

Title: On the perturbation theory for spectra in quantum mechanics

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Series: Mathematical Physics

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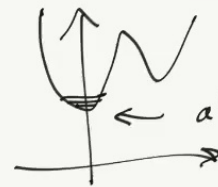
Abstract: Consider a polynomial differential operator in one variable, depending on a small parameter (Planck constant). Under appropriate conditions, the low-energy spectrum admits an asymptotic expansion in $\hbar$.

I will present a way to calculate such a series via a purely "commutative problem", a mixture of variations of Hodge structures and of the Stirling formula. This result came from discussions with A. Soibelman. It seems that we obtain an explanation of an old observation by J. Zinn-Justin of the

universal appearance of Bernoulli numbers.

$$H_q := -\frac{\hbar^2}{2} \left(\frac{d}{dx} \right)^2 + V(x)$$

polynomial



$$H_q: L^2(\mathbb{R}) \ni$$

$V_{\min}$

low-energy spectrum as $\hbar \rightarrow +0$.
 $E_0(\hbar) < E_1(\hbar) < \dots$ admit asymptotic expansion

$$E_k(\hbar) = V_{\min} + \sum_{n \geq 0} c_{k,n} \hbar^n$$

$c_{k,n} \in \mathbb{R}$

Goal of the talk : calculate $(c_{k,n})$
 without noncomm. algebras!

$$[a, a^*] = i\hbar$$

Birkhoff

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Classical case: sympl. mfd (M, ω) $\dim_{\mathbb{R}} M = 2$
(domain in $\mathbb{R}^4$)

$$H_{cl} : M \rightarrow \mathbb{R}$$

$m_0 \in M$ local minimum

$$H''_{cl}|_{m_0} > 0$$

$$H_{cl}(m_0) = 0$$

? Normal form of (M, ω, H_{cl}) near m_0

Without 2-form ω $\exists$ local coord. (x, y)

$$H_{cl} = x^2 + y^2$$



With 2-form ω

$$\int \omega =$$

$$\pi t = \pi z^2 \quad \text{for usual case.}$$

$$x^2 + y^2 = z^2$$

$$z^2 \geq t$$

$$\{H_{cl} \leq t\}$$

$$\leq t$$

$$= a_1 t + a_2 t^{\frac{3}{2}} + \dots \quad \text{in general} \quad \mathbb{R} \text{ still}$$

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? Normal form of (M, ω, H_{cl}) near m_0

Without 2-form ω $\exists$ local coord. (x, y)

$$H_{cl} = \frac{x^2 + y^2}{4}$$



With 2-form ω

$$\int \omega = \pi t = \pi z^2$$

for usual case. $x^2 + y^2 = z^2$
 $z^2 \geq t$

$$\{H_{cl} \leq t\}$$

$$= a_1 t + a_2 t^{\frac{3}{2}} + \dots \sim$$

in general $\mathbb{R}^2$ st

$$\Leftrightarrow H_{cl} = b_1 H_0 + b_2 H_0^2 + \dots$$

$H_0 = \frac{1}{2}(x^2 + y^2)$

Same is true in quantum case:

Instead of $(M, \omega) \rightsquigarrow$ assoc. algebra $A / \llbracket \hbar \rrbracket$
 flat $A / \hbar A \simeq C^\infty(M)$
 $\alpha, b \in A \quad \left. \frac{[\alpha, b]}{i\hbar} \right|_{\hbar=0} = \{ \alpha|_{\hbar=0}, b|_{\hbar=0} \}$
 expression $i\hbar \frac{\partial}{\partial x}, \propto \llbracket \hbar \rrbracket$

E.g. Moyal product: $A = C^\infty(\mathbb{R}^2_{x,y}) \llbracket \hbar \rrbracket$

$$a * b := ab + \frac{i\hbar}{2} \{a, b\} + \dots$$

$$= \sum_{n \geq 0} \frac{(i\hbar)^n}{n!} \left(\frac{\partial}{\partial y} \right)^n a \cdot \left(\frac{\partial}{\partial x} \right)^n b$$

Instead of $H_{cl} \rightsquigarrow$ element $H_q \in A$

Spectrum of H_q at local minimum?
 $\{ \sum_{n \geq 0} c_{k,n} \hbar^n \mid k=0,1,2,\dots \}$

