Title: Twisted Character Varieties and Quantization via Factorization Homology

Speakers: Corina Keller

Series: Mathematical Physics

Date: October 22, 2021 - 1:30 PM

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Abstract: Factorization homology is a local-to-global invariant which "integrates" disk algebras in symmetric monoidal higher categories over manifolds. In this talk I will discuss how to compute categorical factorization homology on oriented surfaces with principal D-bundles, for D a finite group, in terms of categories of modules over algebras defined in purely combinatorial terms. This is an extension of the work of Ben-Zvi, Brochier and Jordan to D-decorated surfaces. The main example for us comes from an action of Dynkin diagram automorphisms on representation categories of quantum groups associated to a reductive group G. We will see that in this case factorization homology gives rise to a quantization of character varieties which are twisted by the group of outer automorphisms of G.

This talk is based on joint work with L. Müller.

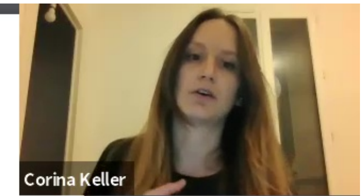
Zoom Link: https://pitp.zoom.us/j/93950433494?pwd=WXI2VE9IdnRweEh5RmZsZ21BV1BQQT09

Factorization Homology on Surfaces with Principal Bundles and Quantization of $\mathrm{Out}(G)$-Twisted Character Varieties

Corina Keller
based on joint work with Lukas Müller
arXiv:2107.12348

Université Montpellier

October 22, 2021



1. Introduction and Background on Factorization Homology

2. Computation of D-Structured Factorization Homology

2.1 Annuli with D-bundles

2.2 Punctured Surfaces with D-bundles

3. Quantization of $\mathrm{Out}(G)$-Twisted Character Varieties

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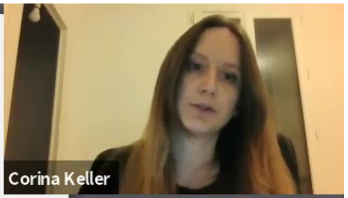
Quantization of G-local Systems

The BZBJ-approach to quantization



Setup:

- $\Sigma = \Sigma_{g,r}$, $r > 0$
- G: Reductive algebraic group
- $\mathcal{M}_G(\Sigma)$: Moduli space of G-local systems on Σ

Theorem (Ben-Zvi-Brochier-Jordan '18)

$$\mathrm{QCoh}(\mathcal{M}_G(\Sigma)) \cong \int_\Sigma \mathrm{Rep}(G)$$

Quantization of G-local Systems

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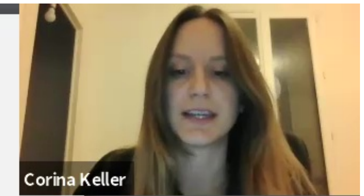

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$\mathrm{QCoh}(\mathcal{M}_G(\Sigma)) \cong \int_\Sigma \mathrm{Rep}(G)$

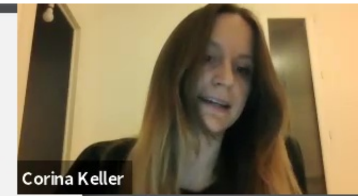
Idea:

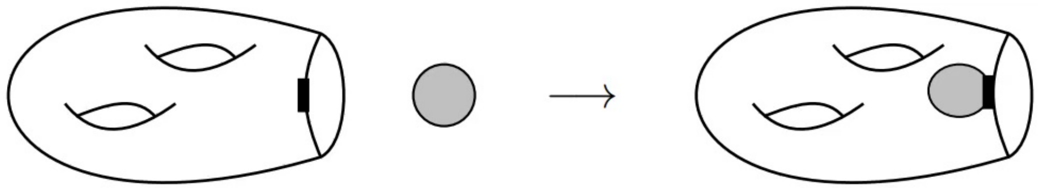
- Quantize locally: $\mathrm{Rep}(G) \rightsquigarrow \mathrm{Rep}_q(G)$
- Glue local data via factorization homology

  $\rightsquigarrow$ Quantum character variety $:= \int_\Sigma \mathrm{Rep}_q(G)$

Quantization of G-local Systems

The BZBJ-approach to quantization





$$\int_\Sigma \mathrm{Rep}_q(G) \quad \boxtimes \quad \mathrm{Rep}_q(G) \quad \xrightarrow{\mathrm{act}} \quad \int_\Sigma \mathrm{Rep}_q(G)$$

