Title: Dynamics of thermalization in isolated quantum many-body systems: A simple solvable example

Speakers: Katja Klobas

Series: Colloquium

Date: November 10, 2021 - 2:00 PM

URL: https://pirsa.org/21110008

Abstract: When a generic isolated quantum many-body system is driven out of equilibrium, its local properties are eventually described by the thermal ensemble. This picture can be intuitively explained by saying that, in the thermodynamic limit, the system acts as a bath for its own local subsystems. Despite the undeniable success of this paradigm, for interacting systems most of the evidence in support of it comes from numerical computations in relatively small systems, and there are very few exact results. The situation changed recently, with the discovery of certain solvable classes of local quantum circuits, in which finite-time dynamics is accessible and the subsystem-thermalization picture can be verified. After introducing the general picture I will present a recent example (arXiv:2012.12256) of a simple interacting integrable circuit, for which the finite-time dynamics can be exactly described, and the model can be shown to exhibit generic thermalization properties.

Dynamics of thermalization in isolated quantum many-body systems: A simple solvable example

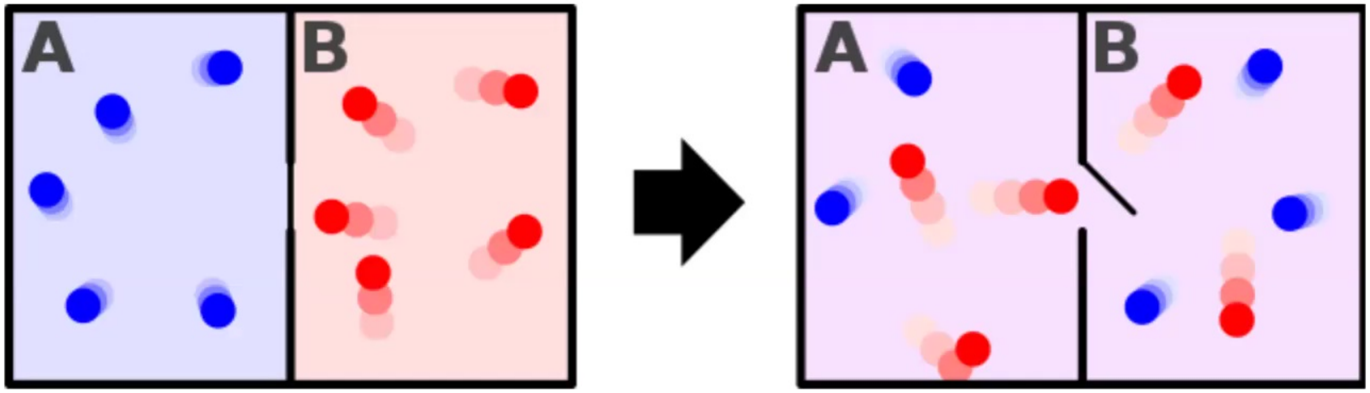
Katja Klobas

Perimeter Institute (online)

10th November, 2021

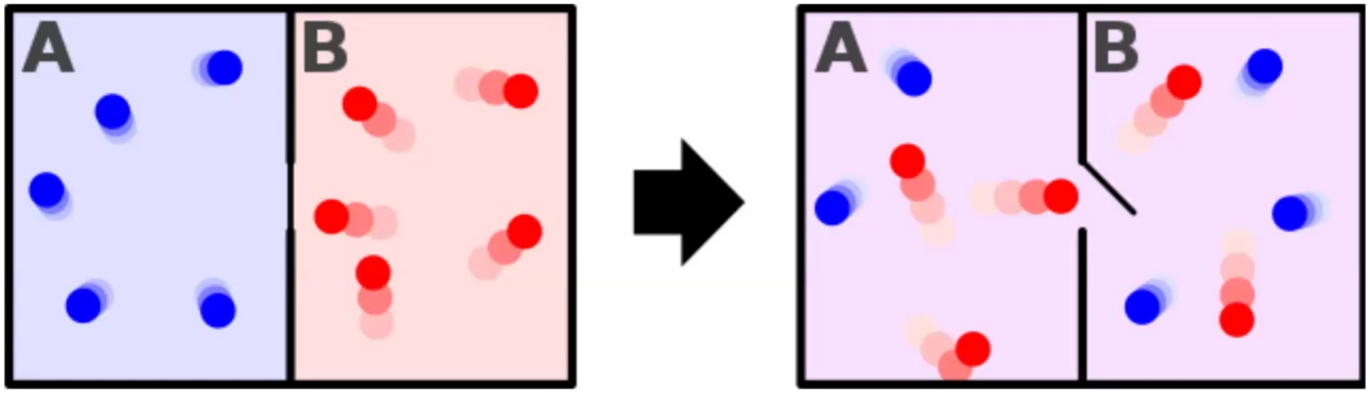
Fundamental question of statistical mechanics

Why does the assumption of being in equilibrium work so well in everyday experience?



Fundamental question of statistical mechanics

Why does the assumption of being in equilibrium work so well in everyday experience?



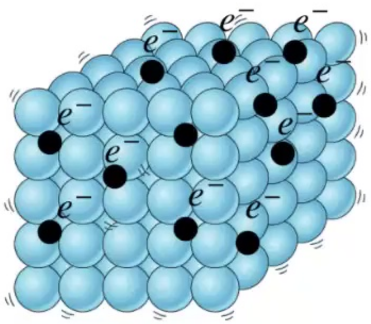
Good effective understanding, but deriving from microscopic (deterministic) dynamics is hard!

Conceptually the cleanest setting: isolated quantum many-body dynamics

Why now?

Until recently: mostly of academic interest

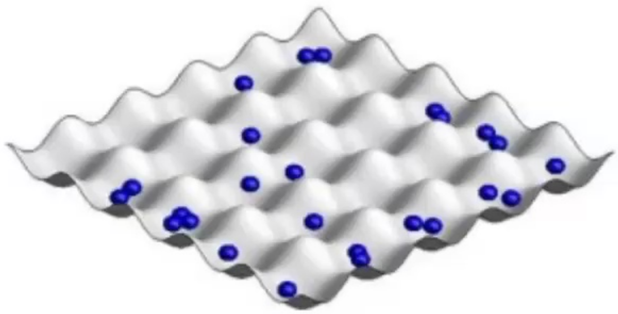
Traditional quantum many-body systems: electrons in solids



Strong interactions with phonons and impurities

Dynamical timescales: $\approx 10^{-15}s$

Last 20 years: synthetic quantum materials (e.g. cold atoms in optical lattices)



Good isolation from the environment (time-scales for heating $\approx 10^{-1}s$)

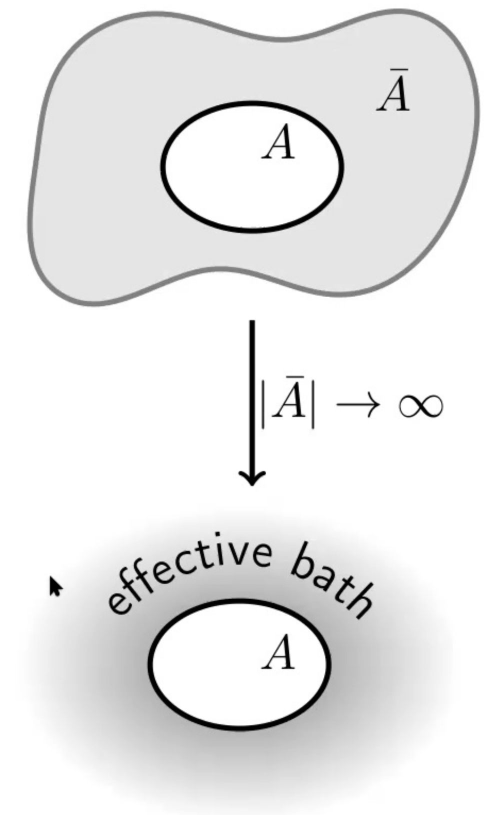
Dynamical timescales: $\approx 10^{-3}s$

Outline

The quantum quench protocol

The system is initialized in a state $|\Psi\rangle$ and let to evolve.

$$|\Psi(t)\rangle = \mathrm{e}^{-iHt}\,|\Psi\rangle$$



$|\Psi\rangle$ is not an eigenstate of H or a finite superposition of eigenstates.

