Title: Quantum Theory needs complex numbers

Speakers: Marc-Olivier Renou

Series: Quantum Foundations

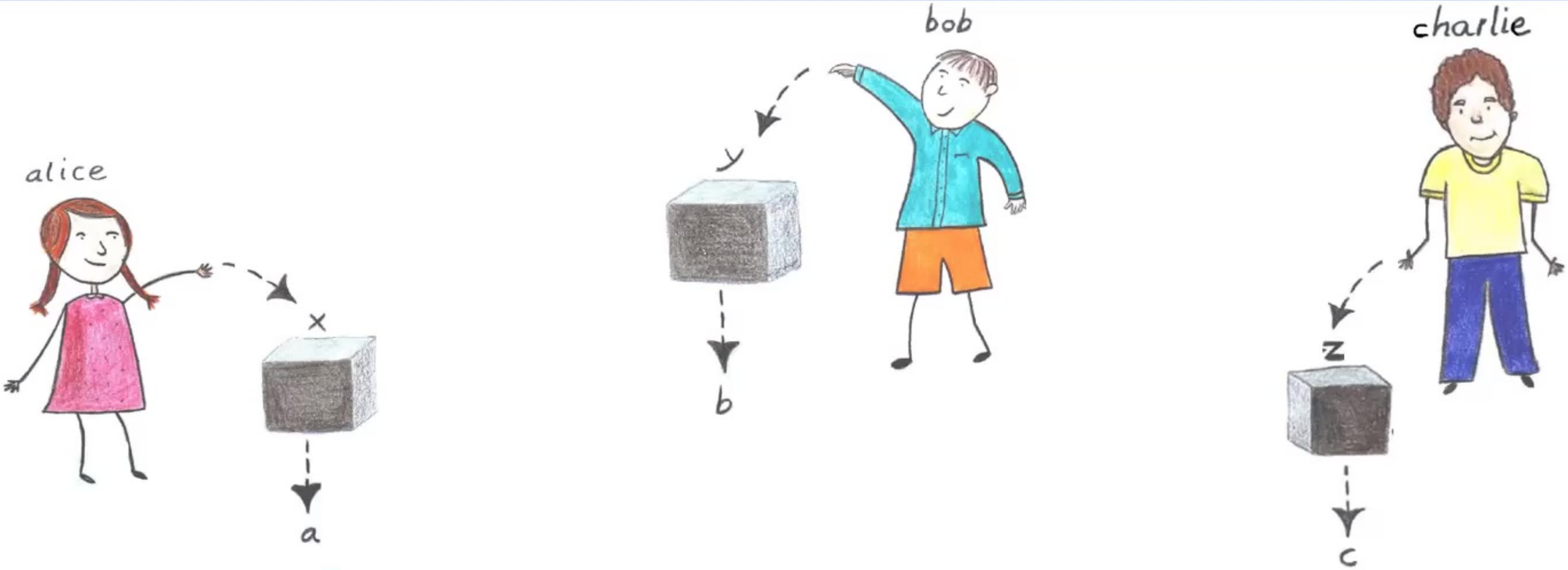
Date: November 19, 2021 - 2:00 PM

URL: https://pirsa.org/21110017

Abstract: While complex numbers are essential in mathematics, they are not needed to describe physical experiments, expressed in terms of probabilities, hence real numbers. Physics however aims to explain, rather than describe, experiments through theories. While most theories of physics are based on real numbers, quantum theory was the first to be formulated in terms of operators acting on complex Hilbert spaces. This has puzzled countless physicists, including the fathers of the theory, for whom a real version of quantum theory, in terms of real operators, seemed much more natural. Are complex numbers really needed in the quantum formalism? Here, we show this to be case by proving that real and complex quantum theory, understood in terms of operators in Hilbert spaces and tensor products to represent independent systems, make different predictions in network scenarios comprising independent states and measurements. This allows us to devise a Bell-like experiment whose successful realization would disprove real quantum theory, in the same way as standard Bell experiments disproved local physics.

Quantum Theory needs complex numbers

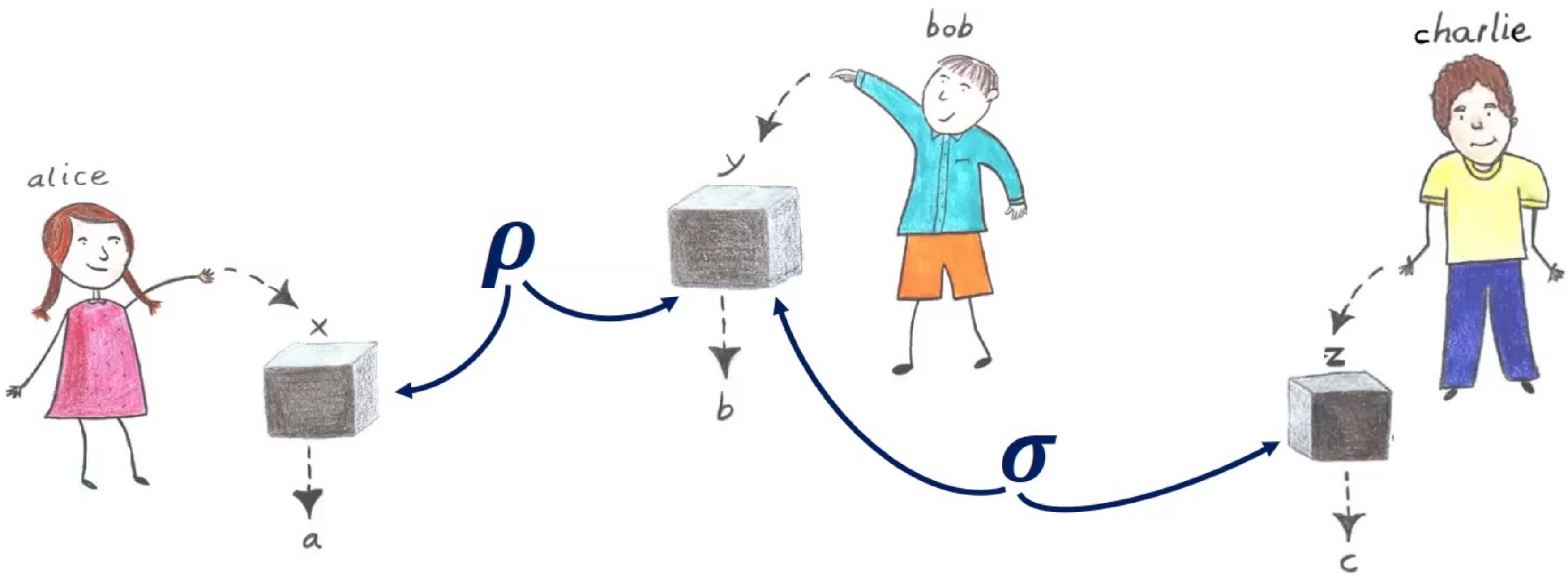
"Real-number Hilbert space Quantum Theory with Tensor Products" is experimentally falsifiable