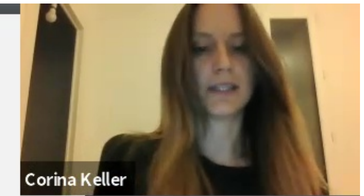
Strategy: Describe category $\int_\Sigma \mathrm{Rep}_q(G)$ internally in terms of $\mathrm{Rep}_q(G)$ (Barr-Beck monadic reconstruction)

Theorem (Ben-Zvi-Brochier-Jordan '18)

Given a ribbon graph presentation of Σ with one vertex, there is an equivalence

$\int_\Sigma \mathrm{Rep}_q(G) \cong a_q^{\mathsf{AGS}}\text{-}\mathrm{Mod}_{\mathrm{Rep}_q(G)}$

with $a_q^{\mathsf{AGS}} =$ Alekseev-Grosse-Schomerus algebra.

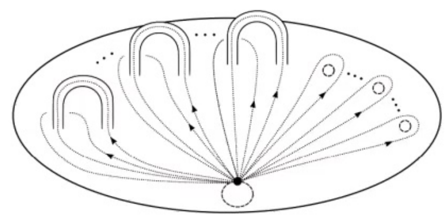


Quantization of G-local Systems

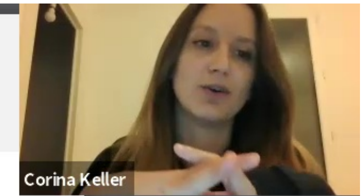
Combinatorial quantization

Alekseev-Grosse-Schomerus algebras:

- 'Building blocks':
  - Reflection equation algebra $\mathcal{O}_q(G)$
    ($=$ braided dual $\widetilde{H^\circ}$ for $H = U_q(\mathfrak{g})$)
  - Algebra of quantum differential operators $D_q(G)$
    $D_q(G) = \mathcal{O}_q(G) \otimes \mathcal{O}_q(G)$ + certain cross-relations
- $a_q^{\mathsf{AGS}} = D_q(G)^{\widetilde{\otimes} g} \widetilde{\otimes} \mathcal{O}_q(G)^{\widetilde{\otimes} r-1}$ twisted tensor product of REAs and quantum differential operators
- a_q^{AGS} is a quantization of the Fock-Rosly Poisson structure on G-representation variety G^{2g+r-1}



Factorization Homology on Surfaces with D-Bundles



K-Müller

For a finite group D, we compute categorical factorization homology on oriented surfaces equipped with principal D-bundles

For $D = \mathrm{Out}(G)$ we use factorization homology to quantize $\mathrm{Out}(G)$-twisted character varieties

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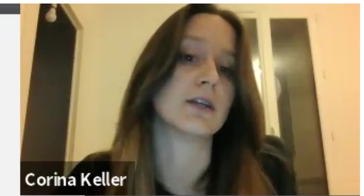
Categorical Factorization Homology

Linear categories



Setup:

- $(\mathrm{Pr}_c, \boxtimes)$: (2,1)-category of $\mathbb{K}$-linear locally finitely presentable categories
  - **Objects**: $\mathbb{K}$-linear categories that are
    - Cocomplete
    - Finitely generated by compact objects
  - **1-Morphisms**: Cocontinuous $\mathbb{K}$-linear functors
  - **2-Morphisms**: Natural isomorphisms

with $\boxtimes$ the Deligne-Kelly tensor product

$$\mathrm{Pr}_c[\mathcal{C} \boxtimes \mathcal{D}, \mathcal{E}] \cong \mathrm{Bil}_{\mathrm{cocont}}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$$

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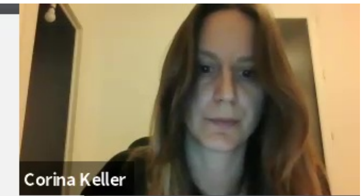
Categorical Factorization Homology

Linear categories



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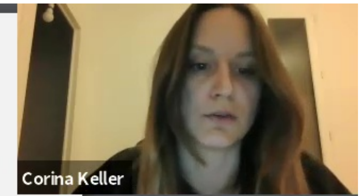
Main example:

- H: Ribbon Hopf algebra

  $\rightsquigarrow H\text{-}\mathrm{Mod}_{lf} \in \mathrm{Pr}_c$

Categorical Factorization Homology

Disk-algebras



A framed E_2-algebra is a symmetric monoidal functor

$$\mathcal{A}\colon \mathrm{Disk}_{\mathrm{or}}^{2,\sqcup} \to \mathrm{Pr}_c^{\boxtimes}$$

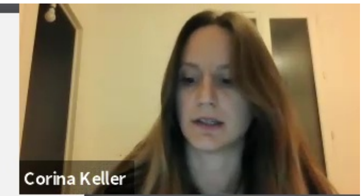
$\mathrm{Man}_{\mathrm{or}}^{2,\sqcup}$:

- **Objects**: Oriented surfaces
- **1-Morphisms**: Oriented embeddings
- **2-Morphisms**: Equivalence classes of isotopies of embeddings

$\mathrm{Disk}_{\mathrm{or}}^{2,\sqcup} \subset \mathrm{Man}_{\mathrm{or}}^{2,\sqcup}$: Full subcategory of disks $\mathbb{R}^2$ and finite disjoint unions thereof

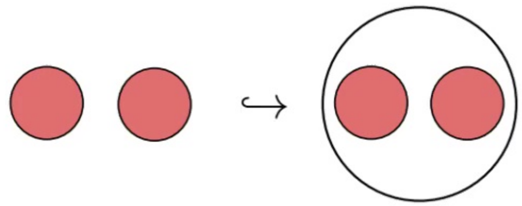
Salvatore-Wahl '03

A framed E_2-algebra in Pr_c is a balanced braided tensor category.

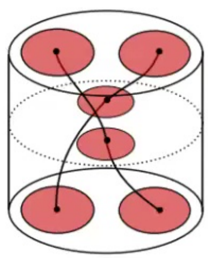


Categorical Factorization Homology

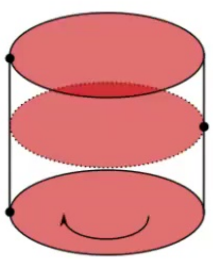
Framed E_2-algebras in $\mathrm{Pr}_c^{\boxtimes}$



$\rightsquigarrow$ $\otimes : \mathcal{A} \boxtimes \mathcal{A} \longrightarrow \mathcal{A}$



$\rightsquigarrow$ $\sigma_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$

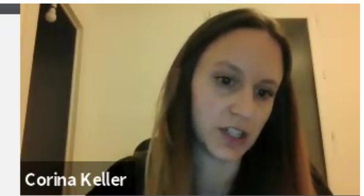


$\rightsquigarrow$ $\theta \colon \mathrm{id}_{\mathcal{A}} \Longrightarrow \mathrm{id}_{\mathcal{A}}$

$$\theta_{X \otimes Y} = \sigma_{Y,X} \circ \theta_Y \otimes \theta_X \circ \sigma_{X,Y}$$

Categorical Factorization Homology

Lurie, Ayala-Francis, Ayala-Francis-Tanaka



Factorization homology $\int_\bullet \mathcal{A}$ with coefficients in a framed E_2-algebra $\mathcal{A}$ is the left Kan extension

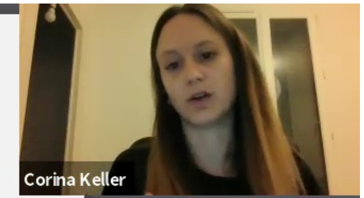
$$\begin{array}{ccc} \mathrm{Disk}^{2,\sqcup}_{\mathrm{or}} & \xrightarrow{\mathcal{A}} & \mathrm{Pr}^{\boxtimes}_c \\ \downarrow & \nearrow_{\int_\bullet \mathcal{A}} & \\ \mathrm{Man}^{2,\sqcup}_{\mathrm{or}} & & \end{array}$$

$$\int_M \mathcal{A} = \underset{\mathbb{D}^{\sqcup n} \to M}{\mathrm{colim}} \, \mathcal{A}(\mathbb{D}^{\sqcup n})$$

Theorem (Ayala-Francis '15)

The functor $\int_\bullet \mathcal{A}$ satisfies (and is characterized by)

- $\int_{\mathbb{D}} \mathcal{A} \cong \mathcal{A}$
- If $M \cong N \times (-1,1)$, then $\int_M \mathcal{A}$ is E_1-algebra in Pr_c
- $\int_\bullet \mathcal{A}$ satisfies $\boxtimes$-excision