How does the system reach equilibrium?

- equilibrium: mixed state ρ_{th}
- state $|\Psi(t)\rangle$ is pure

Thermalization is *local*: the system acts as its own bath.

Expectation values of *local* observables reach equilibrium values:

$$\lim_{t\to\infty}\lim_{|\bar{A}|\to\infty} \langle\psi(t)|\mathcal{O}_A|\psi(t)\rangle = \mathrm{tr}\left(\rho_{th}\mathcal{O}_A\right)$$

What is ρ_{th}?

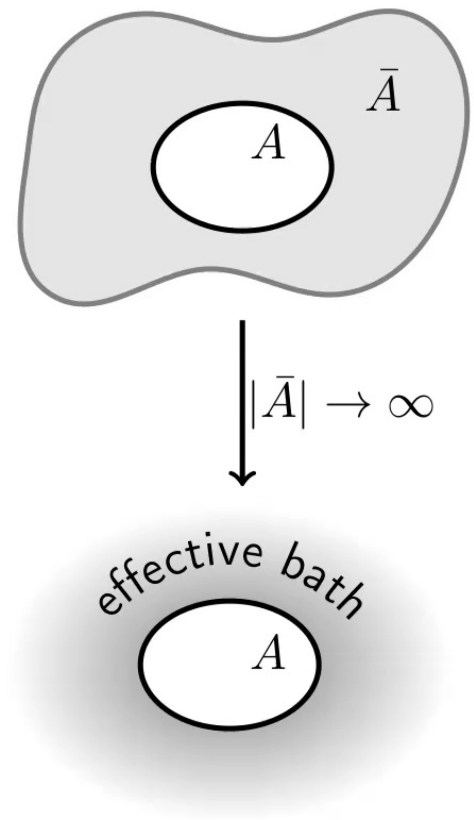
Dynamical constraints: $[H, Q] = 0 \Rightarrow \langle \Psi(t)|Q|\Psi(t)\rangle = \langle \Psi|Q|\Psi\rangle$

In the regime $\bar{A} \to \infty$, only $Q = \int \mathrm{d}\mathbf{r} q(\mathbf{r})$ with local densities matter

Constrained maximisation of thermodynamic entropy:

$$\rho_{th} = \frac{1}{Z}\mathrm{e}^{-\sum_j \beta_j Q^{(j)}}, \qquad \langle \Psi|q^{(j)}(\mathbf{r})|\Psi\rangle = \mathrm{tr}\left(q^{(j)}(\mathbf{r})\rho_{th}\right)$$

- "Generic" systems: only H is conserved
  $\Rightarrow$ Gibbs ensemble $\rho_{th} = \frac{1}{Z}\mathrm{e}^{-\beta H}$
- Integrable systems: infinitely many $Q^{(j)}$
  $\Rightarrow$ Generalised Gibbs ensemble
- Time-dependent Hamiltonian: no conservation laws
  $\Rightarrow$ infinite-temperature state $\rho_{th} = \frac{1}{Z}\mathbb{1}$

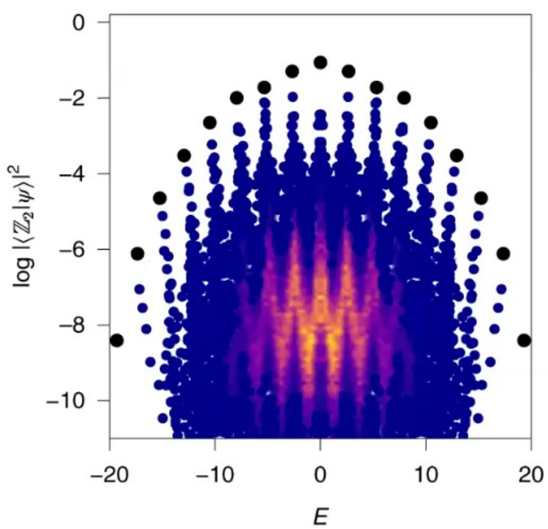


Going beyond the qualitative picture

What are the precise microscopic mechanisms and typical timescales?

Sometimes does not occur

Example: quantum many-body scars



M. Serbyn, D. A. Abanin, Z. Papić, Nat. Phys. 17, 675 (2021)

We need the description of dynamics on long but finite timescales.

Desired regime: $\lim_{t\to\infty} \lim_{|\bar{A}|\to\infty}$

Can be done for non-interacting models

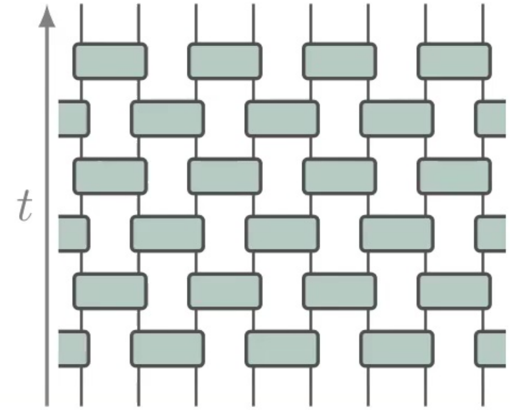
F. H. L. Essler, M. Fagotti, J. Stat. Mech. 2016, 064002 (2016)

Hard for numerics

- Exact diagonalization: Arbitrarily large times, small system sizes
- Tensor-network methods: Arbitrary system sizes, but short times

Solvable *interacting* examples

This calculation can be performed exactly for a class of solvable *chaotic* circuit models. L. Piroli et al., Phys. Rev. B **101**, 094304 (2020)



The subsystem dynamics is non-generic (too simple).

Bruno Bertini

(Oxford → Nottingham)

Lorenzo Piroli

(Munich → ENS Paris)

KK, B. Bertini, L. Piroli, PRL **126**, 160602 (2021):
A solvable circuit with "generic" dynamical properties

(see also arXiv:2104.04511, arXiv:2104.04513)

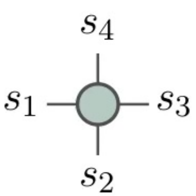
Quench dynamics in circuits

The effective-bath idea is very explicit in quantum circuits.

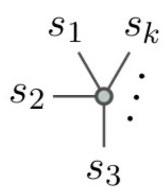
Time-evolution is given in discrete time-steps

$$|\psi(t+1)\rangle = U_{\mathrm{o}} U_{\mathrm{e}} |\psi(t)\rangle \,,$$

and defined by a tensor network.