Marc-Olivier Renou, David Trillo, Mirjam Weilenmann, Le Phuc Thinh, Armin Tavakoli, Nicolas Gisin, Antonio Acín and Miguel Navascués,

arXiv:2101.10873

Marc-Olivier Renou
FNS Early Mobility Fellow
QIT group (Antonio Acín)
ICFO - Castelldefels (Barcelona)



Quantum Theory uses $\mathbb{C}$-numbers operators

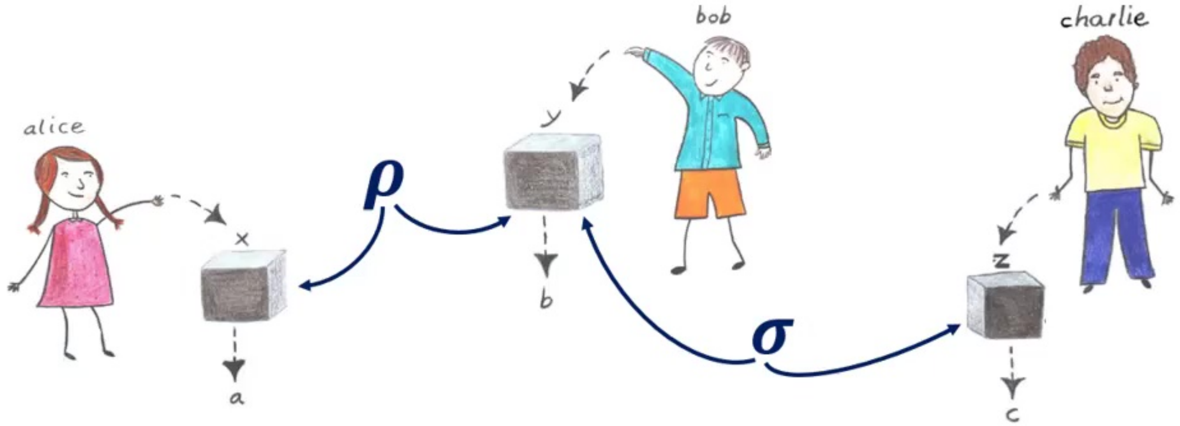
Why $\mathbb{C}$? Why not $\mathbb{R}$?

What about replacing $\mathbb{C}$ with $\mathbb{R}$?

Letter from Schrödinger to Lorentz (1926):

'What is unpleasant here, and indeed directly to be objected to, is the use of complex numbers.
ψ is surely fundamentally a real function'

First answer



As such, question does not make sense:

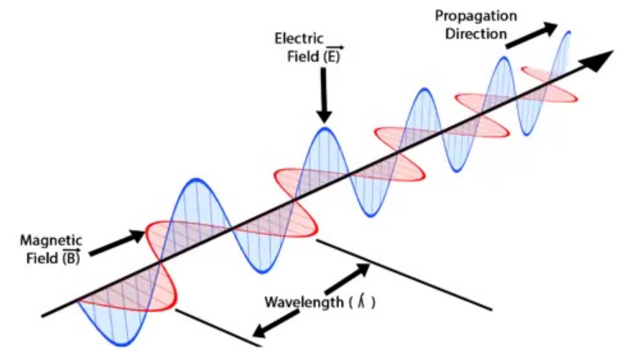
- Any experiment described by statistics it produces
- Statistics are understood in terms of $\mathbb{R}$ −numbers probabilities $\boldsymbol{P(abc|xyz)}$
  - ➢ A "Book" containing all probabilities of all potential experiement would give a $\mathbb{R}$ − description of Quantum Theory

ℂ-numbers in Physical Theories

Before QT, ‘all Physical Theories expressed with ℝ-numbers’

- ℂ-numbers are sometimes used
- In the end, the fondamental formalism / laws are in terms of ℝ-numbers
  - ➢ ℂ is used to ‘simplify computations’

