Title: The onset of quantum chaos in disordered systems

Speakers: Adar Sharon

Series: Quantum Fields and Strings

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Abstract: We study the Lyapunov exponent in disordered quantum field theories. Generically the Lyapunov exponent can only be computed in isolated CFTs, and little is known about the way in which chaos grows as we deform the theory away from weak coupling. In this talk we describe families of theories in which the disorder coupling is an exactly marginal deformation, allowing us to follow the Lyapunov exponent from weak to strong coupling. We find surprising behaviors in some cases, including a discontinuous transition into chaos. We also describe a new method allowing for computations in nontrivial CFTs deformed by disorder at leading order in 1/N.

The Onset of Quantum Chaos

Adar Sharon

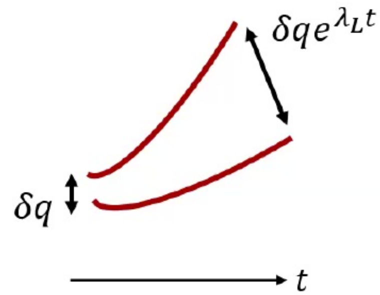
Weizmann Institute of Science

November 2, 2021

Work with: M. Berkooz, N. Silberstein, E. Urbach (to appear (soon))

Classical Chaos

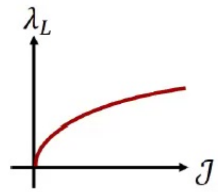
Classical chaos is familiar:



Basic question: how does λ_L behave as we go from weak to strong coupling?

Classical Chaos

Basic question: how does λ_L behave as we go from weak to strong coupling?
This is well-studied (and still actively researched). Naively, expect:



BUT! Not always generic. Phenomenologically we find:

$$\text{integrable} + \text{small deformation} = \text{“basically integrable”}$$

Many examples of this [Kolmogorov,Arnold,Moser; Fermi,Pasta,Ulam...]

what about QFTs?

Quantum Chaos

Useful diagnostic/definition of quantum chaos: the out-of-time-ordered correlator (OTOC).
Basic idea - measure a commutator $[V(0), W(t)]$:

$$C = \frac{1}{\langle VV\rangle\langle WW\rangle}\langle [V(0), W(t)]^2\rangle_\beta$$
$$= \frac{1}{\langle VV\rangle\langle WW\rangle}\left(\langle V(0)W(t)V(0)W(t)\rangle_\beta + \langle W(t)V(0)W(t)V(0)\rangle_\beta - \langle V(0)W(t)W(t)V(0)\rangle_\beta - \langle W(t)V(0)V(0)W(t)\rangle_\beta\right)$$

out of time order!

In chaotic theories with $N \gg 1$ dof, expect

$$C \sim 1 - \frac{1}{N}e^{\lambda_L t}$$

λ_L is the Lyapunov exponent (or chaos exponent).

Quantum Chaos

$$C \sim 1 - \frac{1}{N} e^{\lambda_L t}$$

In unitary theories, bounded [Maldacena,Shenker,Stanford]:

$$\lambda_L \leq 1$$

The upper bound is saturated by semiclassical gravity and by the SYK model.

Computing λ_L

The "standard" method for computing λ_L requires three assumptions:

- Disorder
- Large-N
- Scale invariance

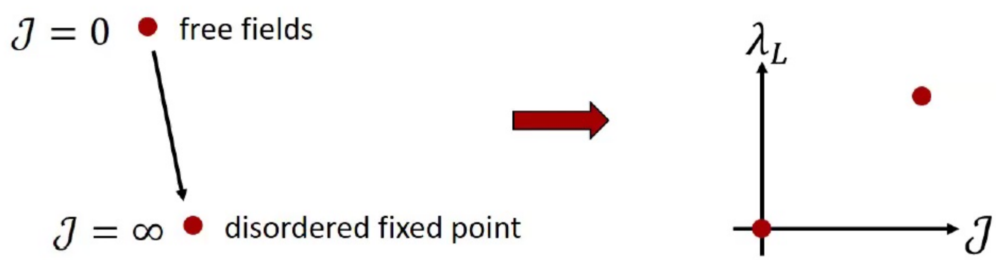
Schematically:

$$V \propto J_{i_1 \dots i_q} \phi_{i_1} \dots \phi_{i_q} \; ,$$

with J random and taken from an ensemble:

$$\langle J_{i_1 \dots i_q} J_{i_1 \dots i_q} \rangle \sim \mathcal{J}^2 \; .$$

As a result, λ_L is usually only computed in isolated points:



Computing λ_L

Examples:

1. 0+1d SYK model, with N fermions ψ_i and interaction

   $$H \sim J_{i_1 i_2 i_3 i_4} \psi_{i_1} \psi_{i_2} \psi_{i_3} \psi_{i_4} ,$$

   or more generally

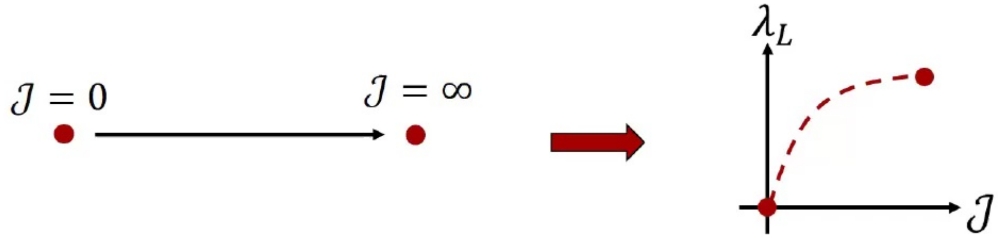
   $$H \sim J_{i_1 \ldots i_q} \psi_{i_1} \ldots \psi_{i_q} , \quad q > 2 .$$

2. Examples also exist in 1+1d: [Murugan,Stanford,Witten] discussed the 2d $\mathcal{N} = (1,1)$ model with superpotential

   $$W = J_{i_1 \ldots i_q} \Phi_{i_1} \ldots \Phi_{i_q} .$$

Computing λ_L

Our trick: consider disorder deformations $\mathcal{J}$ which are exactly marginal. Scale invariance is preserved, and we can compute $\lambda_L(\mathcal{J})$:



We will discuss two main classes of examples:

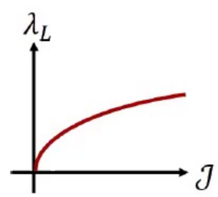
1. Generalized free fields in 0+1d and 1+1d.
2. The conformal manifold emanating from the $\mathcal{N} = (2,2)$ minimal models in 1+1d.

Computing λ_L

Computation of $\lambda_L(\mathcal{J})$ has been done in some cases, e.g.:

- Large q SYK [Maldacena,Stanford]
- ϕ^4 theory in 4d [Stanford] (computed space-averaged λ_L)
- Chiral 2d SYK [Lian,Sondhi,Yang; Hu,Lian]

So far, all cases found continuous behavior:



Review: Chaos in the SYK Model

The SYK model:

$$H \sim J_{i_1 \dots i_q} \psi_{i_1} \cdots \psi_{i_q}$$

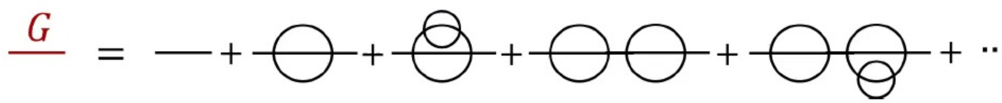
with $J_{i_1 \dots i_q}$ chosen from a Gaussian distribution with variance $\mathcal{J}^2$.

Take the large-N limit and compute λ_L at $\mathcal{J} = \infty$. Proceeds in two steps:

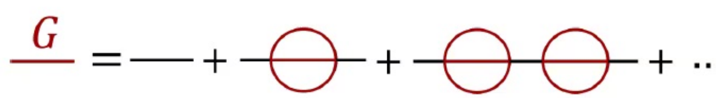
1. Compute full 2-point function $G = \langle \psi(t) \psi(0) \rangle$.
2. Compute the OTO 4-point function.

SYK: 2-point

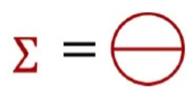
First the full 2-point function G. Diagramatically, the contributions are (for $q = 4$):



This can be written as a self-consistency equation:



Defining



We can rewrite this as an algebraic Schwinger-Dyson equation:

$$G = \frac{1}{G_0^{-1} - \Sigma}, \qquad \Sigma = \mathcal{J}^2 G^{q-1}\,.$$

SYK: 2-point

$$G = \frac{1}{G_0^{-1} - \Sigma}, \qquad \Sigma = \mathcal{J}^2 G^{q-1} \,.$$

At large $\mathcal{J}$, we assume scale invariance:

$$G = \frac{b}{|x|^{2\Delta}} \,,$$

for some b, Δ. Plugging into the equation, we find

$$b^q = \frac{1}{\mathcal{J}^2} \frac{(1-2\Delta)\tan \pi\Delta}{2\pi}, \quad \Delta = 1/q \,.$$

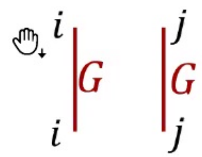
We thus have the full 2-point function at $\mathcal{J} \to \infty$.