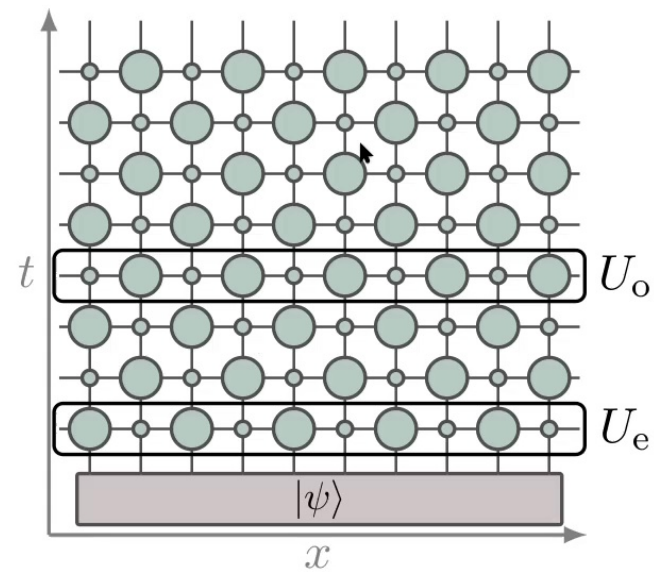


$$= f(s_1, s_2, s_3, s_4)$$



$$= \prod_{j=1}^{k-1} \delta_{s_j, s_{j+1}}$$

$s_j \in \{0, 1, \ldots, d-1\}$ $\quad$ $U_{\mathrm{e}},\ U_{\mathrm{o}}$ are unitary

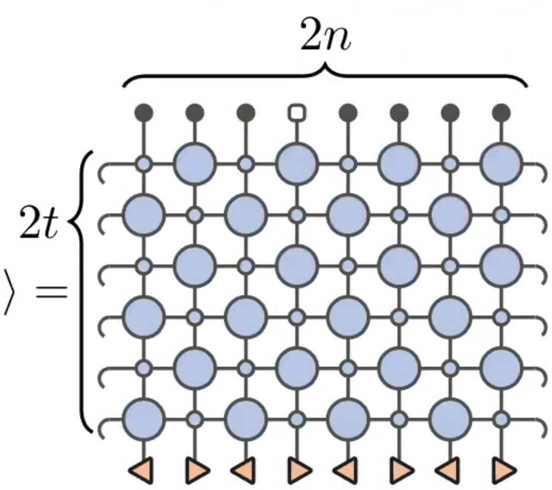


The upcoming discussion can be rephrased for a standard geometry with minimal changes.

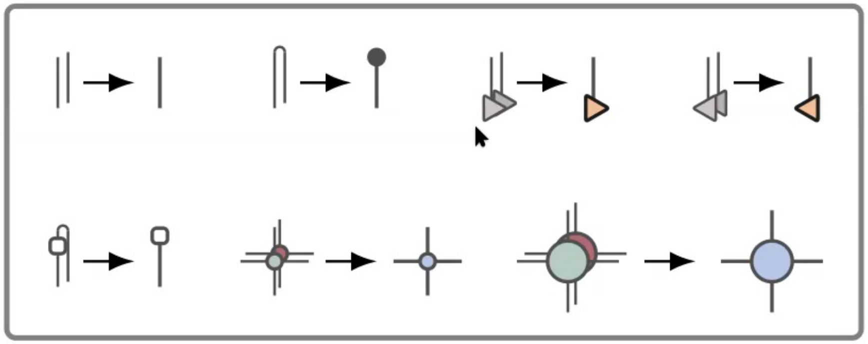
Space-time duality transformation

This tensor network can be equivalently understood as propagation in space

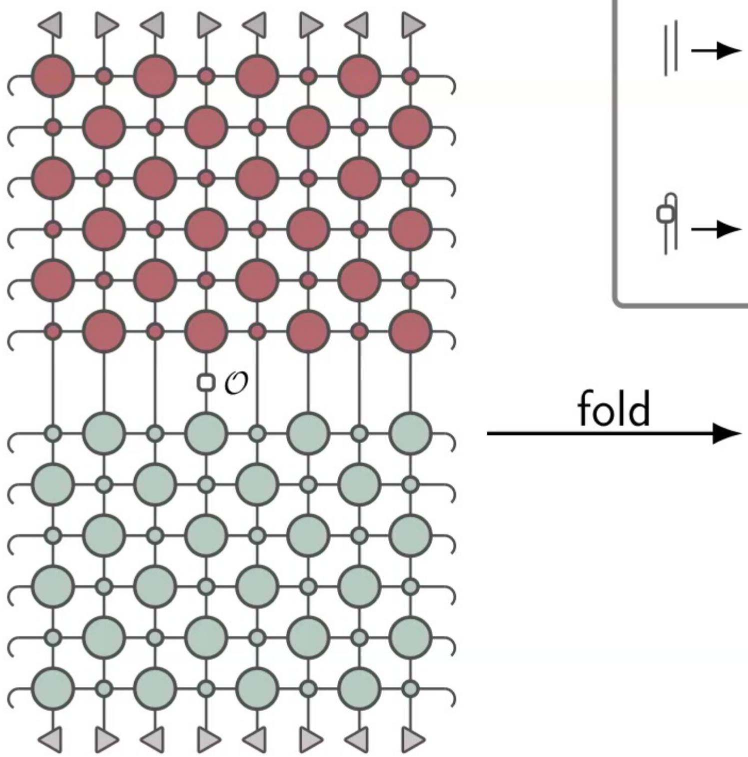
$$\langle \Psi(t)|\mathcal{O}|\Psi(t)\rangle =$$

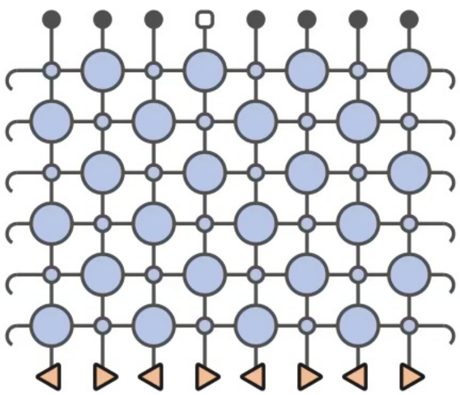


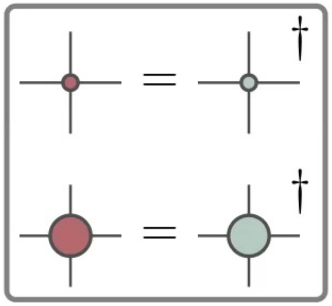
Expectation value of a local observable at time t



$$\langle \Psi(t)|\mathcal{O}|\Psi(t)\rangle =$$



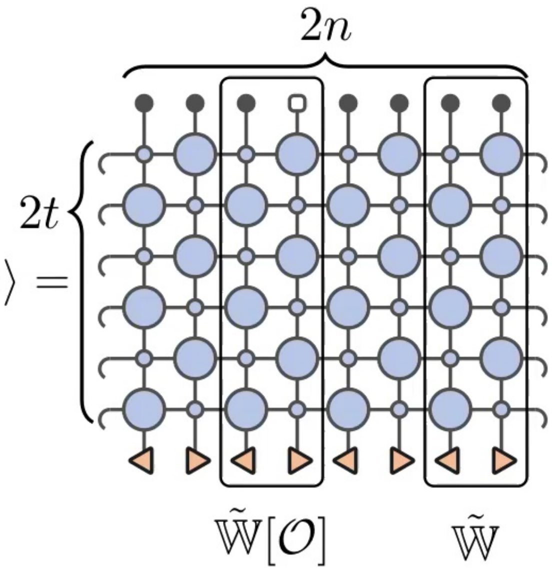




Space-time duality transformation

This tensor network can be equivalently understood as propagation in space: *transverse* transfer matrices $\tilde{\mathbb{W}}$, $\tilde{\mathbb{W}}[\mathcal{O}]$

$$\langle \Psi(t)|\mathcal{O}|\Psi(t)\rangle =$$



$$= \mathrm{tr}\left(\tilde{\mathbb{W}}^{n-1}\tilde{\mathbb{W}}[\mathcal{O}]\right) \xrightarrow{n\to\infty} \langle L|\tilde{\mathbb{W}}[\mathcal{O}]|R\rangle$$