Electromagnetic Wave



$$\boldsymbol{E} = \boldsymbol{E_0} e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})} \quad \boldsymbol{B} = \boldsymbol{B_0} e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})}$$

$$\nabla . E = \frac{\rho}{\epsilon_0} \qquad \nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla . B = 0 \qquad \nabla \times H = J + \frac{\partial D}{\partial t}$$

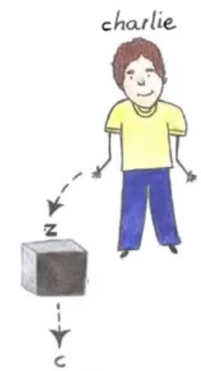
13

Quantum theory needs complex numbers

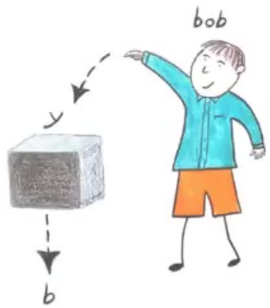
Real-number Hilbert space quantum theory with tensor products is experimentally falsifiable

I. $\mathbb{C}$-Quantum Theory and $\mathbb{R}$-Quantum Theory

II. A $\mathbb{C}$-QT Theorist challenge to a $\mathbb{R}$-QT Experimentalist



III. $\mathbb{R}$-QT Experimentalist wins any standard Bell challenge



IV. A quantum network unfeasible challenge

'(Standard) Quantum Theory' formalism

$\mathbb{C}$-Hilbert space Quantum Theory with tensor products

1 Particle S

$(i)_{\mathbb{C}}$ To any system S corresponds a $\mathbb{C}$-Hilbert Space $\mathcal{H}_S$, and its state is represented by an operator ρ_S of $\mathcal{H}_S$ with $\rho_S \succcurlyeq 0,\ \mathrm{Tr}(\rho_S) = 1$

(ii) A measurement M of S corresponds to a set $\{M_r\}$ of operators of $\mathcal{H}_S$ with $M_r \succcurlyeq 0,\ M_r^2 = M_r,\ \sum_r M_r = \mathrm{Id}$

(iii) Born rule: Measuring M over S in state ρ_S we obtain result r with probability $P(r) = \mathrm{Tr}(\rho_S \cdot M_r)$

2 Particles $\{\mathrm{S}, \mathrm{T}\}$

(iv) The Hilbert Space of the composition of two particles $\{\mathrm{S}, \mathrm{T}\}$ is the tensor product $\mathcal{H}_{ST} = \mathcal{H}_S \otimes \mathcal{H}_T$. Independent preparations of two systems ρ_S, σ_T correspond to a state $\rho_S \otimes \sigma_T$

‘$\mathbb{R}$-QT’ VS ‘$\mathbb{C}$-QT’

Do $\mathbb{R}$-QT and $\mathbb{C}$-QT have the same experimental predictions?

$\mathbb{C}$-QT includes $\mathbb{R}$-QT. Can we falsify $\mathbb{R}$-QT in an experiment?

- **IF NO:** $\mathbb{C}$ does not play a fondamental role, just here to simplify calculations
- **IF YES:** $\mathbb{C}$ is essential to the tensor-product Hilbert space formalism of Quantum Theory

Two important remarks

There exists alternative formulations of QT with $\mathbb{R}$:

- With a 'universal qubit' (Stueckelberg)
- Bohm's formulation
- ...

➢Have the same predictive power of QT. But do not satisfy $(\boldsymbol{i}) - (\boldsymbol{iv})$

There exists arguments rejecting $\mathbb{R}$-QT:

- Based on dimension

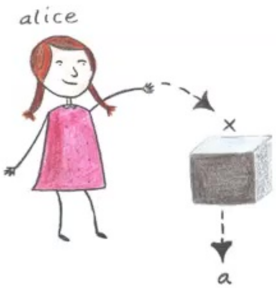
➢Not testable: all experimental systems have infinite dimension

- Based on local tomography

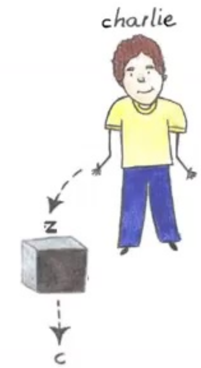
➢New axiom added to $(\boldsymbol{i}) - (\boldsymbol{iv})$ (we prove: not needed)

Quantum theory needs complex numbers

Real-number Hilbert space quantum theory with tensor products is experimentally falsifiable

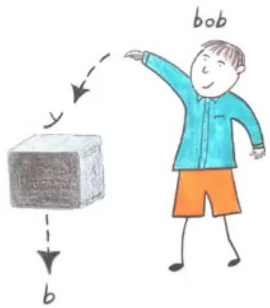


I. $\mathbb{C}$-Quantum Theory and $\mathbb{R}$-Quantum Theory



II. A $\mathbb{C}$-QT Theorist challenge to a $\mathbb{R}$-QT Experimentalist

III. $\mathbb{R}$-QT Experimentalist wins any standard Bell challenge



IV. A quantum network unfeasible challenge

$\mathbb{C}$-QT theorist and $\mathbb{R}$-QT experimentalist

Game

$\mathbb{C}$-QT Theorist

Proposes a challenge:

- A $\mathbb{C}$-QT experiment with
  $\boldsymbol{\rho}, \boldsymbol{M}_{a|x}, \dots \mapsto \boldsymbol{P}(a, \dots | x, \dots)$

$\mathbb{R}$-QT Experimentalist

Takes up the challenge

- A $\mathbb{R}$-QT experiment (with **real operators**)
  $\tilde{\boldsymbol{\rho}}, \tilde{\boldsymbol{M}}_{a|x}, \dots \mapsto \boldsymbol{P}(a, \dots | x, \dots)$