SYK: 4-point

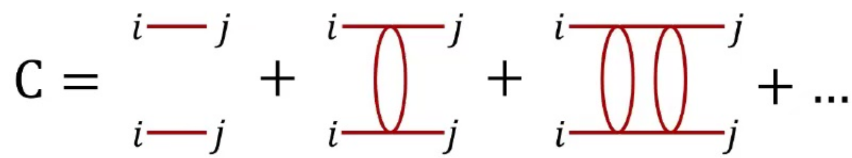
To find chaos, we must compute an out-of-time-ordered 4-point function

$$C = \frac{\langle \psi_i \psi_j \psi_i \psi_j \rangle}{\langle \psi_i \psi_i \rangle \langle \psi_j \psi_j \rangle} \overset{?}{\sim} 1 - \frac{1}{N} e^{\lambda_L t}$$

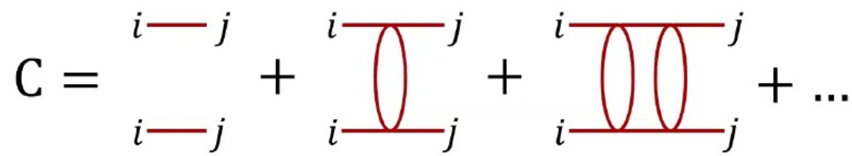
At leading order in $1/N$ we find a "disconnected" contribution:



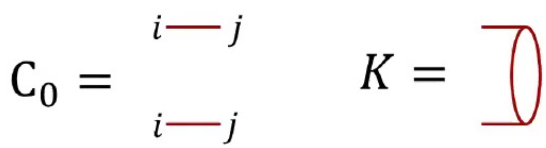
At subleading order, we find the following contributions:



SYK: 4-point



So the 4-point function has a kernel structure: define



then again we have an algebraic relation:

$$C = \sum_{n=0}^{\infty} K^n C_0 = \frac{C_0}{1-K} \, .$$

Can find the full 4-point function of the theory, but to find λ_L a simpler calculation suffices.

Chaos in SYK

$$C = \frac{C_0}{1-K} .$$

Rewrite as

$$C = C_0 + K \cdot C .$$

If the theory is chaotic, at large times $C \sim e^{\lambda_L t}$ grows exponentially. Then we can neglect C_0, and the equation becomes

$$KC = C .$$

We learn that $C \sim e^{\lambda_L t}$ is an eigenfunction of K with eigenvalue 1.

Chaos in SYK

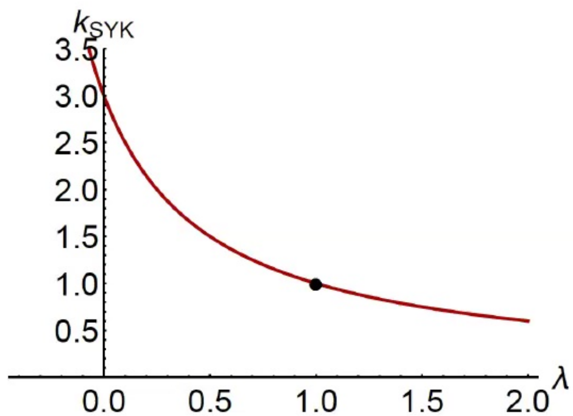
To summarize: finding chaos has been reduced to an algebraic problem. Solve

$$KC = k(\lambda)C, \quad C \sim e^{\lambda t},$$

and λ_L is the maximal λ such that $k(\lambda) = 1$.
In SYK:

$$k_{SYK}(\lambda) = \frac{\Gamma(3-2\Delta)\Gamma(2\Delta+\lambda)}{\Gamma(1+2\Delta)\Gamma(2-2\Delta+\lambda)}.$$

For $q = 4$ ($\Delta = 1/4$):



Class #1: Generalized Free Fields

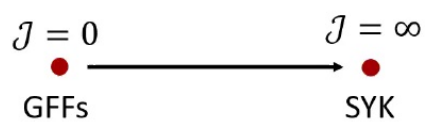
How to get exactly marginal disorder?