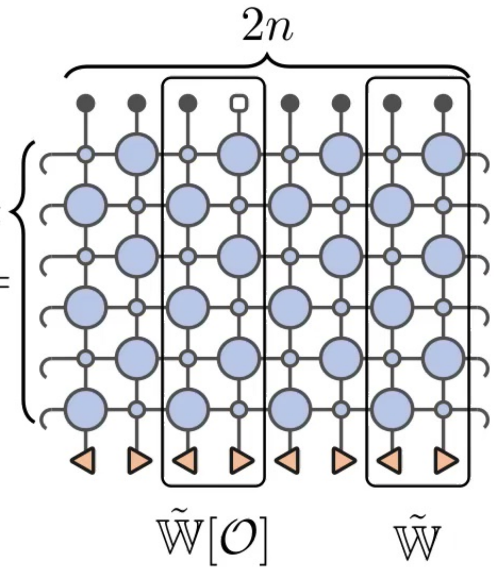
$$\langle L|\,\tilde{\mathbb{W}} = \langle L| \quad \tilde{\mathbb{W}}\,|R\rangle = |R\rangle$$
$$\langle L|R\rangle = 1$$

As $n \to \infty$, behaviour is determined by leading eigenvectors (*fixed points*) of $\mathbb{W}$.

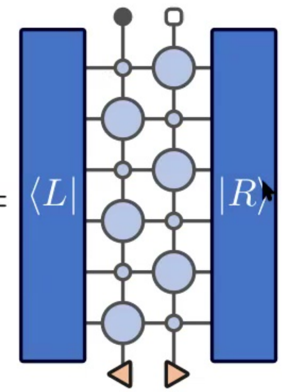
Space-time duality transformation

This tensor network can be equivalently understood as propagation in space: *transverse* transfer matrices $\tilde{\mathbb{W}}$, $\tilde{\mathbb{W}}[\mathcal{O}]$

$$\langle \Psi(t)|\mathcal{O}|\Psi(t)\rangle =$$



$$= \mathrm{tr}\left(\tilde{\mathbb{W}}^{n-1}\tilde{\mathbb{W}}[\mathcal{O}]\right) \xrightarrow{n\to\infty} \langle L|\tilde{\mathbb{W}}[\mathcal{O}]|R\rangle =$$



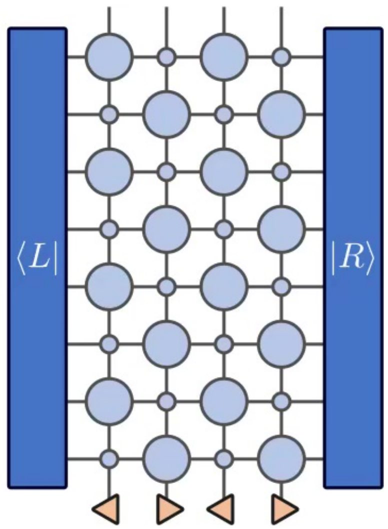
$$\langle L|\,\tilde{\mathbb{W}} = \langle L| \qquad \tilde{\mathbb{W}}\,|R\rangle = |R\rangle$$
$$\langle L|R\rangle = 1$$

As $n \to \infty$, behaviour is determined by leading eigenvectors (*fixed points*) of $\mathbb{W}$.

Effective bath

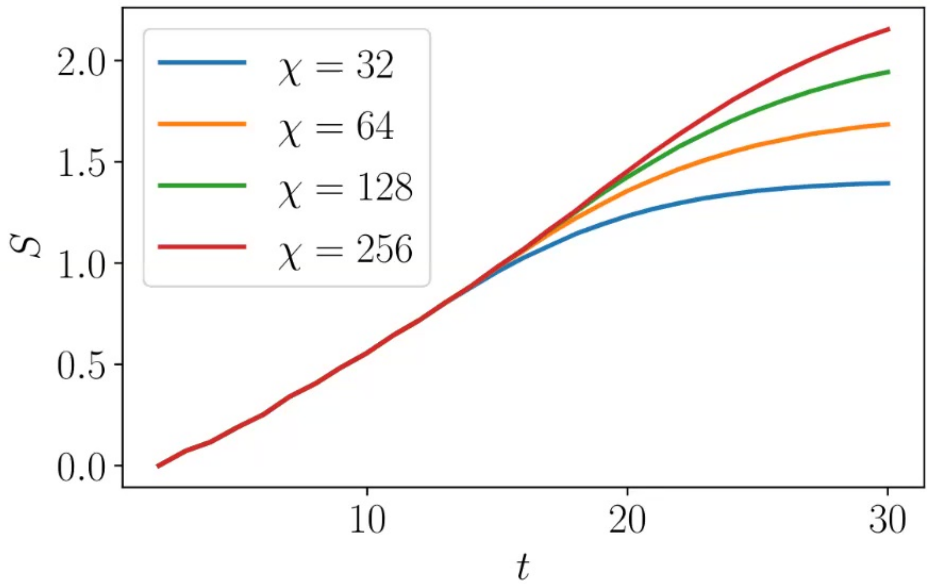
This works for any *finite* observable support!

$$\rho_{\mathcal{S}}(t) = \mathrm{tr}_{\bar{\mathcal{S}}} |\Psi(t)\rangle\langle\Psi(t)| =$$



Problem: complexity of $\langle L|$, $|R\rangle$ generically grows exponentially with t.

A. Lerose, M. Sonner, D. A. Abanin, Phys. Rev. X 11, 021040 (2021)



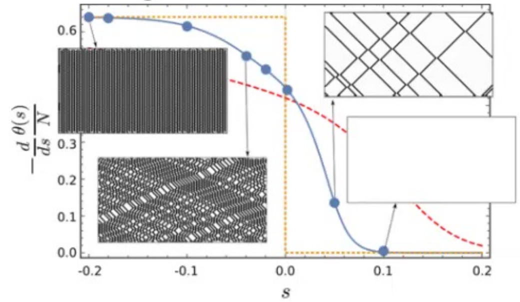
The effect of the rest of the system on $\mathcal{S}$ is fully determined by fixed points $\langle L|$, $|R\rangle$.

Our example: fixed points are MPSs with finite bond-dimension.

Rule 54 reversible cellular automaton

Possibly the simplest *interacting* model — A. Bobenko et al., Commun. Math. Phys. 158, 127–134 (1993)

Exact large-deviation statistics

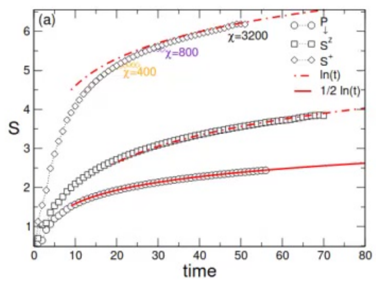


B. Buča et al., Phys. Rev. E 100, 020103(R) (2019)

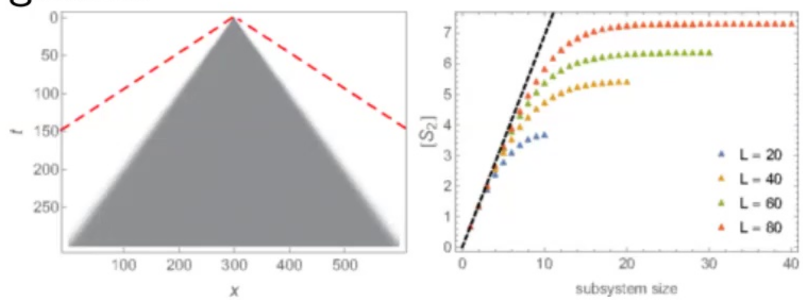
Recent review:

B. Buča, KK, T. Prosen, J. Stat. Mech. 2021, 074001 (2021)

Operator growth



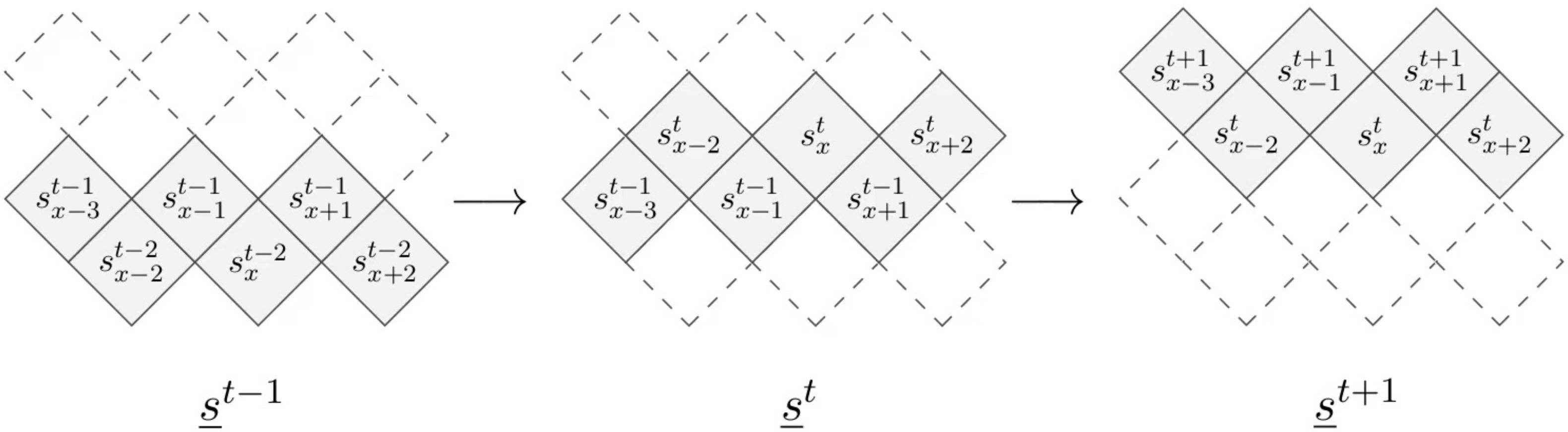
V. Alba, Phys. Rev. B 104, 094410 (2021)



S. Gopalakrishnan, Phys. Rev. B 98, 060302(R) (2018)

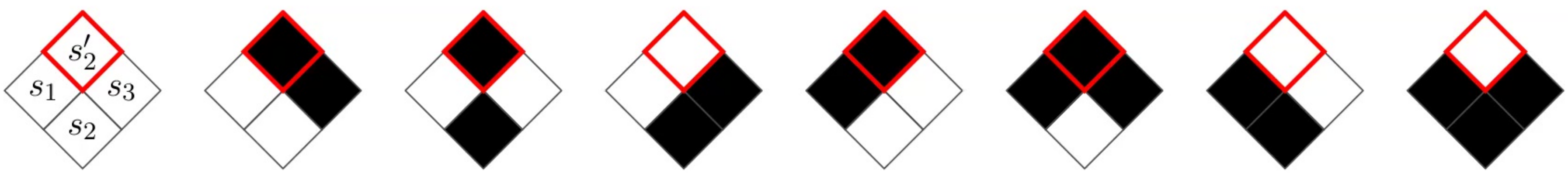
Definition of dynamics

1-dim lattice of binary variables, with staggered time evolution:



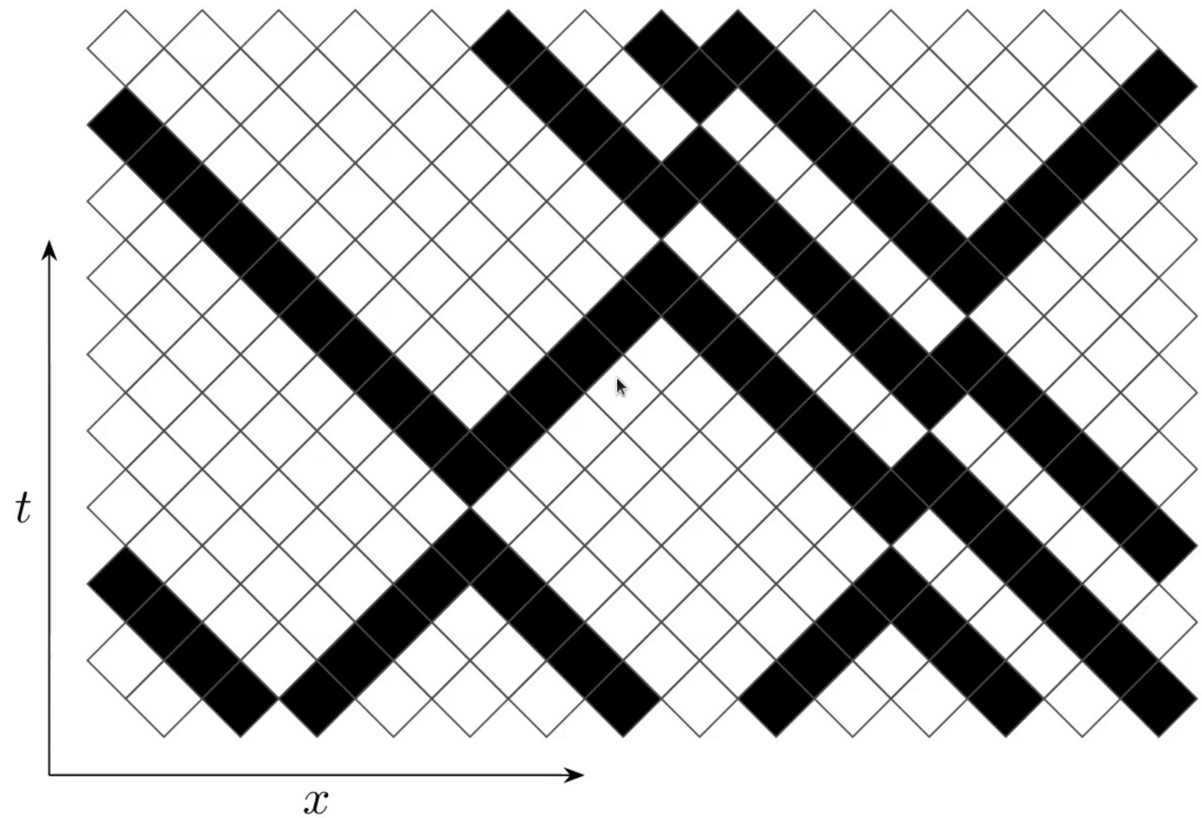
Local time evolution maps:

$$s_2' = \chi(s_1, s_2, s_3) \equiv s_1 + s_2 + s_3 + s_1 s_3 \pmod 2$$



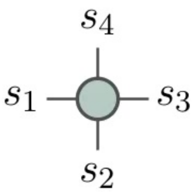
Definition of dynamics

Solitons move with fixed velocities ± 1 and obtain a delay while scattering.

