Title: Twistor action for GR - Atul Sharma, University of Oxford

Speakers:

Series: Quantum Gravity

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Abstract: It has been a long-standing dream of twistor theorists to understand gravity without ever talking about gravitons in space-time. To this end, I will describe the recent discovery of a twistor action formulation of perturbative general relativity. This takes the form of a theory governing complex structure deformations on twistor space. It reduces to Plebanski's formulation of GR on performing the Penrose transform to space-time. Some promising applications include finding on-shell recursion relations like MHV rules for graviton scattering amplitudes, studying the quantum integrability of self-dual GR, etc.

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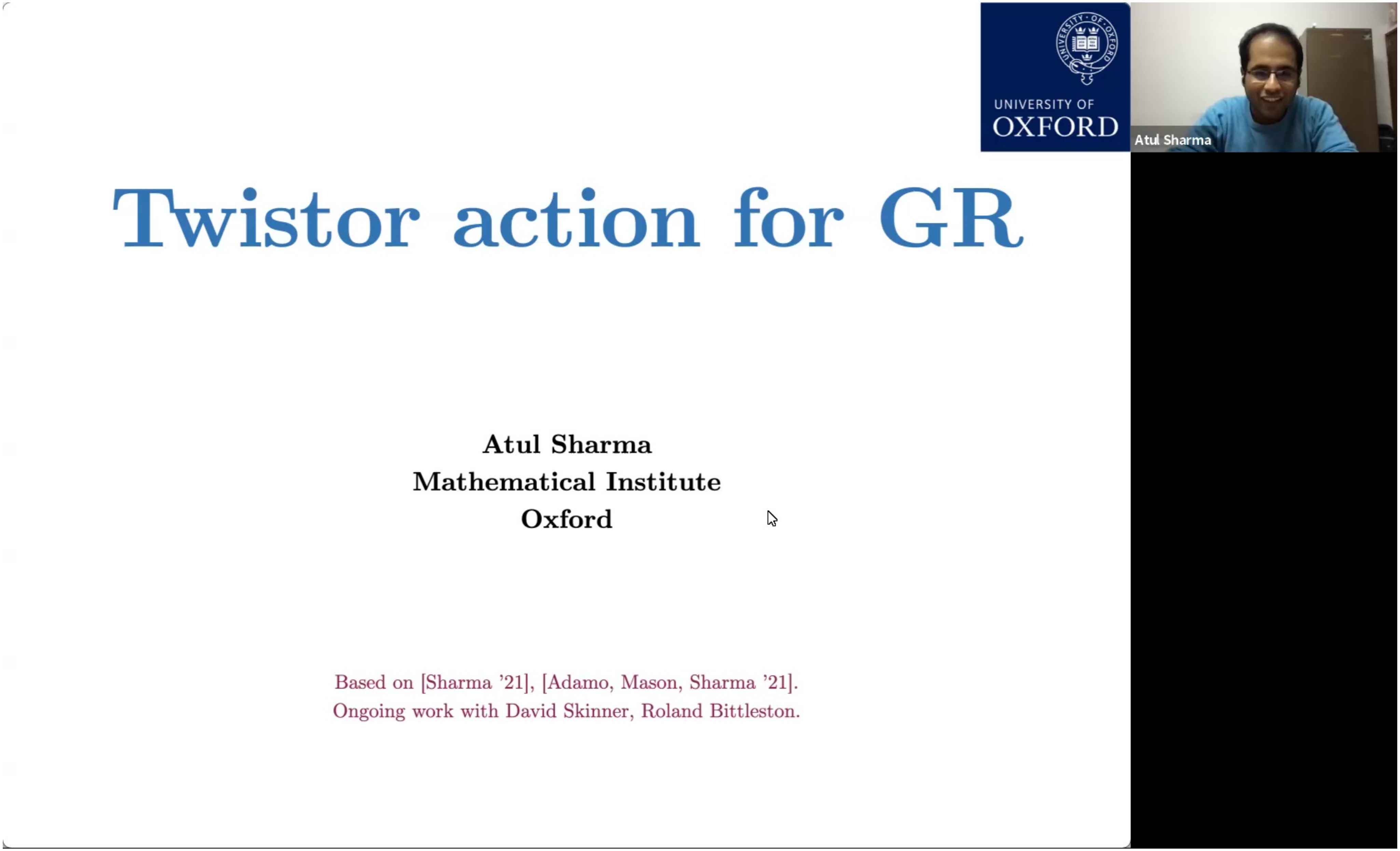

Twistor action for GR

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Based on [Sharma '21], [Adamo, Mason, Sharma '21].
Ongoing work with David Skinner, Roland Bittleston.

MHV amplitudes

- Gluons

$$\frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\cdots\,\langle n\,1\rangle}$$

[Parke, Taylor '86]

- Gravitons

$$\frac{\langle 1\,2\rangle^8}{\langle 1\,2\rangle^2\,\langle 2\,r\rangle^2\,\langle r\,1\rangle^2}\;\det(\mathbb{H}^{12r}_{12r})$$

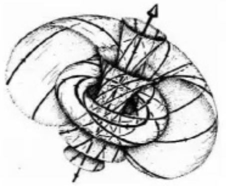
[Hodges '12]

$$\mathbb{H}_{ij} = \begin{cases} \frac{[i\,j]}{\langle i\,j\rangle}\,, & i \neq j \\ -\sum_{k\neq i} \frac{[i\,k]\,\langle 1\,k\rangle\,\langle 2\,k\rangle}{\langle i\,k\rangle\,\langle 1\,i\rangle\,\langle 2\,i\rangle}\,, & i = j \end{cases}$$