Formal definition:

Quantum Birkhoff normal form.

$$\mathcal{Q} = x$$

$$y = -i\hbar \frac{\partial}{\partial x}$$

$$H_{0,q} = \frac{1}{2} (\hat{x}^2 + \hat{y}^2)$$

$$H_q = a_0 + a_1 H_{0,q} + a_2 H_{0,q}^2 + \dots$$

$$a_i \in \mathbb{R} \ll \hbar \gg$$

$$(\sum b_{k,n} H_{0,q}^k \cdot \hbar^n)$$

$$b_{1,0} > a$$

For $H_{0,1}$

Spectrum is $\frac{1}{2}\hbar, \frac{3}{2}\hbar, \frac{5}{2}\hbar, \dots$

$$(l + \frac{1}{2})\hbar \quad l=0,1,\dots$$

Substitute in
 $H_{0,q} \rightsquigarrow (l + \frac{1}{2})\hbar$

Theorem (Sjostrand)

For polynomial diff op.
depend on small parameter $\hbar$

--- Actual spectrum : series expansion
coming from quantum normal form

Many degrees of freedom

$$\omega = \sum_{i=1}^N dx_i \wedge dy_i$$

$$H_{cl} = \sum_1^N \omega_i^2 \frac{1}{2} (x_i^2 + y_i^2) + \text{cubic term}$$

Assume $\oplus \mathbb{Q} \omega_i \hookrightarrow \mathbb{R}$ $\mathbb{Q}$ -linearly independent

$$\rightarrow H_{cl} = \text{formal series in } \left(\frac{x_1^2 + y_1^2}{2}, \dots, \frac{x_N^2 + y_N^2}{2} \right) \quad \text{classical}$$

$$\left(\frac{x_1^2 + y_1^2}{2}, \dots, \frac{x_N^2 + y_N^2}{2}, \left(\frac{\hbar}{2} \right) \right) \quad \text{quantum}$$

$(A_q, \hat{H}_q)$
 $\star$ -algebra hamiltonian

?

(M, ω, H)
 $\dim M = 2$

"integrable system"

lagrangian fibration

(M, ω)
 $\pi \downarrow H$
 $A' \quad \mathbb{R}'/\mathbb{C}$

"quantum SHe system"

$\pi^{-1}(t \in A')$ curve Lagrangian.

Quasi-Thm (M. Gromov, D. van Straten) (complex)

(Complex) Quantum SHe system
 in $\star$ -product language.

$\longleftrightarrow$

classical SHe system
 depends on $\hbar$
 $\mathcal{O}(\hbar)$

$(M, \omega) \rightarrow B$
 $\mathcal{O}(B) \hookrightarrow$ commutative is $\checkmark$



local piece

M, A

$\pi \downarrow$
 B

$d\pi \neq 0$

The same is true
in quantum case

$\omega(B)$ func in x $\mathbb{R}^n$

A $\frac{\hbar}{2\pi} \frac{\partial}{\partial x}, x, \mathbb{R}^n$

it on B

classical limit

2d domain, ω

foliation

by curves
(level sets of π)

all such strata are stan in $T^*\mathbb{R}$

$\omega = dx \wedge dy$

$\pi : (x, y) \rightarrow x$

Ambiguity:

1) $x \sim x' = c_0 + c_1 x + c_2 x^2$
 $y \rightarrow y \frac{dx'}{dx}$
 $c, \neq 0$

2) $(x, y) \rightarrow (x, y + \varphi(x))$

Lie alg of infinit. symmetric

$\text{Vect}(\mathbb{R}) \times \mathfrak{sl}_2(\mathbb{R})$

$\varphi(x) dx$

Quanta int. system.
with coordinates
on the base



cover by standard
 domains.
 transition functions
 in Abelian Sheaf

Catch: everything works
 outside critical points in M

$$P(x, y) \in \mathbb{C}(x, y)$$



Let $P \rightarrow$ link subset of $\mathbb{P}^2$
 Assume: Morse

$$Z, Z_2$$

Conclusion: quantum sys in 1 d. of freedom
 cons. polynomial diff. eq. ($\hbar$)
 or family of class. sys (M'_h, ω)
 $\downarrow$ coord. line
 A'_t
 $M'_0 = M - \text{Crit pts}$
 $\{', \cdot, \circ\}$