[Gross,Rosenhaus]: consider an SYK-like QM model with interaction

$$H \sim J_{i_1 \dots i_q} \psi_{i_1} \cdots \psi_{i_q} \ ,$$

where ψ_i are generalized free fields with dimension $\Delta = 1/q$.

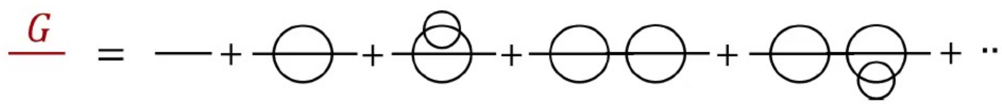
The interaction $\mathcal{J}$ is thus classically marginal. There is evidence that it is exactly marginal at large N.



What is $\lambda_L(\mathcal{J})$?

GFF: 2-point function

Compute the 2-point function. Diagrams are the same:



only difference is G_0 propagator.

So the SD equations are similar to SYK:

$$G = \frac{1}{G_0^{-1} - \Sigma}, \qquad \Sigma = \mathcal{J}^2 G^{q-1} \, .$$

Class #1: Generalized Free Fields

How to get exactly marginal disorder?

[Gross,Rosenhaus]: consider an SYK-like QM model with interaction

$$H \sim J_{i_1 \dots i_q} \psi_{i_1} \cdots \psi_{i_q} \ ,$$

where ψ_i are generalized free fields with dimension $\Delta = 1/q$.

GFF: 2-point function

SD equations:

$$G = \frac{1}{G_0^{-1} - \Sigma}, \qquad \Sigma = \mathcal{J}^2 G^{q-1} \, .$$

Now we can solve for *any* $\mathcal{J}$ using the conformal ansatz:

$$G = \frac{b(\mathcal{J})}{|x|^{2\Delta}} \, .$$

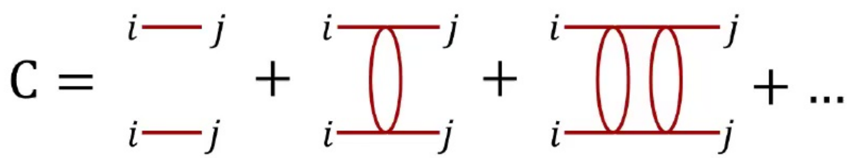
find

$$\frac{b^q(\mathcal{J})}{1 - 2b(\mathcal{J})} = \frac{1 - 2\Delta}{2\pi \mathcal{J}^2} \tan \pi\Delta, \qquad \Delta = 1/q \, .$$

At $\mathcal{J} = 0$, reproduce GFF 2-point function. At $\mathcal{J} = \infty$, reproduce SYK 2-point function.

GFF: 4-point function

Next we find the 4-point function. Diagrams are the same as SYK:



So it is easy to find K_{GFF}:

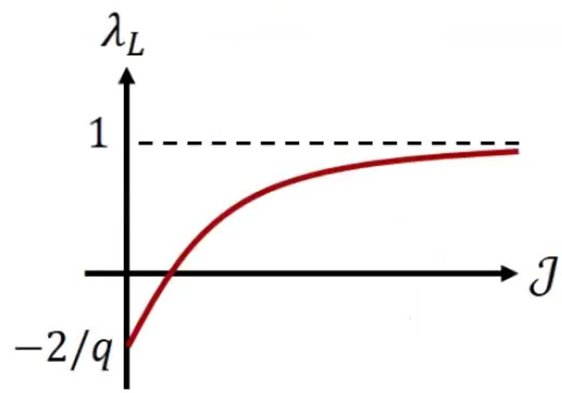
$$K_{GFF} = \left(\frac{b(\mathcal{J})}{b(\mathcal{J} = \infty)} \right)^q K_{SYK}$$

and the eigenvalues:

$$k_{GFF}(\lambda) = \left(\frac{b(\mathcal{J})}{b(\mathcal{J} = \infty)} \right)^q k_{SYK}(\lambda) \, .$$

GFF: Chaos

Now we can finally find λ_L. Solve $k_{GFF}(\lambda) = 1$ as a function of $\mathcal{J}$, and plot the result for some arbitrary q:



At large $\mathcal{J}$, we get maximal chaos, as expected in SYK.
The more interesting situation is at low $\mathcal{J}$.