Similar to Bell Games

34

$\mathbb{C}$-QT theorist and $\mathbb{R}$-QT experimentalist

Game

$\mathbb{C}$-QT Theorist

Proposes a challenge:

- A $\mathbb{C}$-QT experiment with
  $\boldsymbol{\rho}, \boldsymbol{M}_{a|x}, \dots \mapsto \boldsymbol{P}(a, \dots | x, \dots)$

$\mathbb{R}$-QT Experimentalist

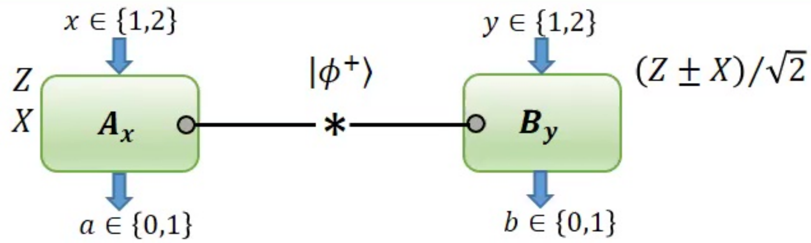
Takes up the challenge

- A $\mathbb{R}$-QT experiment (with **real operators**)
  $\tilde{\boldsymbol{\rho}}, \tilde{\boldsymbol{M}}_{a|x}, \dots \mapsto \boldsymbol{P}(a, \dots | x, \dots)$

Look for a 'good game' i.e.
a game where the $\mathbb{R}$-QT Experimentalist fails

33

Similar to Bell Games



$\mathbb{C}$-QT Theorist

Proposes a challenge:

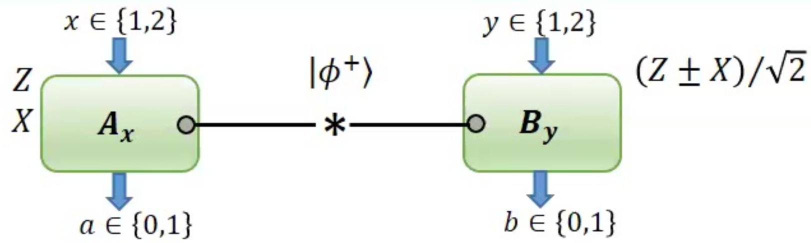
➢The CHSH experiment

$|\boldsymbol{\phi}^+\rangle, \boldsymbol{A}_{a|x}, \boldsymbol{B}_{b|y} \mapsto \boldsymbol{P}(\boldsymbol{a}, \boldsymbol{b}|\boldsymbol{x}, \boldsymbol{y})$ such that

$\mathbf{CHSH} = \mathbf{2}\sqrt{\mathbf{2}}$

Local Hidden Variable Model Experimentalist

Similar to Bell Games



$\mathbb{C}$-QT Theorist

Proposes a challenge:

- The CHSH experiment

$|\boldsymbol{\phi}^+\rangle, \boldsymbol{A}_{a|x}, \boldsymbol{B}_{b|y} \mapsto \boldsymbol{P}(\boldsymbol{a}, \boldsymbol{b}|\boldsymbol{x}, \boldsymbol{y})$ such that $\mathbf{CHSH} = \mathbf{2}\sqrt{\mathbf{2}}$

Local Hidden Variable Model Experimentalist

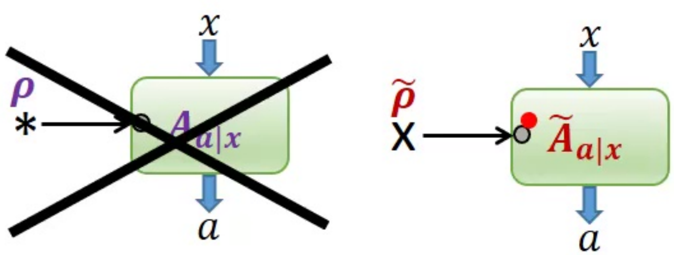
Cannot take up the challenge

- Any LHV model is limited by $\mathbf{CHSH} \leq \mathbf{2}$

- CHSH is a ‘good game’ in which the LHV model Experimentalist fails

Theorist's Challenge 1

$\mathbb{C}$-QT experiment n°1:



➢ $P(a|x)$

$\rho, A_{a|x}$ well chosen such that

$P(a|x)$ **unsimulable**?

$\mathbb{R}$-QT Simulation n°1:

The experimentalist introduces:

- $|i\rangle := \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$
- $|-i\rangle := \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) = |i\rangle^*$

Simulation with:

$$\tilde{\rho} := \frac{1}{2}(\rho \otimes |i\rangle\langle i| + \rho^* \otimes |-i\rangle\langle -i|) = \tilde{\rho}^*$$