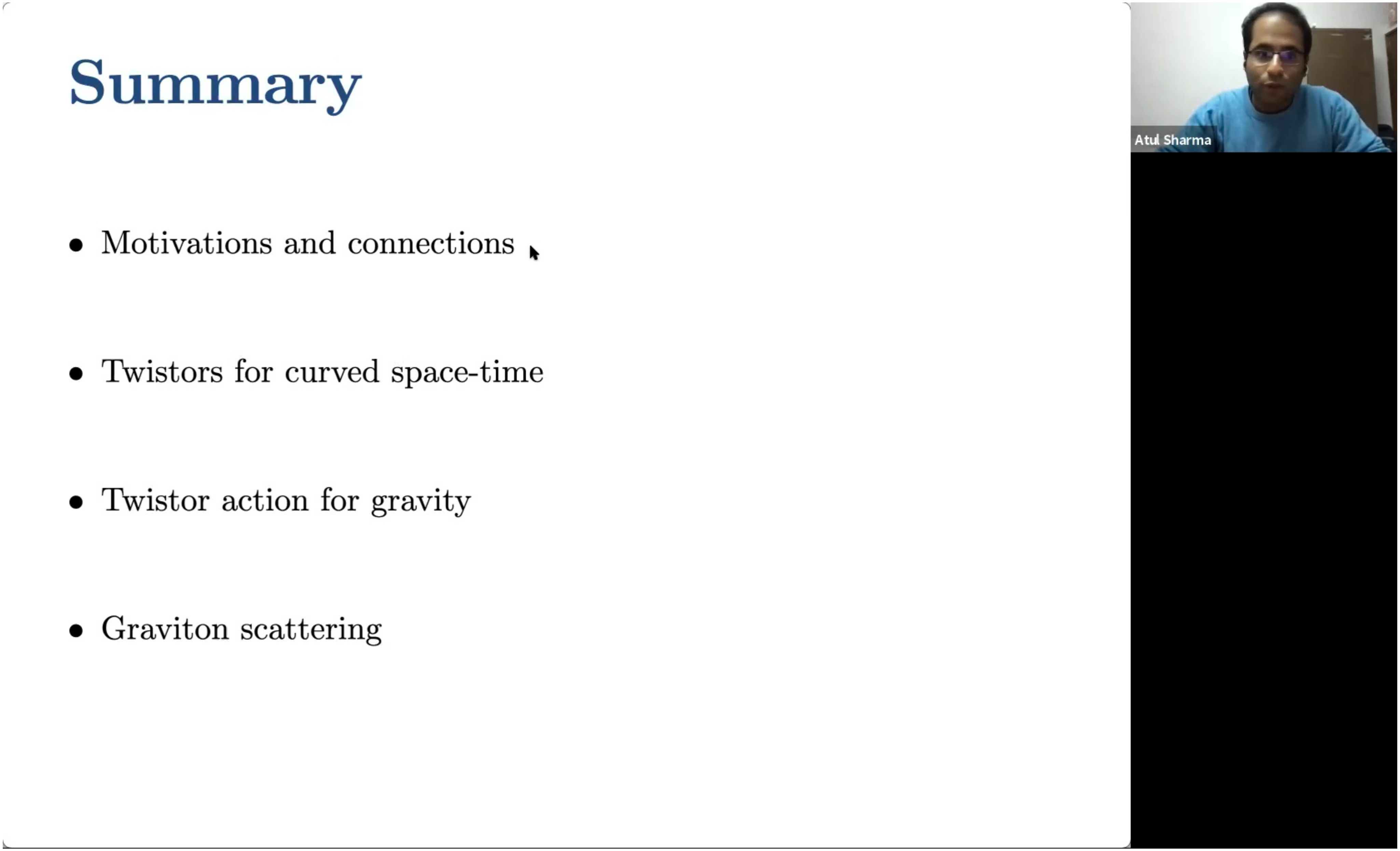
- With great simplicity comes great duality.

Twistor strings and actions

Twistor actions provide target space descriptions of twistor strings.

	**Gauge theory + Conformal gravity**	**General relativity**
Twistor strings	Witten, Berkovits, Roiban, Spradlin, Volovich,...	Skinner, Cachazo, Mason, Geyer,...
Twistor actions	Mason, Skinner, Boels, Bullimore, Adamo,..., Costello, Bittleston,...	**Today!**

Constructive origins of Parke-Taylor formula



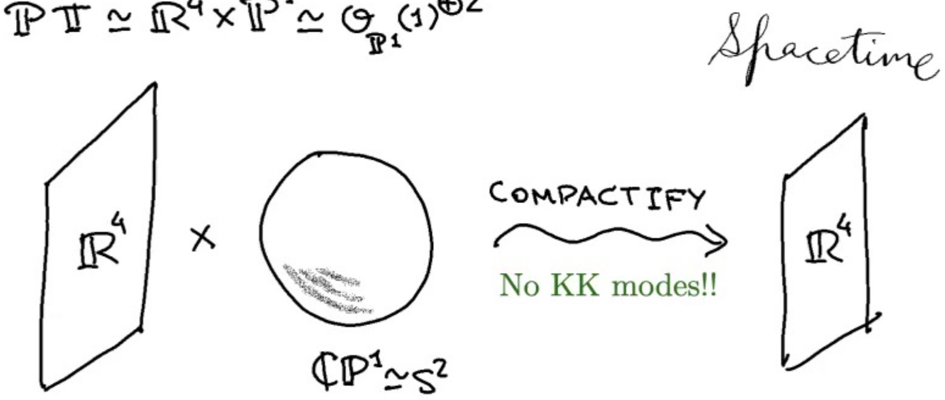
Summary

- Motivations and connections
- Twistors for curved space-time
- Twistor action for gravity
- Graviton scattering

Motivations

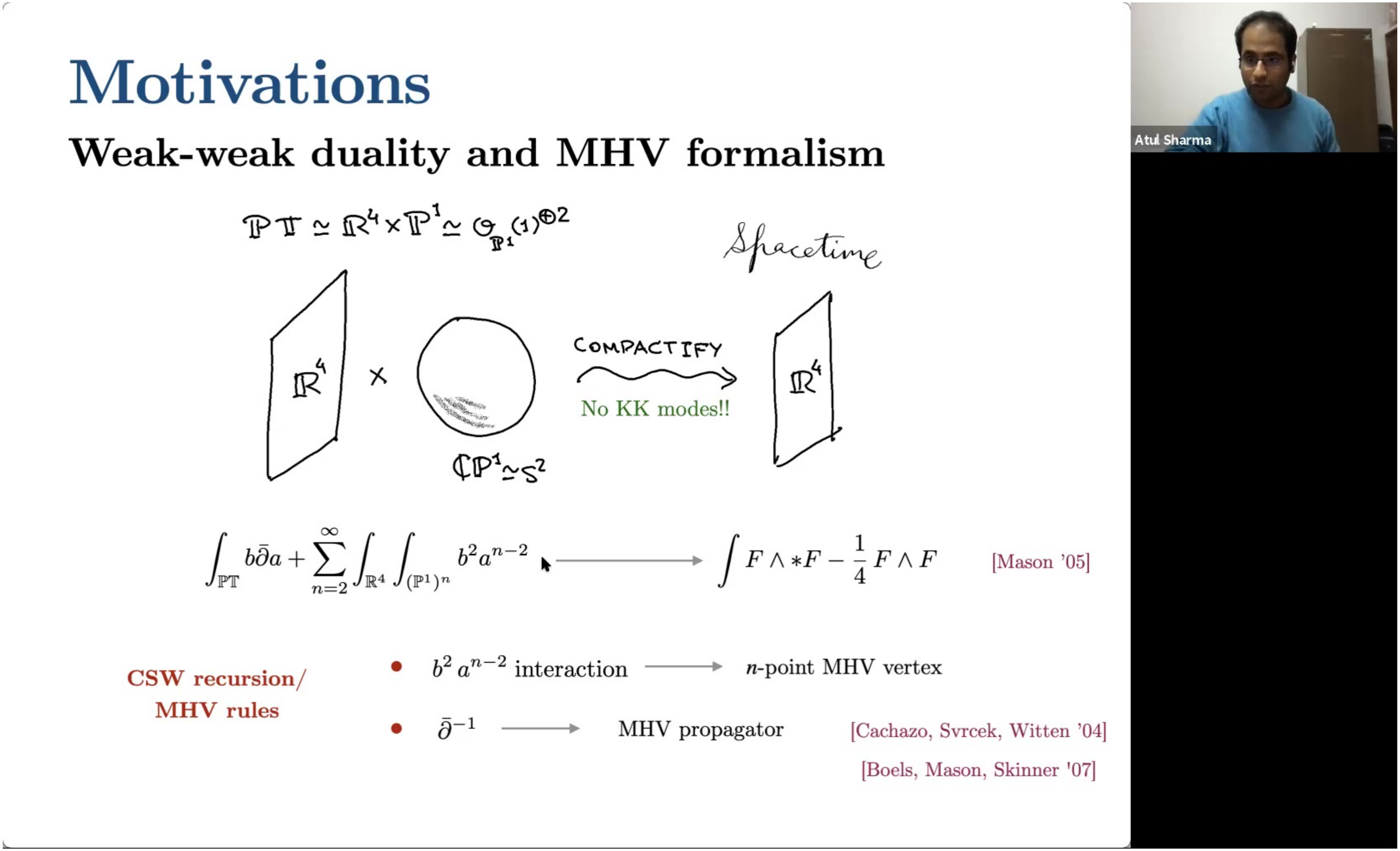
Weak-weak duality and MHV formalism

$\mathbb{PT} \simeq \mathbb{R}^4 \times \mathbb{P}^1 \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$



$$\int_{\mathbb{PT}} b\bar{\partial}a + \sum_{n=2}^{\infty} \int_{\mathbb{R}^4} \int_{(\mathbb{P}^1)^n} b^2 a^{n-2} \longrightarrow \int F \wedge *F - \frac{1}{4} F \wedge F$$