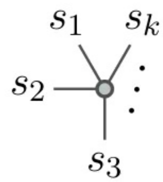


Tensor-network formulation

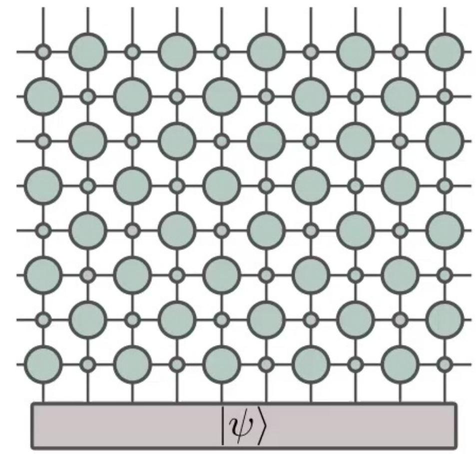
Quantum model: computational-basis states obey CA rules.



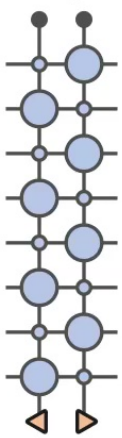
$= \delta_{s_4,\chi(s_1,s_2,s_3)}$



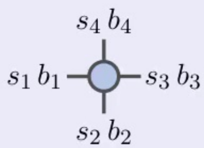
$= \prod_{j=1}^{k-1} \delta_{s_j,s_{j+1}}$



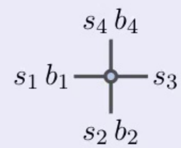
Goal: fixed points of

$\tilde{\mathbb{W}} =$ 

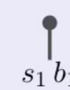
Blue tensors are defined on the *doubled* space $(s_j, b_j \in \{0,1\})$



$= \delta_{s_4,\chi(s_1,s_2,s_3)}\delta_{b_4,\chi(b_1,b_2,b_3)}$



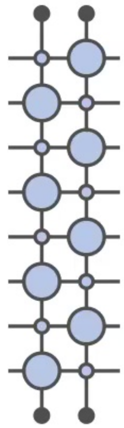
$= \prod_{j=1}^{3} \delta_{s_j,s_{j+1}}\delta_{b_j,b_{j+1}}$



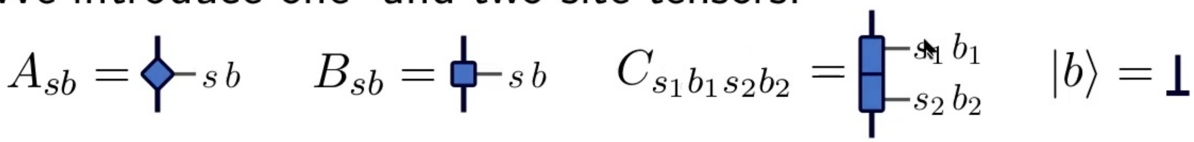
$= \delta_{s_1,b_1}$

Step 1: "Infinite-temperature" transfer matrix

Let us consider the transfer matrix $\tilde{\mathbb{W}}_\infty$ corresponding to the maximum-entropy state.

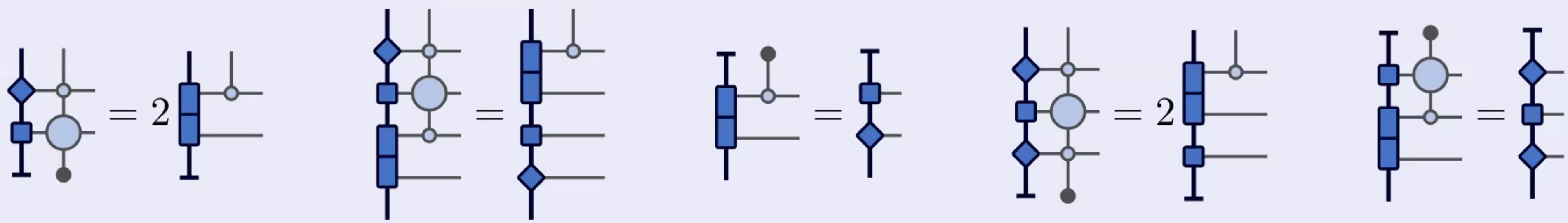
$\tilde{\mathbb{W}}_\infty = \frac{1}{4}$ 

We introduce one- and two-site tensors:



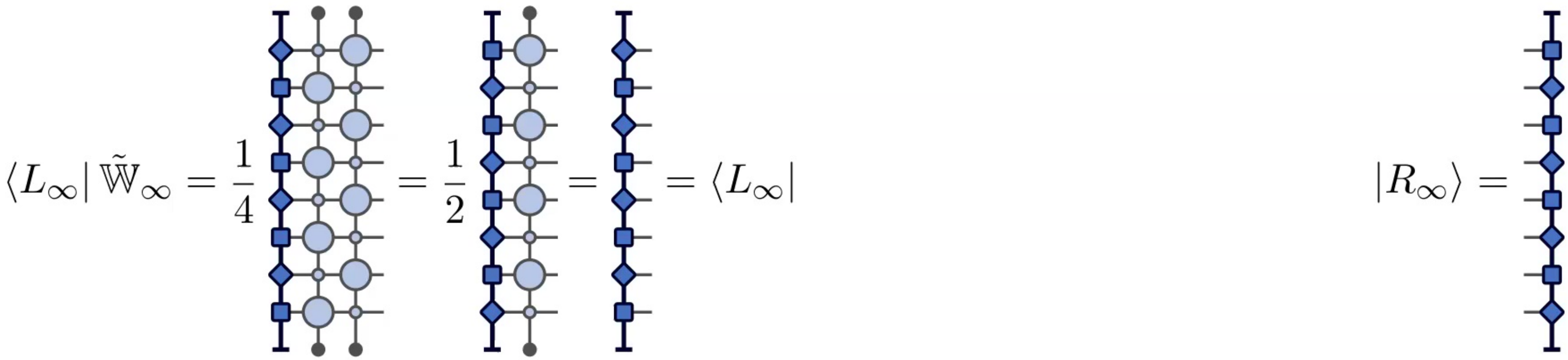
If the set of algebraic relations is fulfilled, we can construct fixed points $\langle L_\infty|$ and $|R_\infty\rangle$.

Algebraic relations

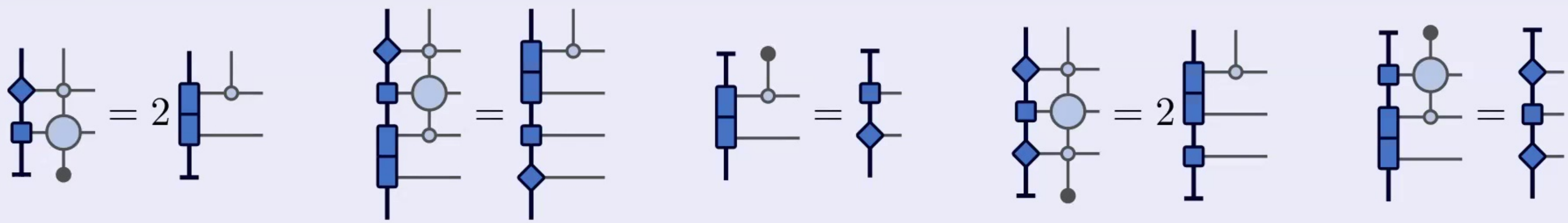


Step 1: "Infinite-temperature" transfer matrix

$\langle L_\infty | \tilde{\mathbb{W}}_\infty = \frac{1}{4} \ \cdots = \frac{1}{2} \ \cdots = \ \cdots = \langle L_\infty |$ $\qquad |R_\infty\rangle = \ \cdots$