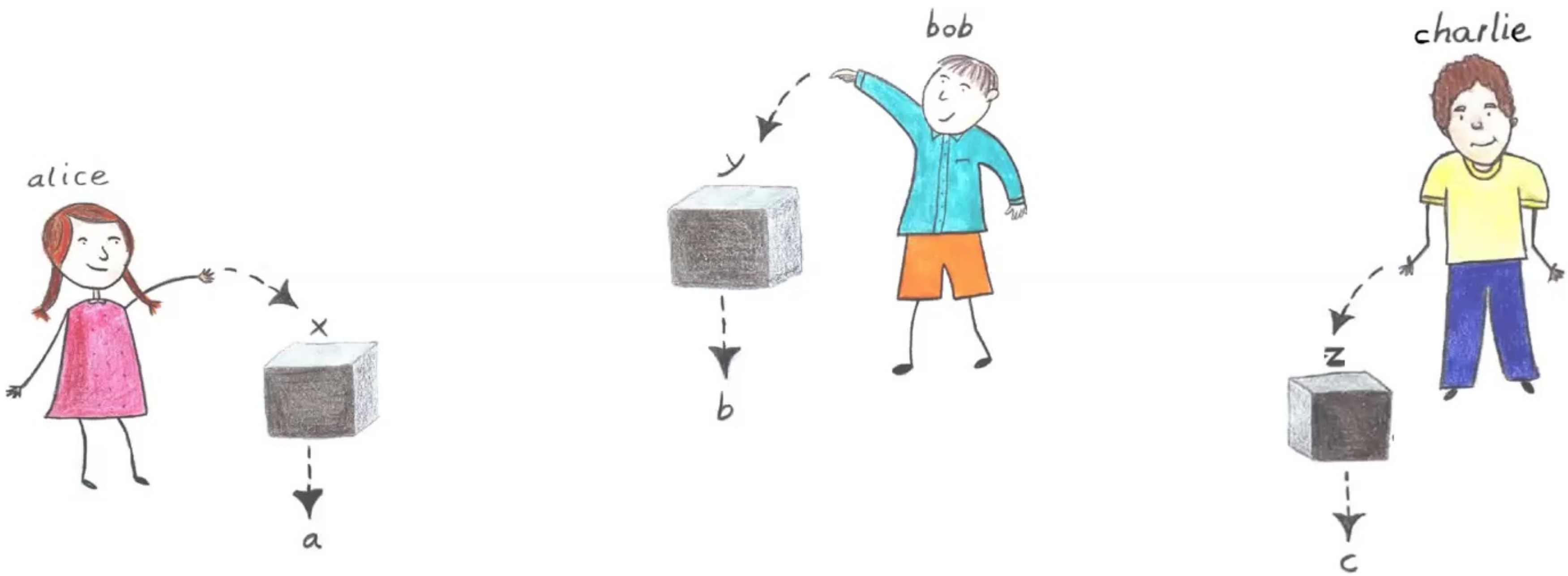
$$\tilde{A}_{a|x} := \frac{1}{2}\left(A_{a|x} \otimes |i\rangle\langle i| + A^*_{a|x} \otimes |-i\rangle\langle -i|\right) = \tilde{A}^*_{a|x}$$

➢Idea:

Theorist point of view:

$|i\rangle, |-i\rangle$ work as a 'flag particle' which indicates 'what is $\sqrt{-1}$'

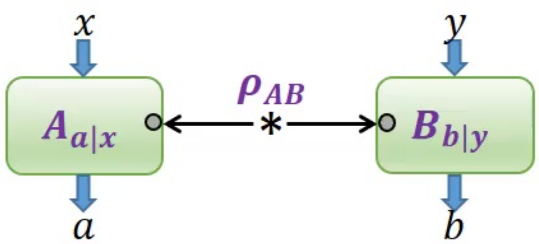
46



Is this trick «‘what is $\sqrt{-1}$’ flag » problem clear?

Theorist's Challenge 2

$\mathbb{C}$-QT experiment n°2:



➢ $P(ab|xy)$

$\rho_{AB}, A_{a|x}, B_{b|y}$ well chosen such that $P(ab|xy)$ **unsimulable**?

$\mathbb{R}$-QT Simulation n°2:

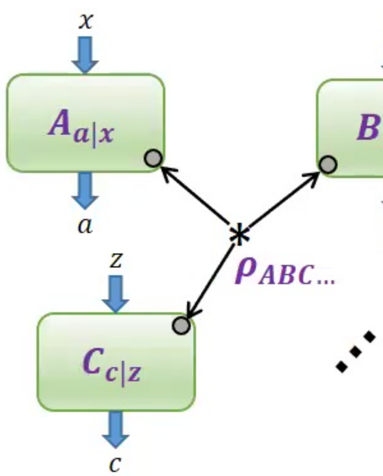
Simulation with:

$$\tilde{\rho}_{AA'BB'} := \frac{1}{2}\left(\rho_{AB} \otimes |ii\rangle\langle ii|_{A'B'} + \rho^*_{AB} \otimes |{-i}{-i}\rangle\langle{-i}{-i}|_{A'B'}\right)$$

Same $\tilde{A}_{a|x}, \tilde{B}_{b|y}$ as in Simulation n°1

Theorist's Challenge 3

$\mathbb{C}$-QT experiment n°3:



➢ $P(abc\ldots|xyz\ldots)$

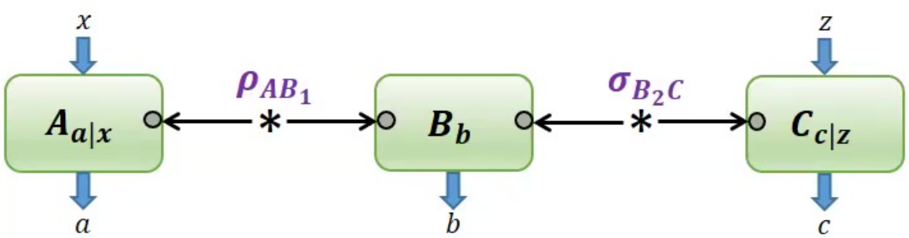
$\mathbb{R}$-QT Simulation n°3:

Same idea

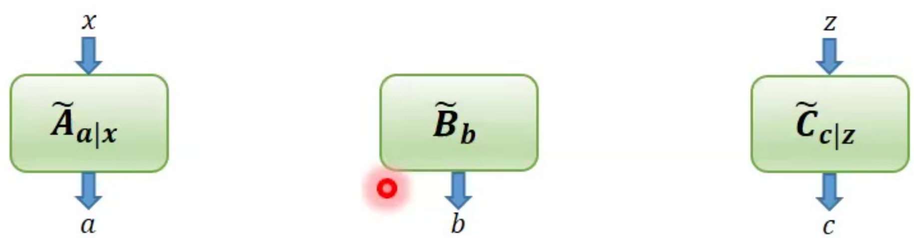
...