[Mason '05]



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[Mason '05]

CSW recursion/ MHV rules

- $b^2\, a^{n-2}$ interaction $\longrightarrow$ n-point MHV vertex
- $\bar{\partial}^{-1}$ $\longrightarrow$ MHV propagator

[Cachazo, Svrcek, Witten '04]

[Boels, Mason, Skinner '07]

Broader connections

Integrable systems and celestial holography

- Holomorphic Chern-Simons twistor actions for self-dual Yang-Mills.

[Bittleston, Skinner '20] [Costello]

$$\int_{\mathbb{PT}} \frac{\Omega}{\langle\lambda\,1\rangle^2\,\langle\lambda\,2\rangle^2} \wedge \operatorname{tr}\left(a\wedge\bar{\partial}a+\frac{2}{3}\,a\wedge a\wedge a\right)$$

$$\downarrow$$

Exactly reduces to SD YM action on compactification.
Related to heterotic $N=2$ strings.

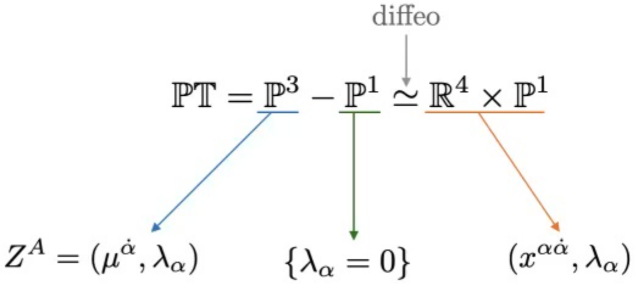
- Costello applied *twisted holography* to relate twistor actions to *celestial holography*.

[Costello, Strings '21]

Twistor space for $\mathbb{R}^4$

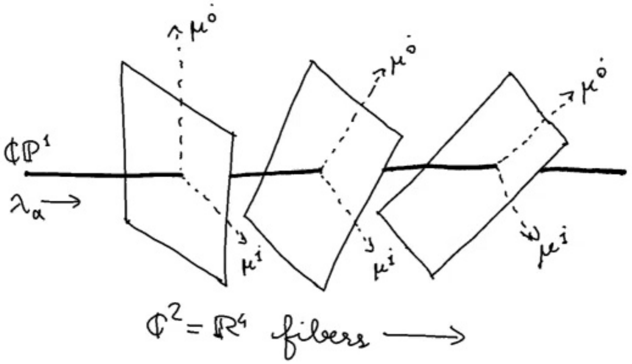
[Woodhouse '85]

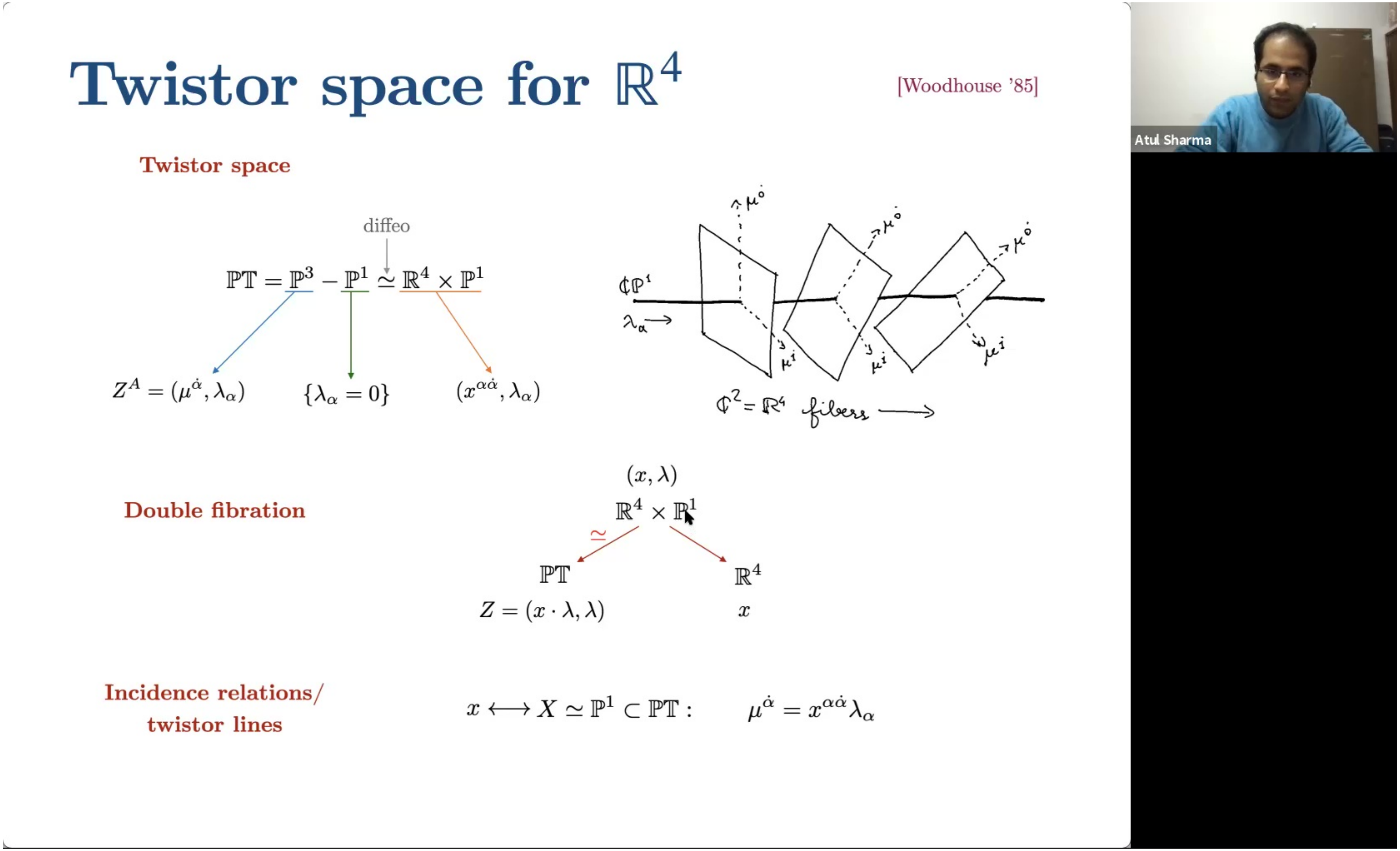
Twistor space



$$\mathbb{PT} = \underline{\mathbb{P}^3} - \underline{\mathbb{P}^1} \overset{\text{diffeo}}{\simeq} \underline{\mathbb{R}^4 \times \mathbb{P}^1}$$

$Z^A = (\mu^{\dot\alpha}, \lambda_\alpha)$ $\{\lambda_\alpha = 0\}$ $(x^{\alpha\dot\alpha}, \lambda_\alpha)$