GFF: Chaos

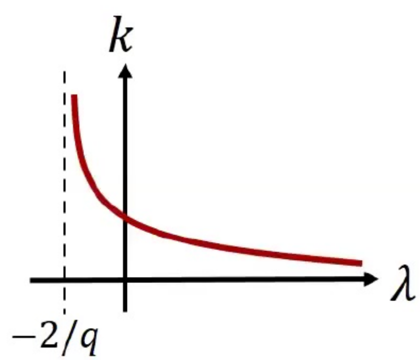
What happened?

$$k_{GFF}(\lambda) = \left(\frac{b(\mathcal{J})}{b(\mathcal{J} = \infty)} \right)^q k_{SYK}(\lambda) \, .$$

Focus on small $\mathcal{J}$. Then

$$\frac{b(\mathcal{J})}{b(\mathcal{J} = \infty)} \to 0 \, .$$

So solutions to $k_{GFF}(\lambda) = 1$ correspond to $k_{SYK}(\lambda) \to \infty$.
Plot k_{SYK}:

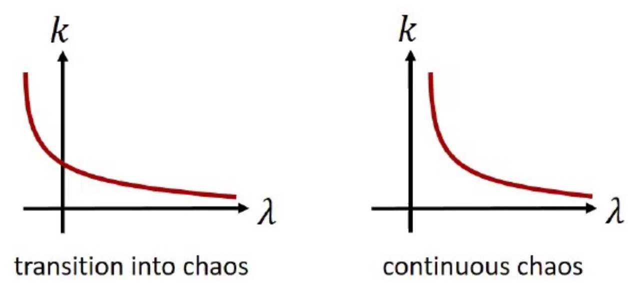


GFF: Chaos

This result is general. At $\mathcal{J} \to 0$, $k(\lambda) \to 0$. Solution to

$$k(\lambda) = 1$$

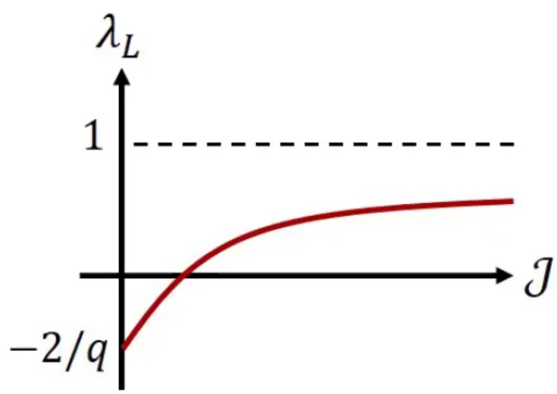
should be possible only if $k(\lambda)$ diverges. Two options:



Summary: chaos at small $\mathcal{J}$ is determined by largest λ such that $k(\lambda)$ diverges.

2d SUSY GFFs

Repeat calculation for a 2d theory: $\mathcal{N}=2$ SUSY GFFs. Computation of λ_L is very similar:



Class #2: The $\mathcal{N} = 2$ Minimal Models

First a toy model: 2d $\mathcal{N} = (2,2)$ theory with 3 chiral superfields

$$W = h_1(X^3 + Y^3 + Z^3) + h_2 XYZ \,.$$

A conformal manifold exists in the h_1-h_2 plane [Leigh,Strassler].

Now consider the "IR" version. First flow to the CFT at $h_2 = 0$:

$$W = X^3 + Y^3 + Z^3 \,.$$

This is 3 copies of the $\mathcal{N} = (2,2)$ A_2 minimal model. Facts:

- A_2 has no continuous global symmetries.
- A_2 includes an operator $\tilde{\mathcal{O}}$ of dimension $\Delta_{\tilde{\mathcal{O}}} = 1/3$.

No global symmetries $\Longrightarrow$ any marginal superpotential deformation is exactly marginal [GKSTW; Kol]:

$$W = (X^3 + Y^3 + Z^3) + J\tilde{X}\tilde{Y}\tilde{Z} \,.$$

Class #2: The $\mathcal{N} = 2$ Minimal Models

Can generalize this: Consider

$$W = \Phi^q \, .$$

For $q > 2$, flows to the A_{q-1} minimal model. Facts:

- no continuous global symmetries.
- includes an operator $\tilde{\Phi}$ of dimension $\Delta_{\tilde{\Phi}} = 1/q$.
- central charge is $c = 3(1 - 2/q)$.