Theorist's Challenge 4

$\mathbb{C}$-QT experiment n°4:



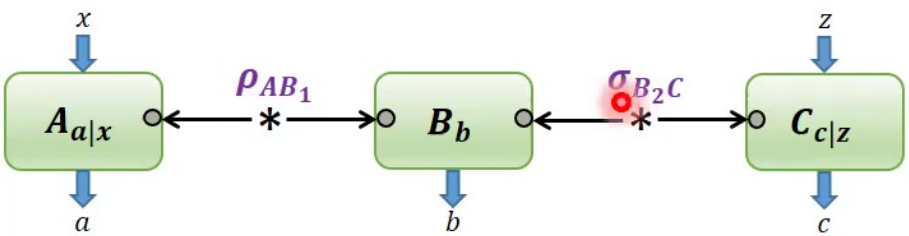
$\mathbb{R}$-QT simulation n°4?



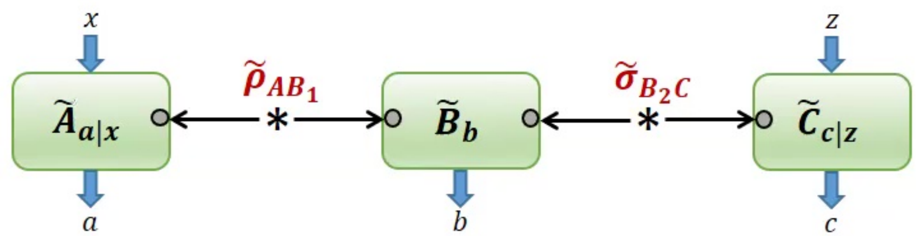
54

Theorist’s Challenge 4

$\mathbb{C}$-QT experiment n°4:



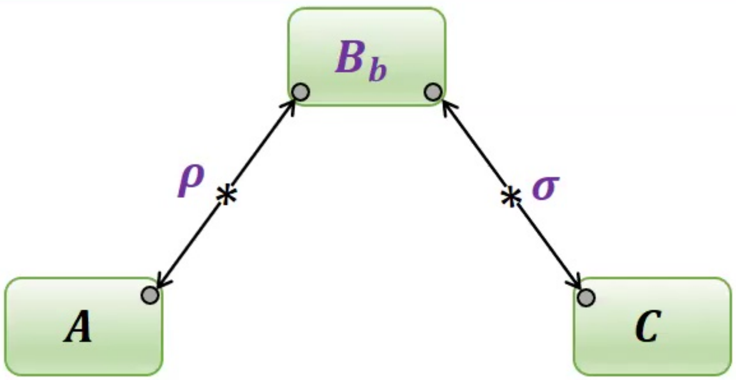
$\mathbb{R}$-QT simulation n°4?

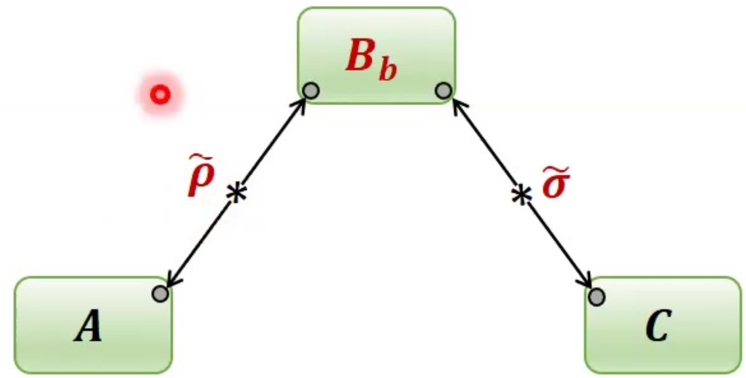


- **Extra assumption**:
the $\mathbb{R}$-QT experimentalist should respect the same experimental network structure

Intuition

Why the trick «'what is $\sqrt{-1}$ flag » is not possible?





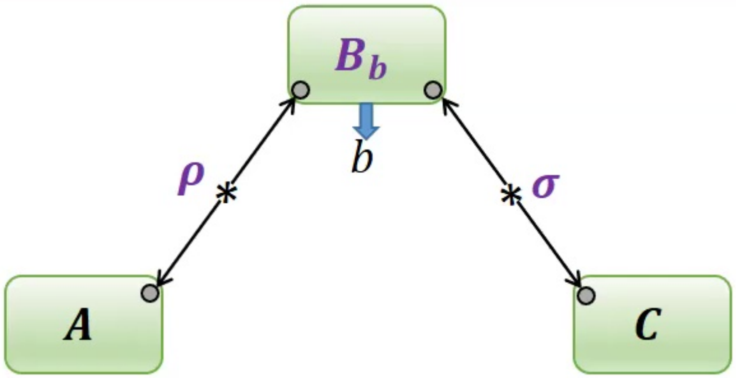
- $\boldsymbol{B}$ has two 'flags', one from $\tilde{\boldsymbol{\rho}}$ and one from $\tilde{\boldsymbol{\sigma}}$