Twistor space for $\mathbb{R}^4$

[Woodhouse '85]

Twistor space

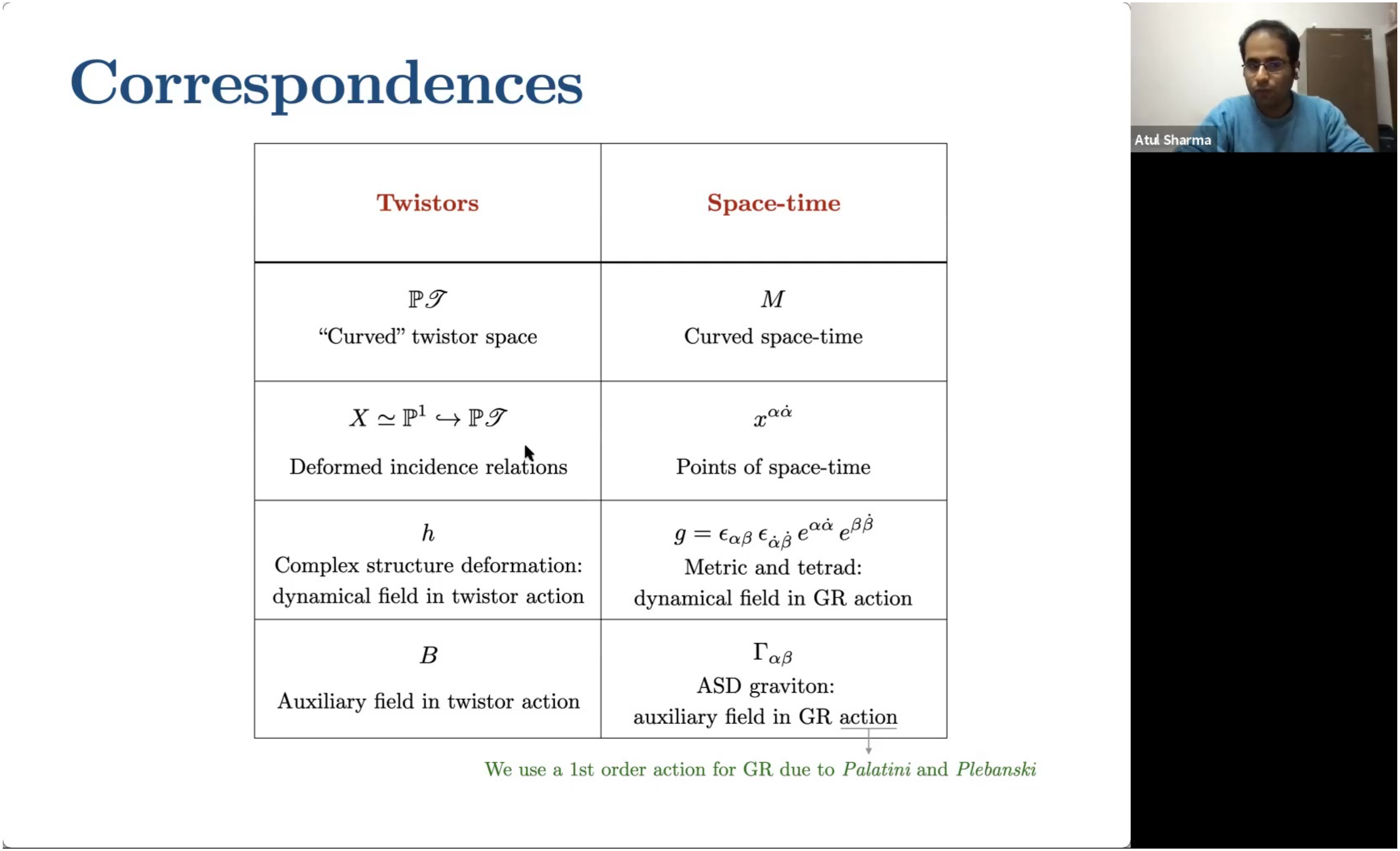
$$\mathbb{PT} = \underbrace{\mathbb{P}^3}_{Z^A = (\mu^{\dot\alpha}, \lambda_\alpha)} - \underbrace{\mathbb{P}^1}_{\{\lambda_\alpha = 0\}} \overset{\text{diffeo}}{\simeq} \underbrace{\mathbb{R}^4 \times \mathbb{P}^1}_{(x^{\alpha\dot\alpha}, \lambda_\alpha)}$$

Double fibration

$$\begin{tikzcd} & \overset{(x,\lambda)}{\mathbb{R}^4 \times \mathbb{P}^1} \arrow[dl, "\simeq"'] \arrow[dr] & \\ \underset{Z = (x\cdot\lambda, \lambda)}{\mathbb{PT}} & & \underset{x}{\mathbb{R}^4} \end{tikzcd}$$

Incidence relations/ twistor lines

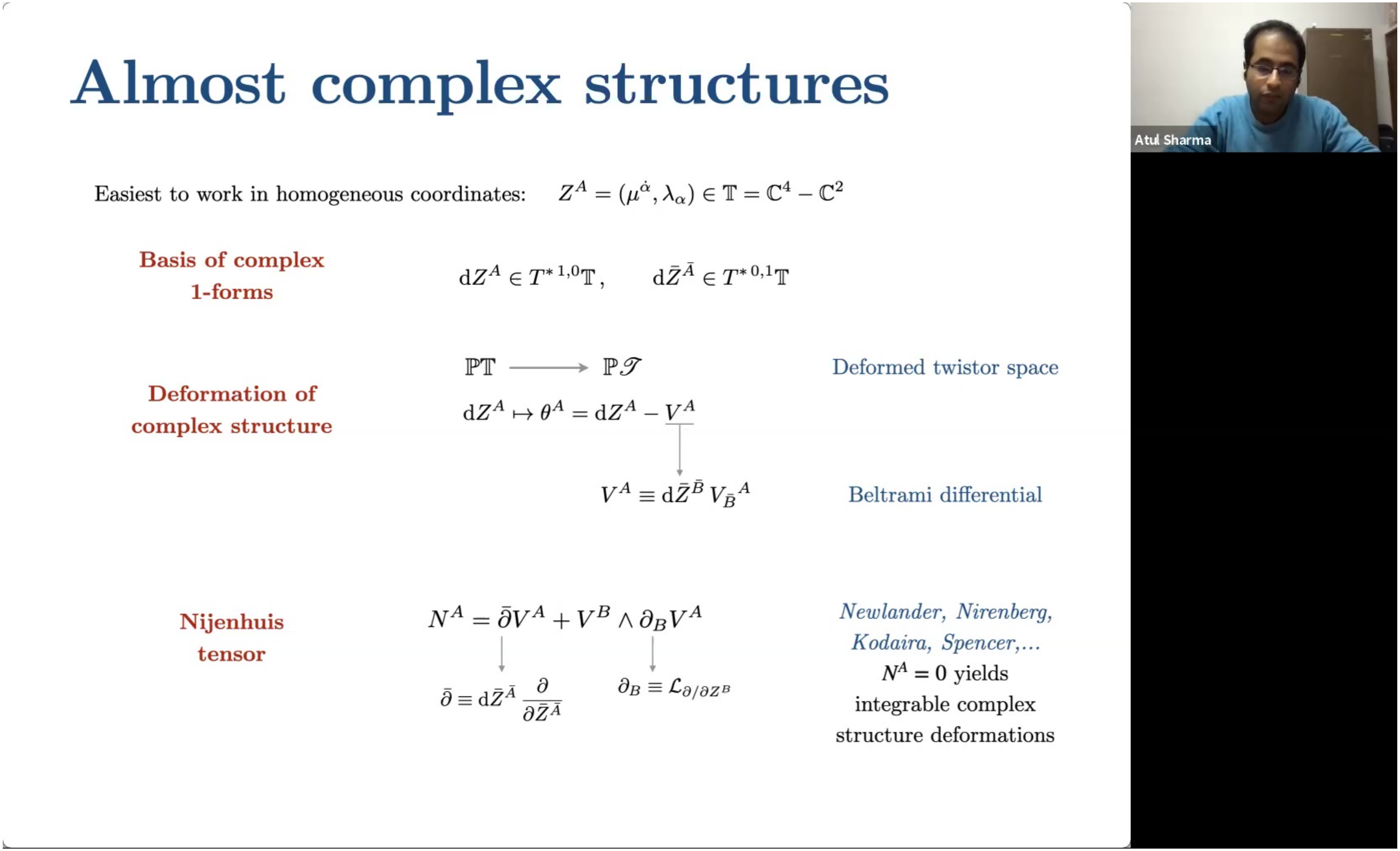
$$x \longleftrightarrow X \simeq \mathbb{P}^1 \subset \mathbb{PT} : \qquad \mu^{\dot\alpha} = x^{\alpha\dot\alpha}\lambda_\alpha$$