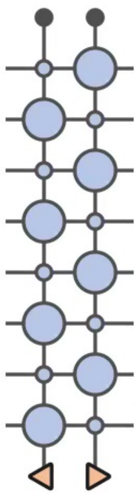


Algebraic relations

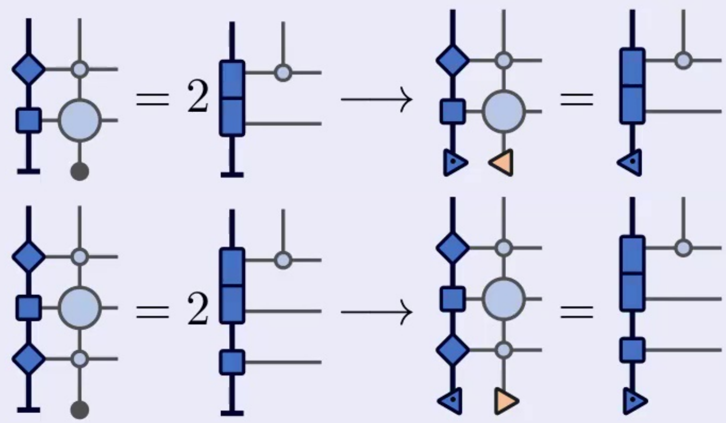


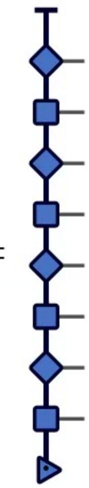
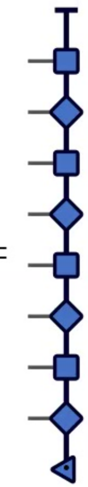
Step 2: Determine compatible *pure* initial states

Ansatz: $\langle L|$ and $|R\rangle$ differ from $\langle L_\infty|$ and $|R_\infty\rangle$ only at the very bottom.

$\tilde{\mathbb{W}} =$ 

New boundary algebraic relations:



$\langle L| =$  $|R\rangle =$ 

For a family of *solvable* initial states we can describe the full subsystem dynamics!

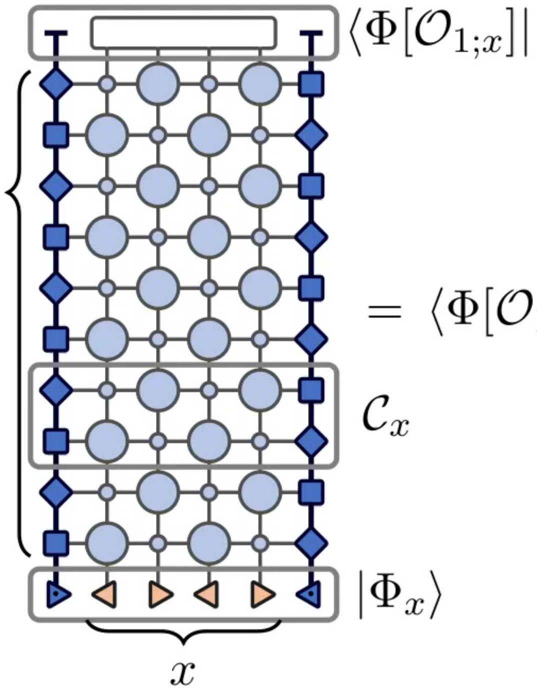
Solution:

 $= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ e^{i\varphi}\end{bmatrix}$  $= \begin{bmatrix}1\\ 0\end{bmatrix}$

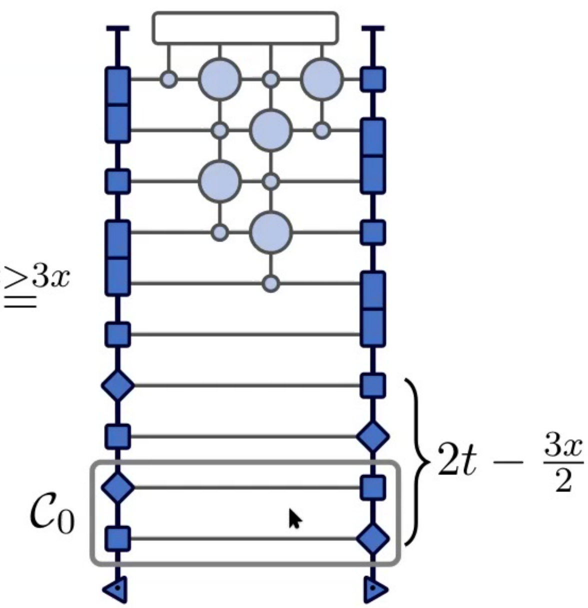
Thermalization dynamics

Expectation values are exponentially costly in support x rather than time t.

$\langle \Psi(t) | \mathcal{O}_{1;x} | \Psi(t) \rangle =$



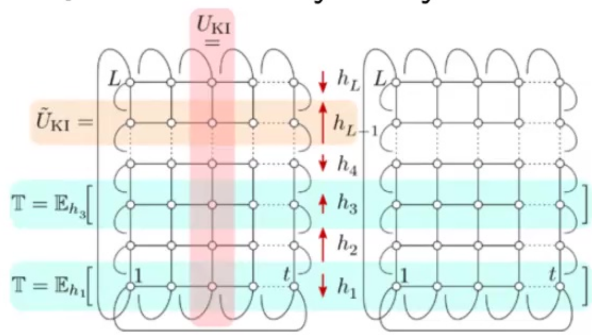
$= \langle \Phi[\mathcal{O}_{1;x}] | \mathcal{C}_x^t | \Phi_x \rangle \overset{4t \geq 3x}{=}$



Excpectation values of *all* local observables decay *exponentially*:
$\langle \Psi(t) | \mathcal{O}_{1;x} | \Psi(t) \rangle - \mathrm{tr}\left(\rho_\infty \mathcal{O}_{1;x}\right) \sim \mathrm{e}^{-t/\tau}, \qquad \tau^{-1} = 2 \log 2$

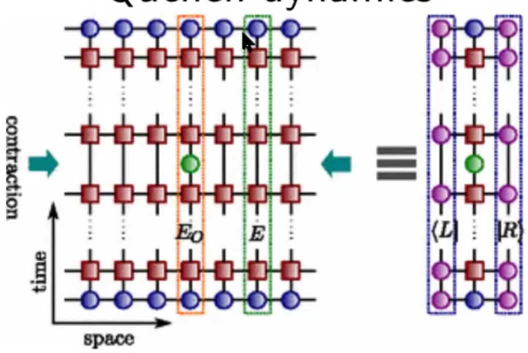
Space-time duality is more general

Quantum many-body chaos



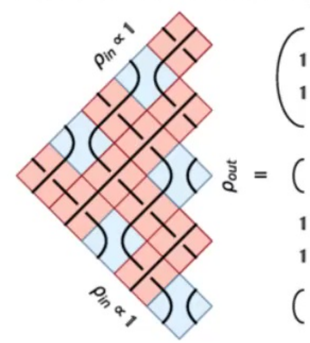
B. Bertini, P. Kos, T. Prosen, Phys. Rev. Lett. 121, 264101 (2018)

Quench dynamics



M. C. Bañuls et al., Phys. Rev. Lett. 102, 240603 (2009)

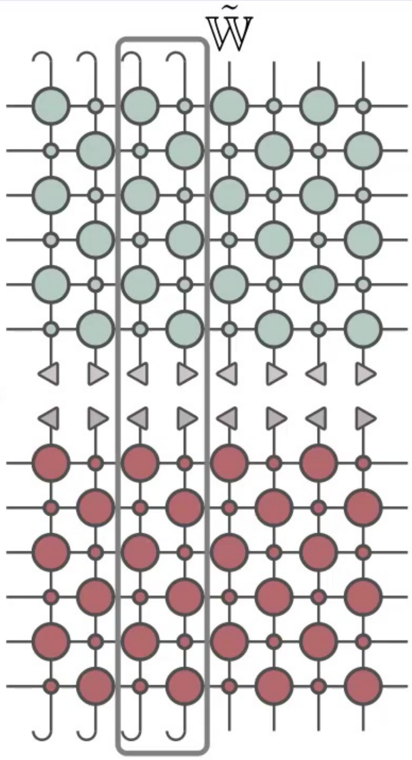
Measurement-induced dynamics



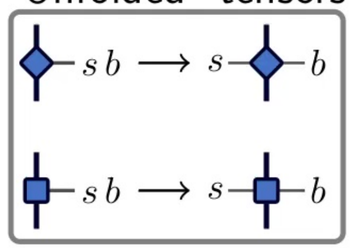
M. Ippoliti, V. Khemani, Phys. Rev. Lett. 126, 060501 (2021)