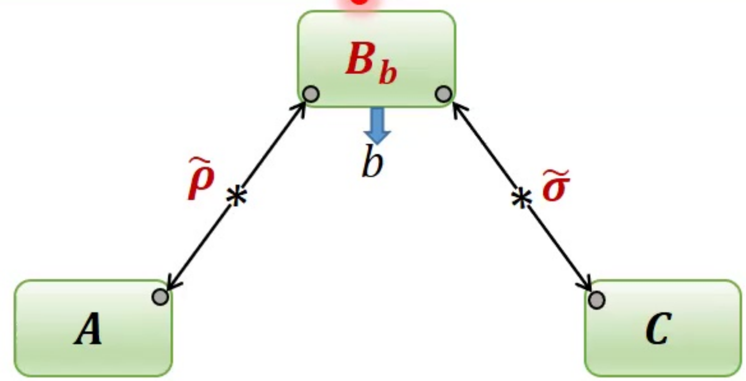
$$\tilde{\rho} = \frac{1}{2}(\rho \otimes |ii\rangle\langle ii| + \rho^* \otimes |{-i}{-i}\rangle\langle{-i}{-i}|)$$
$$\tilde{\sigma} = \frac{1}{2}(\sigma \otimes |ii\rangle\langle ii| + \sigma^* \otimes |{-i}{-i}\rangle\langle{-i}{-i}|)$$

59

Intuition

Why the trick «'what is $\sqrt{-1}$ flag » is not possible?





$$\tilde{\rho} = \frac{1}{2}(\rho \otimes |ii\rangle\langle ii| + \rho^* \otimes |{-i}{-i}\rangle\langle{-i}{-i}|)$$
$$\tilde{\sigma} = \frac{1}{2}(\sigma \otimes |ii\rangle\langle ii| + \sigma^* \otimes |{-i}{-i}\rangle\langle{-i}{-i}|)$$

- $\boldsymbol{B}$ has two 'flags', one from $\tilde{\rho}$ and one from $\tilde{\sigma}$
- Measuring the flags first:
  - ➢ The flag might coincide or not
  - ➢ When not coinciding, $\boldsymbol{A}$ and $\boldsymbol{C}$ have opposite definitions for $\sqrt{-1}$

Intuition

Why the trick «'what is $\sqrt{-1}$ flag » is not possible?

B_b

B_b

A

$\tilde{\sigma}$

C

- Here, the trick «'what is $\sqrt{-1}$' flag » does not work
- Alternative trick could exist?
    - Entangled simulation/flag measure
    - Infinite dimensions
    - …

from $\tilde{\sigma}$

$\tilde{\rho} = \frac{1}{2}(\rho \otimes$

$\tilde{\sigma} = \frac{1}{2}(\sigma \otimes$

$\tilde{\tau} = \frac{1}{2}(\tau \otimes$

ve opposite

neasure $\tilde{\tau}^b$

64

Intuition

Why the trick «'what is $\sqrt{-1}$ flag » is not possible?

B_b B_b

$\tilde{\sigma}$

A C

- Here, the trick «'what is $\sqrt{-1}$' flag » does not work
- Alternative trick could exist?
  - Entangled simulation/flag measure
  - Infinite dimensions
  - …

➢ Our work = well chosen challenge (states, measurements) achieving $P(abc|xz)$ such that:

i. The flag strategy is the only possible one

$\tilde{\rho} = \frac{1}{2}(\rho \otimes$

$\tilde{\sigma} = \frac{1}{2}(\sigma \otimes$

$\tilde{\tau} = \frac{1}{2}(\tau \otimes$

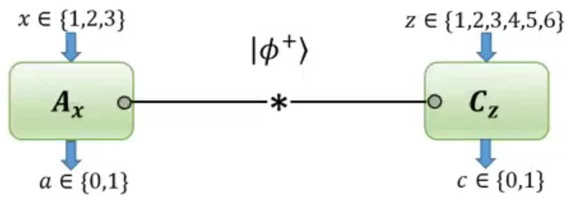
from $\tilde{\sigma}$

ve opposite

easure $\tilde{\tau}^b$

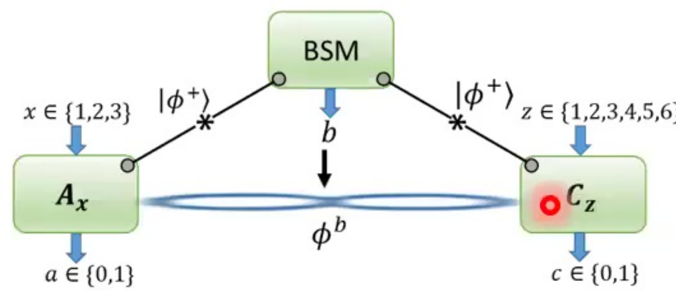
Challenge 4

3-CHSH Self Testing
Impose use of Pauli $\mathbb{C}$ measurements (flagged $\mathbb{R}$ simulation possible)