Correspondences

Twistors	**Space-time**
$\mathbb{P}\mathscr{T}$ "Curved" twistor space	M Curved space-time
$X \simeq \mathbb{P}^1 \hookrightarrow \mathbb{P}\mathscr{T}$ Deformed incidence relations	$x^{\alpha\dot{\alpha}}$ Points of space-time
h Complex structure deformation: dynamical field in twistor action	$g = \epsilon_{\alpha\beta}\,\epsilon_{\dot{\alpha}\dot{\beta}}\,e^{\alpha\dot{\alpha}}\,e^{\beta\dot{\beta}}$ Metric and tetrad: dynamical field in GR action
B Auxiliary field in twistor action	$\Gamma_{\alpha\beta}$ ASD graviton: auxiliary field in GR action

We use a 1st order action for GR due to *Palatini* and *Plebanski*



Almost complex structures

Easiest to work in homogeneous coordinates: $Z^A = (\mu^{\dot\alpha}, \lambda_\alpha) \in \mathbb{T} = \mathbb{C}^4 - \mathbb{C}^2$

Basis of complex 1-forms

$$\mathrm{d}Z^A \in T^{*\,1,0}\mathbb{T}\,, \qquad \mathrm{d}\bar Z^{\bar A} \in T^{*\,0,1}\mathbb{T}$$

Deformation of complex structure

$$\mathbb{PT} \longrightarrow \mathbb{P}\mathscr{T}$$

Deformed twistor space

$$\begin{tikzcd} \mathrm{d}Z^A \mapsto \theta^A = \mathrm{d}Z^A - \underline{V^A} \arrow[d] \\ V^A \equiv \mathrm{d}\bar Z^{\bar B}\, V_{\bar B}{}^A \end{tikzcd}$$

Beltrami differential

Nijenhuis tensor

$$\begin{tikzcd}[column sep=tiny] N^A = & \bar\partial \arrow[d] & V^A + V^B \wedge & \partial_B \arrow[d] & V^A \\ & \bar\partial \equiv \mathrm{d}\bar Z^{\bar A}\,\dfrac{\partial}{\partial \bar Z^{\bar A}} & & \partial_B \equiv \mathcal{L}_{\partial/\partial Z^B} & \end{tikzcd}$$

Newlander, Nirenberg, Kodaira, Spencer,…

$N^A = 0$ yields integrable complex structure deformations

Hamiltonian deformations

Hamiltonian complex structure deformation

$$V^A = I^{AB}\,\partial_B h\,, \qquad h \equiv h_{\bar{A}}\,\mathrm{d}\bar{Z}^{\bar{A}}$$

Infinity twistor

$$I^{AB} = \begin{pmatrix} \epsilon^{\dot{\alpha}\dot{\beta}} & 0 \\ 0 & \Lambda\,\epsilon_{\alpha\beta} \end{pmatrix}$$

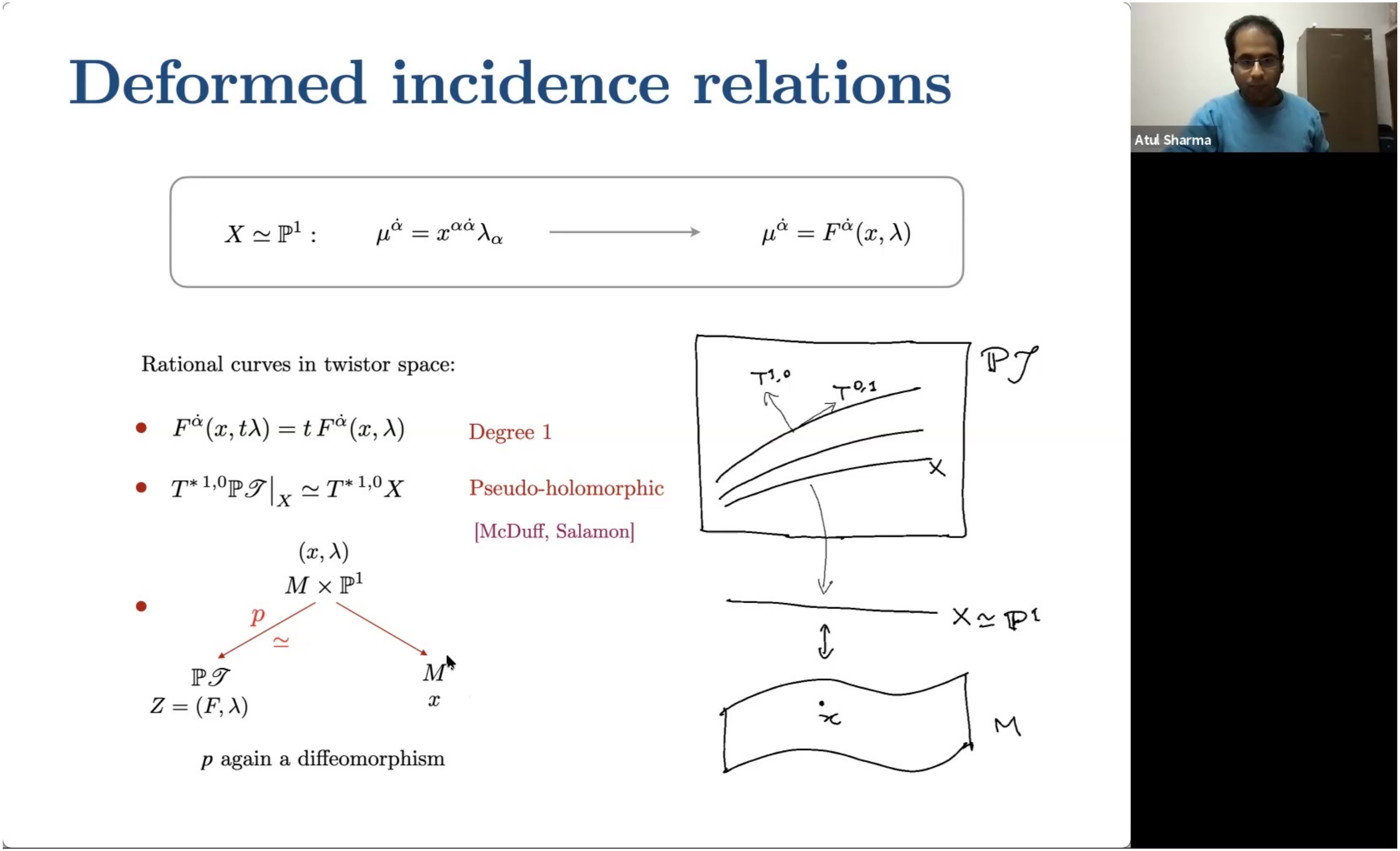
$$\Lambda \longrightarrow \text{Cosmological constant}$$