G-analytic



gem of punct neigh of $(0,0) \in \mathbb{C}^2$
 a_t, d_t

$\downarrow z_1 z_2 + \dots$

$\mathbb{C}_t$

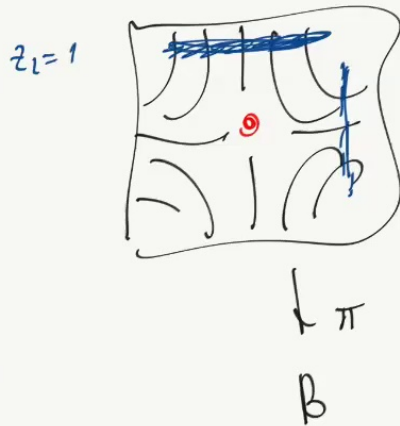
Claim:

$\exists$ canonical coord.
on b

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M, w (dep. on $\|h\|$)

$$\pi^{-1}(b) \approx$$



if $b \neq 0$
vanish. cycle

or



if $b = 0$

$\int w$: closed
vanish. cycle 1-form
on B

$$= 2\pi i \, dt_{can}$$

Defines t_{can} up to $+const$

(a priori two 1-form on B
coincide for $b \neq 0$
 $\Rightarrow$ the same)

$$dz_1, dz_2$$

$$t_i = z_i, z_2$$

$$t_{can} = t$$

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How to fix constant:



point $t \in B$

$\pi^{-1}(t) \supset \gamma_t$ vanishes and $\sim S^1$

$M' = \mathbb{CP}^2 - \bullet \approx S^3$

No problem to find $\mathbb{D}^2 \subset M'$ $\partial \mathbb{D}' = \gamma_t$

How to fix constant:



point $t \in B$

$$\pi^{-1}(t) \supset \gamma_t \quad \text{vanishes} \quad \text{and} \quad \sim S^1$$

$$M' = (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^2 - \bullet \approx S^3$$

Now problem to find $\mathcal{D}_t^2 \subset M' \quad \partial \mathcal{D}_t^2 = \gamma_t$

$$t_{an} = \frac{1}{2\pi i} \int_{\mathcal{D}_t^2} \omega$$

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$k \pi$
 B

M, w (dep. on $\|h\|$)

$\pi^{-1}(b) \approx$

or



if $b \neq 0$
vanishing cycle

if $b = 0$







$\int w$: closed 1-form on B

vanish. cycle

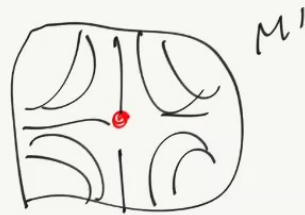
$= 2\pi i \, dt_{can}$

Defines t_{can} up to $+const \|h\|$.

(a priori two 1-forms on B coincide for $b \neq 0 \Rightarrow$ the same)

dz_1, dz_2
 $= t$

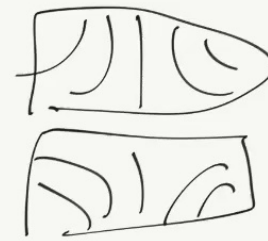
In the canonical coordinate t_{can}



$$\varphi_1 = \log z_1 + \text{regular at } z_1 \neq 0$$

$$\{\varphi_1, t_{can}\} = 1$$

$$U_1$$



$$z_2 = 0$$

$$U_2$$

$$\varphi_2 = \log z_2 + \text{regular}$$

$$\{\varphi_2, t_{can}\} = -1$$

$$\text{Each } \left(\begin{array}{c} U_i, w \\ \downarrow t_{can} \\ \mathbb{C} \end{array} \right) \simeq$$

standard

gluing U_1, U_2
given by 1-form in t_{can}

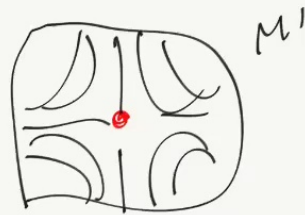
$$\{\varphi_1 + \varphi_2, t_{can}\} = 0 \Rightarrow$$