Growth of entanglement entropy

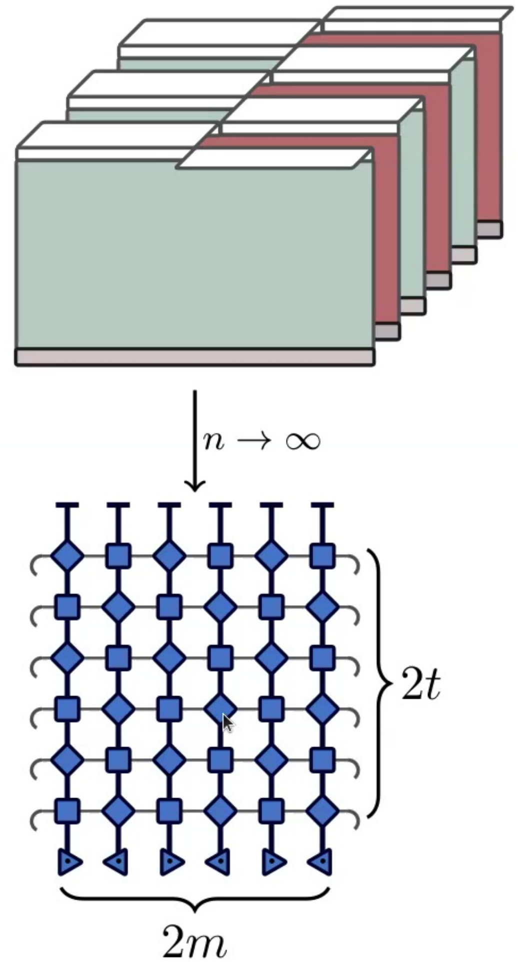
$\rho_H(t) =$



"Unfolded" tensors



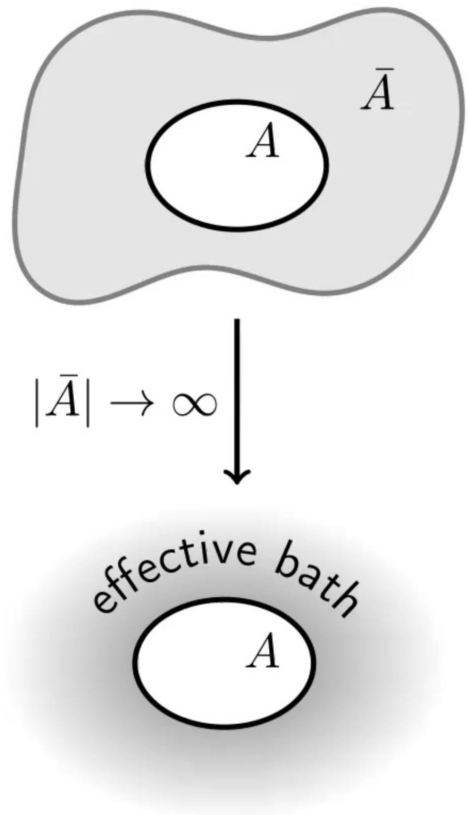
$\mathrm{tr}\big(\rho_H^3(t)\big) =$



Rényi-m entanglement entropy:

$$S_m(t) = \frac{1}{1-m} \log \mathrm{tr}(\rho_H^m(t))$$

Summary and outlook



- Local quantum circuits are very convenient to address questions regarding relaxation.
  A. Lerose, M. Sonner, D. A. Abanin, Phys. Rev. X 11, 021040 (2021)
- Our example: simple MPS for the effective bath.

Open questions:

- Extensions to richer stationary states $\rho_{th} = \frac{1}{Z} - \sum_j \beta_j Q^{(j)}$
  - Possible for $\rho_{th} = \frac{1}{Z} e^{-\beta_+ N_+ - \beta_- N_-}$
    KK, B. Bertini, arXiv:2104.04511 (2021)
  - What about more conservation laws?
- Generalisations to other models — Not hopeless!
  J. W. P. Wilkinson et al., Phys. Rev. E 102, 062107 (2020)
  T. Iadecola, S. Vijay, Phys. Rev. B 102, 180302 (2020)
- Existence of approximate solutions to algebraic relations

Tridactyl 1.21.1

Tridactyl has to override your new tab page due to WebExtension limitations. You can learn how to change it at the bottom of the page, otherwise please read on for some tips and tricks.

- You can view the main help page by typing `:help`, and access the tutorial with `:tutor`. There's a wiki too - feel free to add to it. You may find `:apropos` useful for finding relevant settings and commands.
- You can view your current configuration with `:viewconfig`.
- Tridactyl funding ••: donate via GitHub sponsors here. All GitHub and Patreon donors get a nice little newsletter once every few months; people who donate at least 10USD a month get a "tips & tricks" newsletter roughly once a month (see an example here). You can also donate via PayPal, but they charge fairly high fees and you won't get any newsletters. Donations currently go towards ensuring that bovine3dom can afford to work one day a week on Tridactyl. Previously the donations have funded an in-person developer retreat.
- If Tridactyl breaks a website or is broken by a website, trying the steps in the troubleshooting guide might help.
- You can contact the developers, other users and contributors for support or whatever on Matrix, Gitter, or IRC.
- If you're enjoying Tridactyl (or not), please leave a review on addons.mozilla.org.

Changelog

Highlighted features:

- `f`/`F` — enter the "hint mode" to select a link to follow. `F` to open it in a

Appendix C: MPS representation of the leading eigenvectors

Here we report an explicit representation for the 3×3 matrices A_{sr}, B_{sr}, and $C_{s_1 r_1 s_2 r_2}$ fulfilling relations (6a)–(6b) and (9). The non identically vanishing matrices in the set are given by

$$
\begin{aligned}
&A_{00} = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}, \quad A_{01} = A_{10} = \frac{1}{2}\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_{11} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
&B_{00} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_{11} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
&C_{0001} = C_{0010} = \frac{1}{4}\begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad C_{0101} = C_{1010} = \frac{1}{4}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad C_{0110} = C_{1001} = \frac{1}{4}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \\
&C_{1101} = C_{1110} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C_{0000} = \frac{1}{4}\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}, \quad C_{0011} = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
&C_{1100} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad C_{1111} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\end{aligned}
\tag{C1}
$$

while the appropriate boundary vectors $|b\rangle$, $|v_1\rangle$, $|w_1\rangle$ are

$$
|b\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad |v_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \qquad |w_1\rangle = \sqrt{2}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{C2}
$$

Note that this choice of magnitude of boundary vectors implies the normalisation of leading eigenvectors of the transverse transfer matrix

$$
\langle L_\infty | R_\infty \rangle = \langle b | b \rangle^2 = 1, \qquad \langle L | R \rangle = \langle b | v \rangle \langle b | w \rangle = 1. \tag{C3}
$$

The latter equations follow from Eq. (13) and $\langle b| \otimes \langle b| \mathcal{C}_0 = \langle b| \otimes \langle b|$.