Today: stay in the $\Lambda = 0$ case.

$$V^A \equiv (V^{\dot{\alpha}}, V_\alpha) = \left(\frac{\partial h}{\partial \mu_{\dot{\alpha}}}, 0\right)$$

$$\theta^A \equiv \mathrm{d}Z^A - V^A = (\mathrm{d}\mu^{\dot{\alpha}} - V^{\dot{\alpha}}, \mathrm{d}\lambda_\alpha)$$

$$N^A \equiv (N^{\dot{\alpha}}, N_\alpha) = \left(\frac{\partial}{\partial \mu_{\dot{\alpha}}}\left(\bar{\partial} h + \frac{1}{2}\,\frac{\partial h}{\partial \mu^{\dot{\beta}}} \wedge \frac{\partial h}{\partial \mu_{\dot{\beta}}}\right), 0\right)$$



Deformed incidence relations

$$X \simeq \mathbb{P}^1 : \qquad \mu^{\dot\alpha} = x^{\alpha\dot\alpha}\lambda_\alpha \xrightarrow{\qquad\qquad} \mu^{\dot\alpha} = F^{\dot\alpha}(x,\lambda)$$

Rational curves in twistor space:

- $F^{\dot\alpha}(x, t\lambda) = t\, F^{\dot\alpha}(x,\lambda)$ Degree 1
- $T^{*\,1,0}\mathbb{P}\mathscr{T}\big|_X \simeq T^{*\,1,0}X$ Pseudo-holomorphic

  [McDuff, Salamon]
- $$\begin{tikzcd} & \substack{(x,\lambda)\\ M\times\mathbb{P}^1} \arrow[dl, "p"', "\simeq"] \arrow[dr] & \\ \substack{\mathbb{P}\mathscr{T}\\ Z=(F,\lambda)} & & \substack{M\\ x} \end{tikzcd}$$

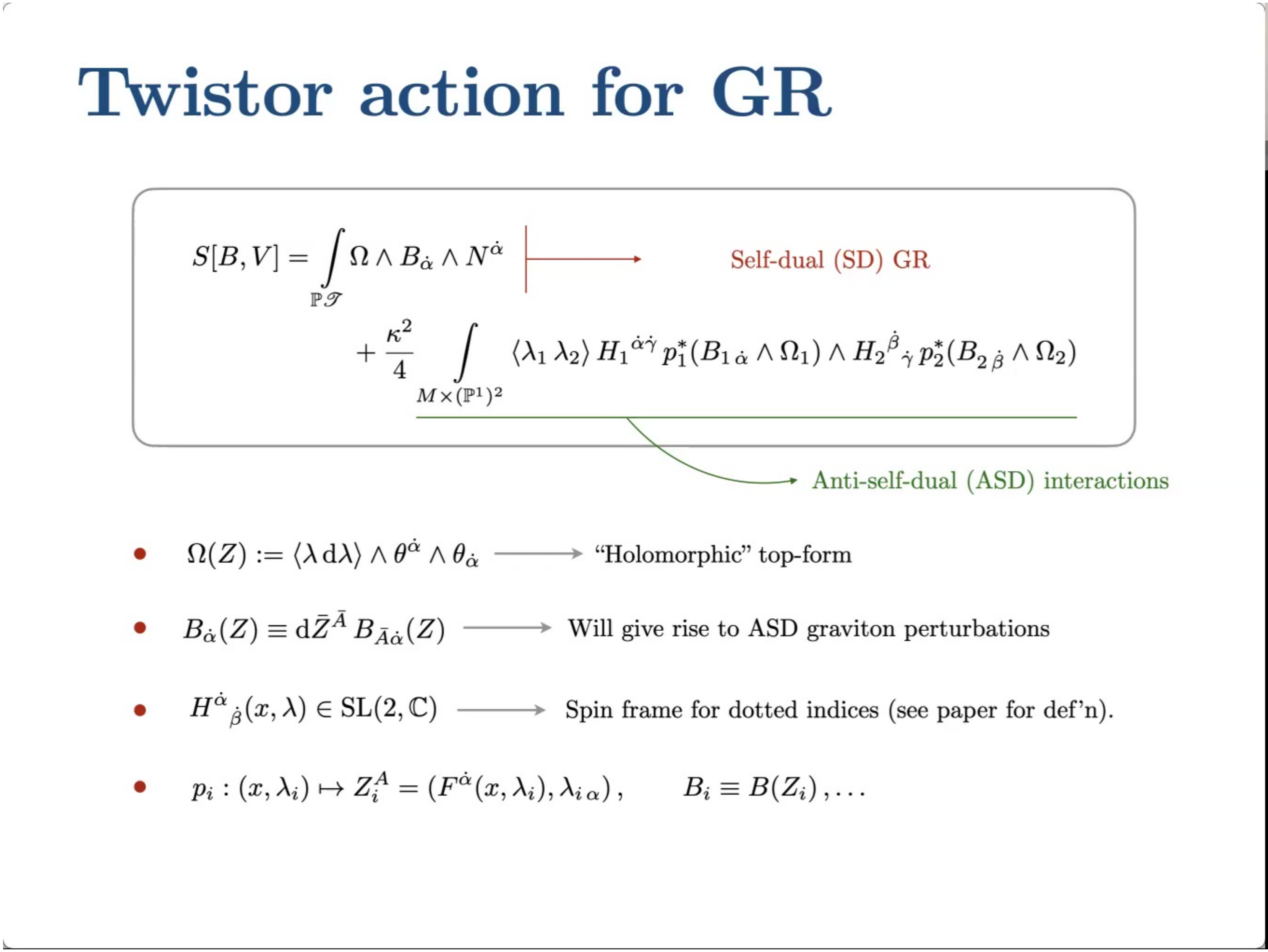
  p again a diffeomorphism

Twistor action for GR

$$S[B,V] = \int_{\mathbb{P}\mathscr{T}} \Omega \wedge B_{\dot\alpha} \wedge N^{\dot\alpha} \;\; \longmapsto \;\; \text{Self-dual (SD) GR}$$

$$+ \underbrace{\frac{\kappa^2}{4} \int_{M\times(\mathbb{P}^1)^2} \langle\lambda_1\,\lambda_2\rangle\, H_1{}^{\dot\alpha\dot\gamma}\, p_1^*(B_{1\,\dot\alpha}\wedge\Omega_1)\wedge H_2{}^{\dot\beta}{}_{\dot\gamma}\, p_2^*(B_{2\,\dot\beta}\wedge\Omega_2)}_{\text{Anti-self-dual (ASD) interactions}}$$