$$\text{Invariant } (\tau_k)_1$$

$$\varphi_1 + \varphi_2 = \log t_{can} + \text{1-valued function}$$

$$\sum_{k=-\infty}^{+\infty} \tau_k (t_{can})^k$$

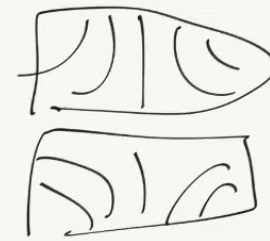
In the canonical coordinate t_{can}



$$\varphi_1 = \log z_1 + \text{regular at } z_1 \neq 0$$

$$\{\varphi_1, t_{can}\} = 1$$

$$U_1$$



$$z_2 = 0$$

$$U_2$$

$$\varphi_2 = \log z_2 + \text{regular}$$

$$\{\varphi_2, t_{can}\} = -1$$

Fact:

$$\tau_{-1} = 0$$

$$\text{Each } \left(\begin{array}{c} U_i, w \\ \downarrow t_{can} \\ \mathbb{C} \end{array} \right) \simeq$$

standard

gluing U_1, U_2
given by 1-form in t_{can}

$$\{\varphi_1 + \varphi_2, t_{can}\} = 0$$

$$\Rightarrow$$

$$\text{Invariant } (\tau_k)_{k \leq -1}$$

$$\varphi_1 + \varphi_2 = \log t_{can} + \text{1-valued function}$$

$$\sum_{k=-\infty}^{+\infty} \tau_k (t_{can})^k$$

Using quantum normal form

$$H_{0,1} = i\hbar \left(z \frac{d}{dz} + \frac{1}{2} \right)$$

$$H_0 = z \frac{d}{dz} + \frac{1}{2}$$

$$[\hat{Q}_1, H_0] = -1$$

$$[\hat{Q}_2, H_0] = +1$$

$$e^{\hat{Q}_1}, e^{\hat{Q}_2} \quad 1\text{-valued}$$

$$[\log(z), H_0] = -1$$

$$[\log\left(\frac{d}{dz}\right), H_0] = +1$$

$$[\log(z) + \log\left(\frac{d}{dz}\right), H_0] = 0$$

$$\log(z) + \log\left(\frac{d}{dz}\right)$$

$$= \log H_0 + \text{1-valued term in } H_0$$

$$= \log H_0 + \frac{1}{24 H_0^2} \Rightarrow \frac{z}{960 H_0^4} + \dots$$

=

$$H_0 \rightsquigarrow s + \frac{1}{2}$$

$$H_0 = z \frac{d}{dz} + \frac{1}{2}$$

$$H_0(z^s) = \left(s + \frac{1}{2}\right) z^s$$

$$\left(\log z + \log \frac{1}{\sqrt{z}}\right) (z^s)$$

Fourier transform

$$\psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}$$

$$\psi\left(s + \frac{1}{2}\right) = \log s + \frac{1}{24s^2} - \frac{7}{960s^4} + \dots$$

$$= \log s + \sum_{n \geq 1} \underbrace{\left(1 - \frac{1}{2^{2n+1}}\right) \cdot \frac{B_{2n}}{2n}}_{\text{Tr } H_0^{2n-1}} s^{-2n}$$

ζ -regularized.

$$H_0 \rightsquigarrow s + \frac{1}{2}$$

$$H_0 = z \frac{d}{dz} + \frac{1}{2}$$

$$H_0(z^s) = \left(s + \frac{1}{2}\right) z^s$$

$$\left(\log z + \log \frac{1}{z}\right) (z^s)$$

Fourier transform

$$\psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}$$

$$\psi\left(s + \frac{1}{2}\right) = \log s + \frac{1}{24s^2} - \frac{7}{960s^4} + \dots$$

$$= \log s + \sum_{n \geq 1}^{\uparrow}$$

$$\left(1 - \frac{1}{2^{2n-1}}\right) \cdot \frac{B_{2n}}{2n} s^{-2n}$$

$$\log H_0 + \sum_{n \geq 1} \frac{\text{Tr } H_0^{2n-1}}{H_0^{2n}}$$

$\text{Tr } H_0^{2n-1}$
5-regularized.