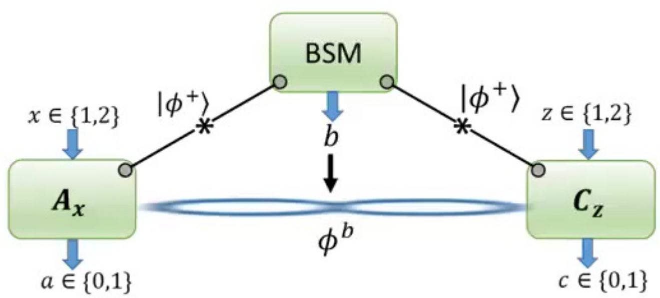
+

=



Challenge 4
- $\boldsymbol{A}, \boldsymbol{C}$: Pauli CHSH like measurements
- $\boldsymbol{B}$: Bell State Measurement
- Shared EPR pairs $\boldsymbol{\phi}^+$

Bell State Measurement Self Testing
Forbids entangled simulation/flag measure (all is in $\mathbb{R}$)

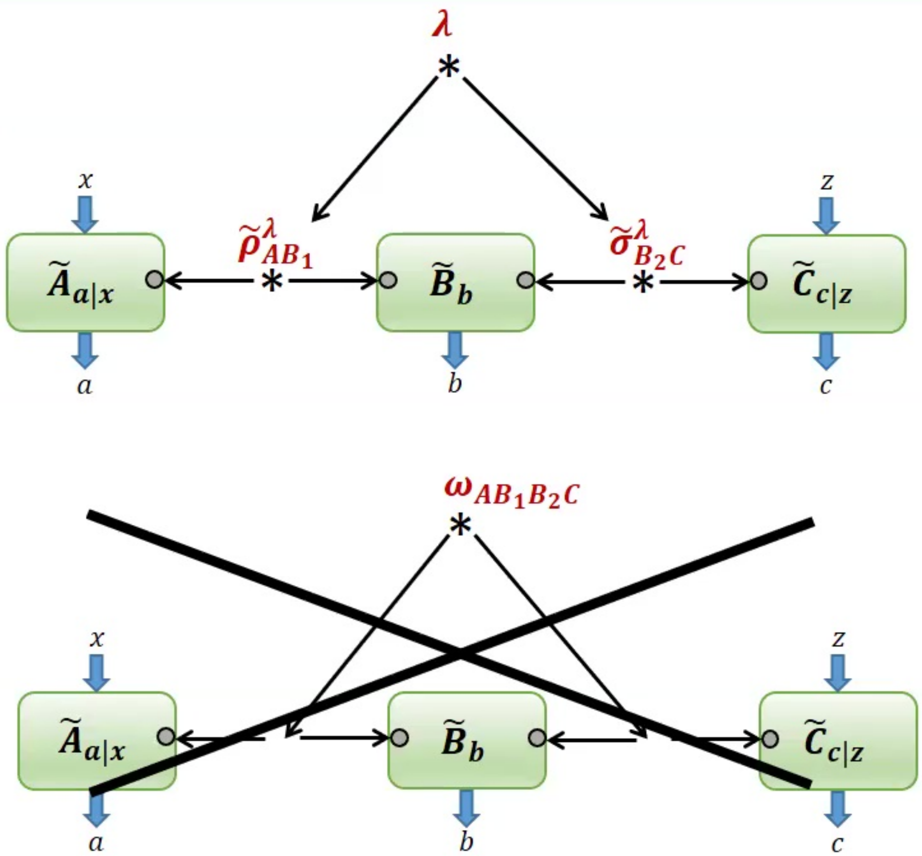


Bowles, Šupić, Cavalcanti, <u>Acín</u>,Phys. Rev. Lett. 121, 180503 & Phys. Rev. A 98, 042336 (2018)
<u>Renou</u>, Kaniewski, Brunner, Phys. Rev. Lett. 121, 250507 (2018)
Bancal, Sangouard, Sekatski Phys. Rev. Lett. 121, 250506 (2018)

Experimental test?

- The independence hypothesis is problematic.
  - Classical memories give access to classical shared randomness
    - ➢ We proved the result still holds in that case
  - Quantum memories give access to global quantum state
    - ➢ The result does not hold: flag simulation

- What about a noisy experiment?
  - We introduce $\mathcal{T}(\vec{P})$, Linear Bell expression summing **CHSH** variants. We show:
    - $\mathcal{T}(P) \leq 7.6605$ for any $\mathbb{R}$-QT strategy
    - $\mathcal{T}(P) = 6\sqrt{2} \approx 8.4852$ is possible with a $\mathbb{C}$-QT strategy



➢ Our theory work: arXiv:2101.10873.

➢ Experimental demonstration: arXiv:2103.08123

➢ More to come!

arXiv:2103.08123: Ruling out real-number description of quantum mechanics, Ming-Cheng Chen, Can Wang, Feng-Ming Liu, Jian-Wen Wang, Chong Ying, Zhong-Xia Shang, Yulin Wu, Ming Gong, Hui Deng, Futian Liang, Qiang Zhang, Cheng-Zhi Peng, Xiaobo Zhu, Adan Cabello, Chao-Yang Lu, Jian-Wei Pan 78

To conclude

We proved

- 'QT needs $\mathbb{C}$-numbers' (more exactly: 'doesn't work with $\mathbb{R}$-numbers')
  - Our proofs rely on the axioms $(\boldsymbol{i}) - (\boldsymbol{iv})$
- This can be witnessed experimentally

Future work?

- $(\boldsymbol{i}) - (\boldsymbol{iv})$ are 'mathematical axioms'
  - ➢ Replace them with physically motivated axioms?
- Bell-like test can reject Local Hidden Variable models, and $\mathbb{R}$-QT.
  - ➢ Can we reject more theories?

Quantum Theory from First Principles: An Informational Approach D'Ariano, Chiribella, Perinotti, Cambridge University Press (2017)

Acknowledgments