- $\Omega(Z) := \langle\lambda\,\mathrm{d}\lambda\rangle\wedge\theta^{\dot\alpha}\wedge\theta_{\dot\alpha} \longrightarrow$ "Holomorphic" top-form
- $B_{\dot\alpha}(Z) \equiv \mathrm{d}\bar{Z}^{\bar{A}}\, B_{\bar{A}\dot\alpha}(Z) \longrightarrow$ Will give rise to ASD graviton perturbations
- $H^{\dot\alpha}{}_{\dot\beta}(x,\lambda) \in \mathrm{SL}(2,\mathbb{C}) \longrightarrow$ Spin frame for dotted indices (see paper for def'n).
- $p_i : (x,\lambda_i) \mapsto Z_i^A = (F^{\dot\alpha}(x,\lambda_i), \lambda_{i\,\alpha})\,, \qquad B_i \equiv B(Z_i)\,, \ldots$



Twistor action for GR

$$S[B,V] = \int_{\mathbb{P}\mathscr{T}} \Omega \wedge B_{\dot\alpha} \wedge N^{\dot\alpha} \longmapsto$$

Self-dual (SD) GR

$$+\,\frac{\kappa^2}{4}\underline{\int_{M\times(\mathbb{P}^1)^2} \langle\lambda_1\,\lambda_2\rangle\, H_1{}^{\dot\alpha\dot\gamma}\, p_1^*(B_{1\,\dot\alpha}\wedge\Omega_1)\wedge H_2{}^{\dot\beta}{}_{\dot\gamma}\, p_2^*(B_{2\,\dot\beta}\wedge\Omega_2)}$$

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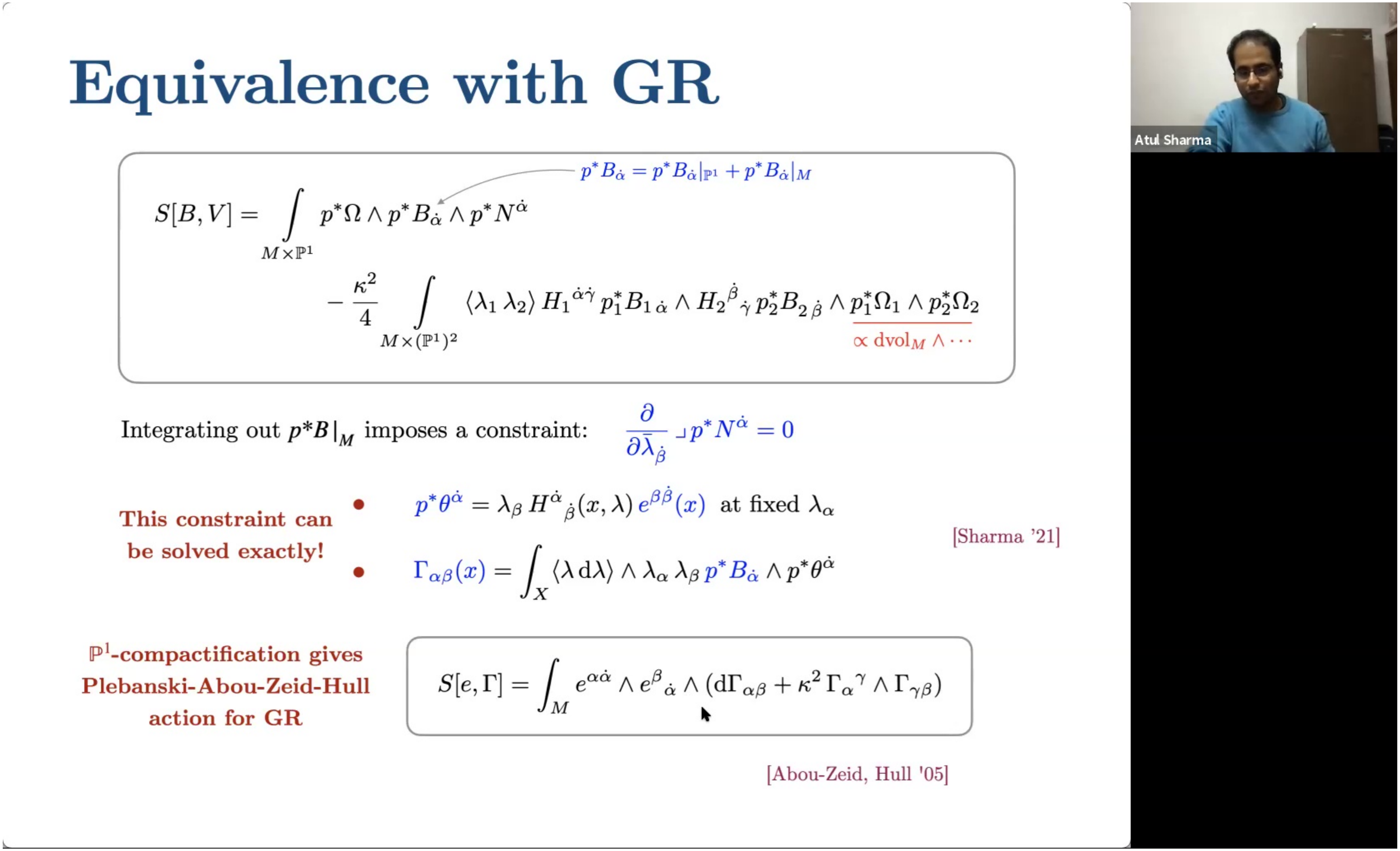


Equivalence with GR

$$p^*B_{\dot\alpha} = p^*B_{\dot\alpha}|_{\mathbb{P}^1} + p^*B_{\dot\alpha}|_M$$

$$S[B,V] = \int_{M\times\mathbb{P}^1} p^*\Omega \wedge p^*B_{\dot\alpha} \wedge p^*N^{\dot\alpha}$$

$$- \frac{\kappa^2}{4} \int_{M\times(\mathbb{P}^1)^2} \langle\lambda_1\,\lambda_2\rangle\, H_1{}^{\dot\alpha\dot\gamma}\, p_1^*B_{1\,\dot\alpha} \wedge H_2{}^{\dot\beta}{}_{\dot\gamma}\, p_2^*B_{2\,\dot\beta} \wedge \underbrace{p_1^*\Omega_1 \wedge p_2^*\Omega_2}_{\propto\, \mathrm{dvol}_M \wedge \cdots}$$

Integrating out $p^*B|_M$ imposes a constraint: $\dfrac{\partial}{\partial\bar\lambda_{\dot\beta}} \lrcorner\, p^*N^{\dot\alpha} = 0$

This constraint can be solved exactly!