Polyn. diff on. dep. on t

classical limit $M = \mathbb{C}^2 \supset M' = \mathbb{C}^2 = \text{cont pt}$

$\downarrow \pi$
 $B = \mathbb{C}$

$\downarrow \pi$
 $B = \text{finite point}$ D_i

Defn: family of curves over B — Discr.
 C_t affine curve.

strut $\int$ hle system;

Fact:

$$H_1(C_t, \mathbb{Z}) \rightarrow \Omega_{cl}^1(B)$$

map descends to

$$H_1(\bar{C}_t, \mathbb{Z}) \rightarrow \Omega_{cl}^1(B)$$

Classical sys on B-Disc.

Bundle with monomorph $Sp(2g, \mathbb{Z})$ $(H_{\mathbb{Z}}^1, \nabla)$
flat conn.

+ section set $(H_{\mathbb{Z}}^1 \otimes \Omega_{A,B}^1)$ holom on B-Disc.

Near sim. on Base: $)))X$

monomorph $\sim \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \\ & & & & 1 \end{pmatrix} 4$

One can find local triu
at t_a of $H_{\mathbb{Z}}^1 \otimes 0$
connection = $dt \begin{pmatrix} 0 & \frac{dt}{t-t_a} & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$

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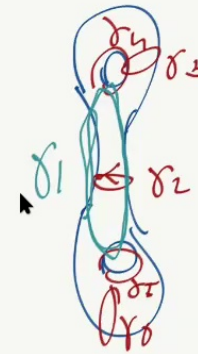
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Constraint for section $S = \begin{pmatrix} s_1(t) \\ \vdots \\ s_q(t) \end{pmatrix} dt$ near t_d ?

Properties

$s_2(t), \dots, s_q(t)$
in

$\begin{pmatrix} s_2 \\ \vdots \\ s_q \end{pmatrix} dt = \int_0^{\infty}$
cycles
invariant
under
monodromy
 $\sigma_2 \dots \sigma_q$



vanishing

$$\sigma_1 \cdot \sigma_2 = 1$$

$$\sigma_1 \rightarrow \sigma_1 + \sigma_2$$



Constraint for section $S = \begin{pmatrix} s_1(t) \\ \vdots \\ s_q(t) \end{pmatrix} dt$ near t_d ?

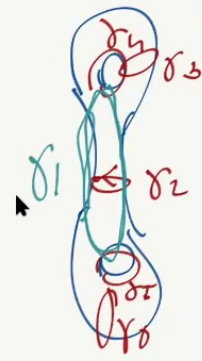
Properties

$s_2(t), \dots, s_q(t)$
have no poles at $t \rightarrow t_0$

$s_1(t)?$

$$s_1(t) dt = \int_{\gamma_1} \omega - \log(t - t_d) s_2(t) dt$$

$\begin{pmatrix} s_2 \\ \vdots \\ s_q \end{pmatrix} dt = \int_{\omega}$
cycles
invariant
under
monodromy
 $\gamma_2 \dots \gamma_q$



vanishing

$\gamma_1 \cdot \gamma_2 = 1$

$\gamma_1 \rightarrow \gamma_1 + \gamma_2$

$\exists$ center. $t_{\text{cen}} = t_{\text{cen},a}$ near t_a center coord. (t)

$$\left(s_1(t) + \log(t - t_a) s_2 \right) dt = \left(\log t_{\text{cen}} + \frac{t^2}{24 t_{\text{cen}}^2} - \frac{7t^4}{960 t_{\text{cen}}^4} + \dots \right) dt_{\text{cen}}$$

+ unknown series in (t_{cen}, t^2) .

$$s_2 dt = dt_{\text{cen}}$$

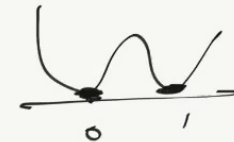
$$\Rightarrow \frac{s_1(t)}{s_2(t)} + \log \frac{t - t_a}{t_{\text{cen}}} = \frac{t^2}{24 t_{\text{cen}}^2} - \frac{7t^4}{960 t_{\text{cen}}^4} + \dots$$

+ unknown regular series.

$s_1, \dots, s_n$ no singularities.

§ Zimm-Suskin

$$-\frac{1}{2}\hbar^2 \left(\frac{d}{dx}\right)^2 + (x(x-1))^2$$



double well

$$\frac{1}{n} e^{-\frac{A}{n} n}$$

$$T(e^{-t\Delta}) \approx \sum c_n t^n \quad c_n \sim n!$$

$$H(y) \approx e^{\frac{iS}{\hbar}} + \text{reson.} = \psi$$