Now deform:

$$W = (\sum_{i=1}^{N} \Phi_i^q) + \sum_{i_1 \neq i_2 \neq \ldots \neq i_q} J_{i_1 \ldots i_q} \tilde{\Phi}_{i_1} \ldots \tilde{\Phi}_{i_q} \, .$$

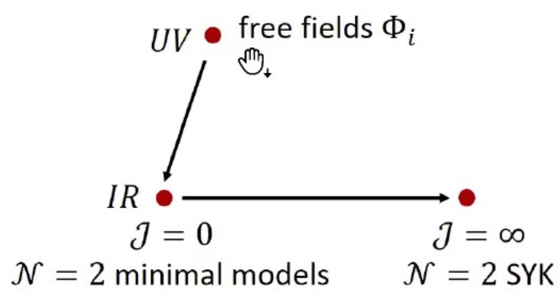
Every $J_{i_1 \ldots i_q}$ is an exactly marginal operator, so we have a conformal manifold.

Class #2: The $\mathcal{N}=2$ Minimal Models

To summarize: we are considering

$$W = \sum_{i=1}^{N} \Phi_i^q + \sum_{i_1 \neq i_2 \neq \ldots \neq i_q} J_{i_1 \ldots i_q} \tilde{\Phi}_{i_1} \ldots \tilde{\Phi}_{i_q} \; .$$

The J's are exactly marginal for every specific realization.

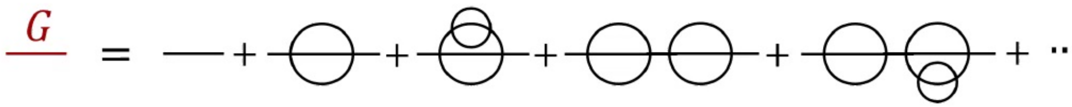


Ensemble average is different from SYK: here we **know** the average is over CFTs.

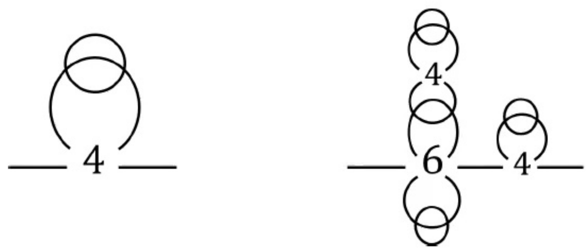
Disorder Around a General CFT

Solve for the two-point function G.

Around a free theory, diagrams were all of the form



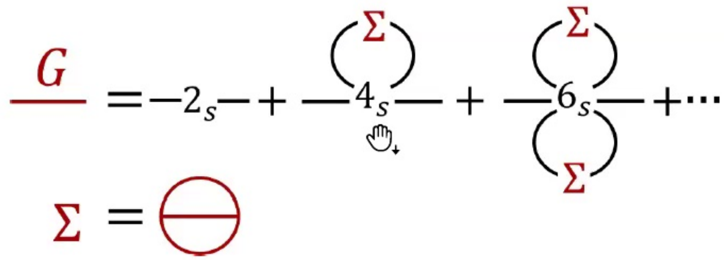
Around a nontrivial CFT, we have more diagrams, e.g.:



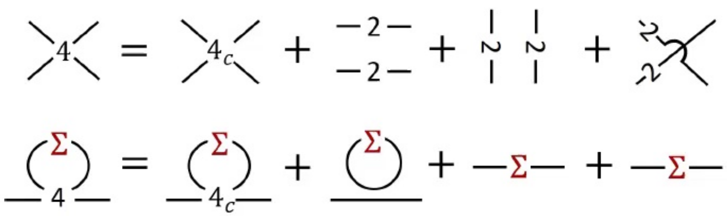
Is there an organizing principle?