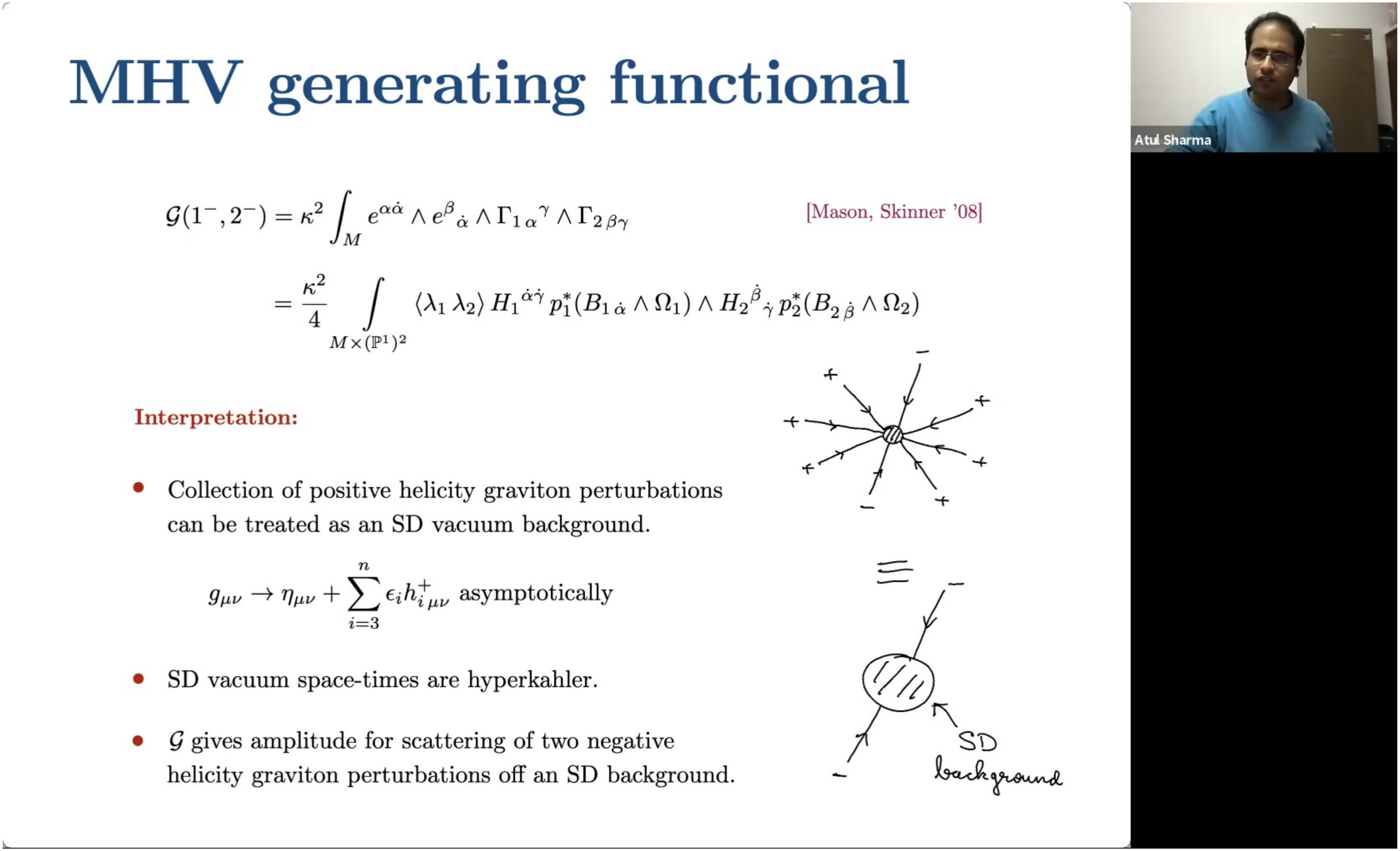
- $p^*\theta^{\dot\alpha} = \lambda_\beta\, H^{\dot\alpha}{}_{\dot\beta}(x,\lambda)\, e^{\beta\dot\beta}(x)$ at fixed λ_α
- $\Gamma_{\alpha\beta}(x) = \int_X \langle\lambda\,\mathrm{d}\lambda\rangle \wedge \lambda_\alpha\,\lambda_\beta\, p^*B_{\dot\alpha} \wedge p^*\theta^{\dot\alpha}$

[Sharma '21]

$\mathbb{P}^1$-compactification gives Plebanski-Abou-Zeid-Hull action for GR

$$S[e,\Gamma] = \int_M e^{\alpha\dot\alpha} \wedge e^{\beta}{}_{\dot\alpha} \wedge (\mathrm{d}\Gamma_{\alpha\beta} + \kappa^2\,\Gamma_\alpha{}^\gamma \wedge \Gamma_{\gamma\beta})$$

[Abou-Zeid, Hull '05]



MHV generating functional

$$\mathcal{G}(1^-,2^-) = \kappa^2 \int_M e^{\alpha\dot\alpha}\wedge e^{\beta}{}_{\dot\alpha}\wedge \Gamma_{1\,\alpha}{}^{\gamma}\wedge\Gamma_{2\,\beta\gamma}$$

[Mason, Skinner '08]

$$= \frac{\kappa^2}{4}\int_{M\times(\mathbb{P}^1)^2} \langle\lambda_1\,\lambda_2\rangle\, H_1{}^{\dot\alpha\dot\gamma}\, p_1^*(B_{1\,\dot\alpha}\wedge\Omega_1)\wedge H_2{}^{\dot\beta}{}_{\dot\gamma}\, p_2^*(B_{2\,\dot\beta}\wedge\Omega_2)$$

Interpretation:

- Collection of positive helicity graviton perturbations can be treated as an SD vacuum background.

$$g_{\mu\nu}\to\eta_{\mu\nu}+\sum_{i=3}^{n}\epsilon_i h^+_{i\,\mu\nu}\ \text{asymptotically}$$

- SD vacuum space-times are hyperkahler.
- $\mathcal{G}$ gives amplitude for scattering of two negative helicity graviton perturbations off an SD background.

Hodges' formula

$$\mathcal{G}(1^-,2^-) = \frac{\kappa^2}{4}\int_{\mathbb{R}^4} \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}(k_1+k_2)\cdot x}\; S_K\big|_{\text{on-shell}}$$

MHV amplitude extracted from the piece in S_K that is multilinear in each ϵ_i

$$\begin{aligned} A(1^-,2^-,3^+,\ldots,n^+) &= \frac{\kappa^2}{4}\int_{\mathbb{R}^4} \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}(k_1+k_2)\cdot x}\left(\prod_{i=3}^{n}\frac{\partial}{\partial\epsilon_i}\right) S_K\big|_{\text{on-shell}}\Big|_{\epsilon_i=0} \\ &= \frac{\kappa^2}{4}\int_{\mathbb{R}^4} \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}(k_1+k_2)\cdot x}\, \langle V_{h_3}V_{h_4}\ldots V_{h_n}\rangle^0_{\text{conn, tree}} \end{aligned}$$

With the "tree-level" connected correlator computed using the free worldsheet theory and vertex operators:

$$S_K^0[m] = \int_X \frac{\langle\lambda\,\mathrm{d}\lambda\rangle}{\langle\lambda\,1\rangle^2\,\langle\lambda\,2\rangle^2}\wedge[m\,\bar\partial|_X m] \qquad V_{h_i} = \int_X \frac{2\,\langle\lambda\,\mathrm{d}\lambda\rangle\wedge h_i|_X}{\langle\lambda\,1\rangle^2\,\langle\lambda 2\rangle^2}$$

Future: what is this good for?

- MHV formalism for gravity???

[Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager ’06] [Adamo, Mason ’13]

- Extending gravitational twistor string technology to *non-supersymmetric*, *loop-level* graviton scattering.

- Covariantization and quantization? [Mason, Wolf ’09]

- SD gravity, quantum integrability, celestial/twisted holography...

$$S[h] = \int_{\mathbb{P}\mathscr{T}} \frac{\Omega}{\langle\lambda\,1\rangle^4\,\langle\lambda\,2\rangle^4} \wedge \left(h\wedge\bar{\partial}h + \frac{1}{3}\,h\wedge\frac{\partial h}{\partial\mu^{\dot\alpha}}\wedge\frac{\partial h}{\partial\mu_{\dot\alpha}}\right)$$

Ongoing work with Skinner and Bittleston