Helv. Phys. Acta* **62** (1989).

as well as the permutation symmetry constraints. Furthermore, the unitaries commute with all of Alice's operators. These constraints should thus be added to the definition of the C^*-algebra with which we describe the causal structure.

The unitaries U_s^j immediately fit nicely in our framework, since they only carry one inflation index j, are bounded by Eq. (57) and obey the symmetry constraints. In this particular example, they can be added as additional generators to the algebra $\mathcal{B}_j \otimes_{\max} \mathcal{C}_j$ of the j'th copy of B and C. Theorem 7 and Lemma 11 are then still applicable and the causal structure can be treated by our model. Theorem 12 and Corollary 14 then show that the SDP finds the correct solution to the causal compatibility problem if one creates an objective function similar to Eq (54) out of equalities of the form of Eq. (56).

In the case where a latent variable has only latent parents, one can define unitaries without the classical control variable and perform the same procedure. For more difficult causal structures, where two or more quantum systems come together to form a non-exogenous quantum system, one can define several such (controlled) unitaries, specifying on which subsystems they act by including their commutation relations as constraints in the SDP.

With this slightly altered version of the formalism, one can thus still solve causal polynomial optimization and the causal compatibility problem. Hence, our approach is applicable to all causal structures that can be represented by a DAG.

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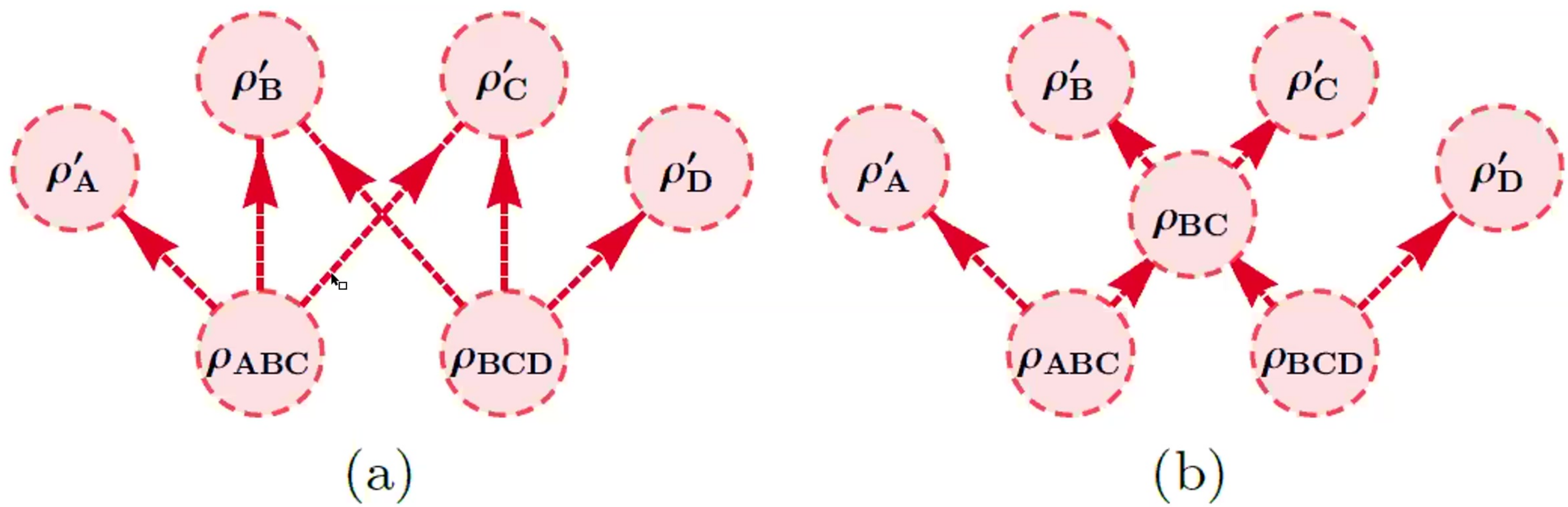

FIG. 19. Inequivalent state generating procedures. (a) depicts a two-layer state generating procedure, whereas (b) allows for the application of an arbitrary Bob-Charlie channel.

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