The 2-point function

A generalized SD equation:

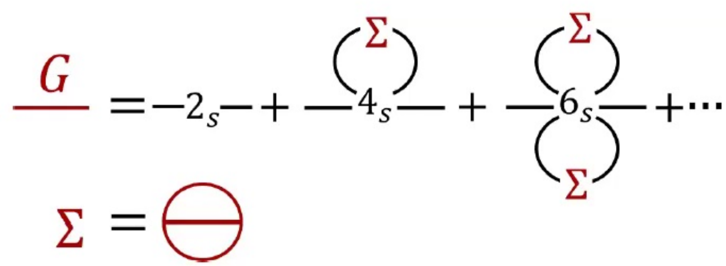


Subtracted n-point functions remove disconnected terms.
For example:

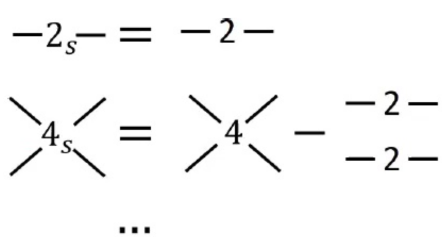


The 2-point function

A generalized SD equation:

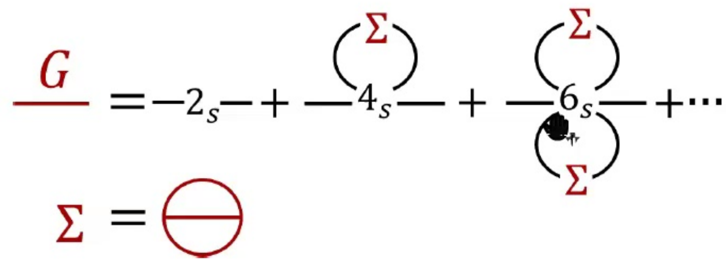


Subtracted n-point functions remove disconnected terms.
Explicitly:



The 2-point function

A generalized SD equation:



Subtracted *n*-point functions remove disconnected terms.

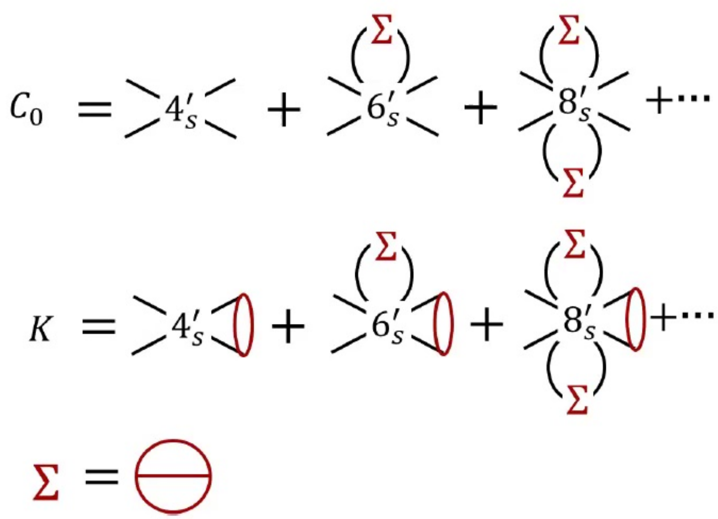
requires summing an infinite number of diagrams, but also allows for a perturbative expansion in $\mathcal{J}$!

The 4-point function

The 4-point function C also has a kernel structure, just like the free theory:

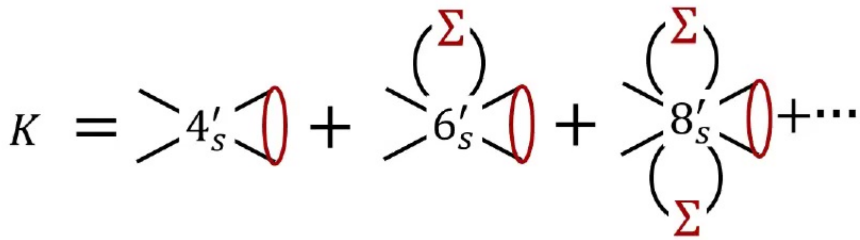
$$C = \frac{\langle \Phi_i \Phi_j \Phi_i \Phi_j \rangle}{\langle \Phi_i \Phi_i \rangle \langle \Phi_j \Phi_j \rangle} = \sum_{n=0}^{\infty} K^n C_0 \, ,$$

where



Chaos

We can compute λ_L using the kernel structure:

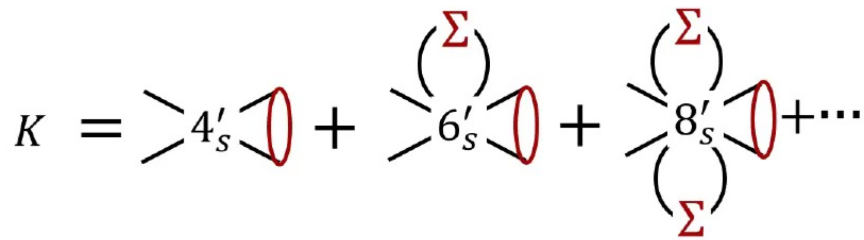


find the eigenvalues:

$$KC = k(\lambda)C, \qquad C \sim e^{\lambda t},$$

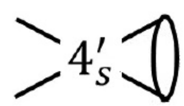
and find solutions for $k(\lambda) = 1$.

Chaos



Can solve in perturbation theory in $\mathcal{J}^2$. Leading order:

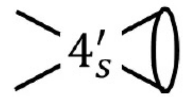
$$K = \mathcal{J}^2 K_0 =$$



very difficult to compute in practice...

Chaos

$$K = \mathcal{J}^2 K_0 =$$



simplest case is $q = 3$:

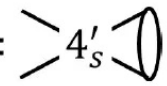
$$W = \sum_{i=1}^{N} \Phi_i^3 + \sum_{i \neq j \neq k} J_{ijk} \tilde{\Phi}_i \tilde{\Phi}_j \tilde{\Phi}_k \ .$$

A_2 minimal model has $c = 1$, so it is the free compact boson $\varphi \sim \varphi + 2\pi R$ with $R = \sqrt{3}/2, \sqrt{3}$.

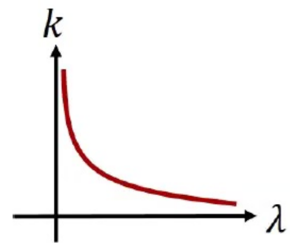
So $\tilde{\Phi}$ is some combination of vertex operators:

$$\tilde{\Phi} = \phi + \theta \psi + \theta^2 F \ .$$

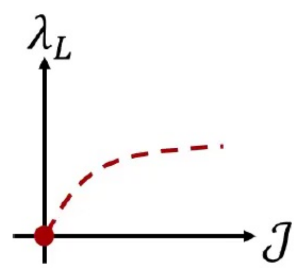
Chaos

$$K = \mathcal{J}^2 K_0 =$$ 

Eigenvalues give a very difficult integral... compute numerically:



Chaos exponent:



Continuity Conjecture

Interesting "coincidence". Four-point function obeys

$$C = \overbrace{C_0}^{\mathcal{J}=0} + K \cdot C$$

So far ignored C_0. However, C_0 also has exponential behavior:

$$C_0 \sim e^{\lambda_L^0 t}\,.$$

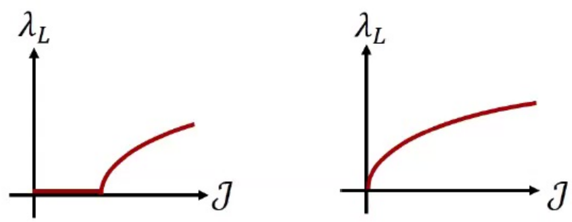
What chaos exponent λ_L^0 does it predict?

	λ_L^0	$\lambda_L(\mathcal{J} \to 0)$
GFF	$-2/q$	$-2/q$
$q = 3$ MM	0	0

"Continuity" at $\mathcal{J} = 0$. Hint at a QFT KAM theorem?

Summary

- On a conformal manifold, can compute $\lambda_L(\mathcal{J})$ for exactly marginal $\mathcal{J}$ from weak to strong coupling. Examples exist in 1d and 2d.
- Can compute chaos exponent even when expanding around nontrivial CFTs using generalized SD equations.
- We find diverse behaviors:
  - For GFFs, discontinuous transition into chaos.
  - For the $\mathcal{N} = (2, 2)$ A_2 minimal model ($c = 1$), continuous chaos exponent.



What is the organizing principle?

Future Directions

- Continuity at $\mathcal{J} = 0$. Can this be proven? Can give concise set of conditions for continuous/discontinuous transition?
- More examples of disordered conformal manifolds. $q = 4$ minimal model ($c = 3/2$), chiral fermions...
- Chaos in higher-d CFTs? Similar conformal manifolds exist in 3d, but computing λ_L is much more complicated. Recent attempt by [Chang,Colin-Ellerin,Peng,Rangamani].

Thank You!