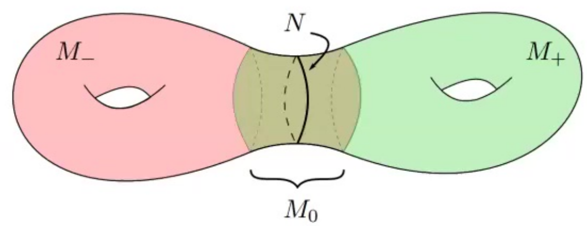
Categorical Factorization Homology

⊠-Excision

Theorem (Ayala-Francis '15 - ⊠-excision)

Factorization homology satisfies excision:

$$\int_{M_- \cup_{M_0} M_+} \mathcal{A} \cong \int_{M_-} \mathcal{A} \underset{\int_{M_0} \mathcal{A}}{\boxtimes} \int_{M_+} \mathcal{A}$$



Relative tensor product $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ is the colimit in Pr_c of the diagram:

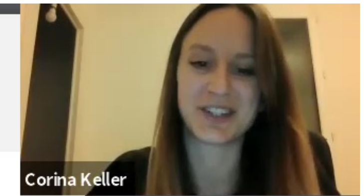
$$\mathcal{M} \boxtimes \mathcal{C} \boxtimes \mathcal{C} \boxtimes \mathcal{N} \rightrightarrows\!\!\!\!\rightarrow \mathcal{M} \boxtimes \mathcal{C} \boxtimes \mathcal{N} \rightrightarrows \mathcal{M} \boxtimes \mathcal{N}$$

D-Structured Categorical Factorization Homology

Setup

- D: Finite group
- $D\text{-}\mathrm{Man}^{2,\sqcup}_{\mathrm{or}}$: (2,1)-category of manifolds with D-bundles
  - **Objects**: $(\Sigma, \varphi\colon \Sigma \to BD)$
  - **1-Morphisms**: $(\Sigma, \varphi) \xrightarrow{f,h} (\Sigma', \varphi')$, with $f\colon \Sigma \to \Sigma'$ an embedding and $h\colon \varphi \to f^*\varphi'$ a homotopy
  - **2-Morphisms**: Equivalence classes of isotopies between embeddings and compatible paths of homotopies

D-Structured Categorical Factorization Homology



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Ayala-Francis '15

D-structured factorization homology

$$\begin{array}{ccc} D\text{-}\mathrm{Disk}_{\mathrm{or}}^{2,\sqcup} & \xrightarrow{\ \mathcal{A}\ } & \mathrm{Pr}_c^{\boxtimes} \\ \downarrow & \nearrow_{\int_\bullet \mathcal{A}} & \\ D\text{-}\mathrm{Man}_{\mathrm{or}}^{2,\sqcup} & & \end{array}$$

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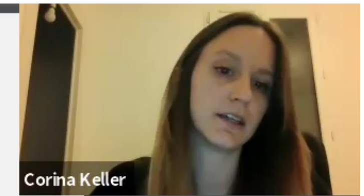


D-Structured Categorical Factorization Homology

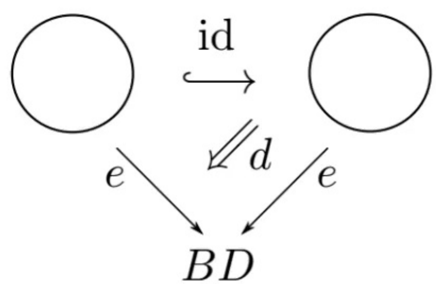
Disk algebras with D-action

Weelinck '20

A $D\text{-}\mathrm{Disk}^{2,\sqcup}_{\mathrm{or}}$-algebra in Pr_c is a balanced braided tensor category $\mathcal{A}$ with a D-action: There are balanced braided automorphisms

$$\vartheta(d)\colon \mathcal{A} \to \mathcal{A}, \quad \vartheta(d)\vartheta(d') \cong \vartheta(dd')$$

for all $d \in D$



$$\rightsquigarrow \quad \vartheta(d)\colon \mathcal{A} \longrightarrow \mathcal{A}$$

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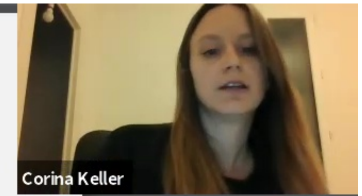
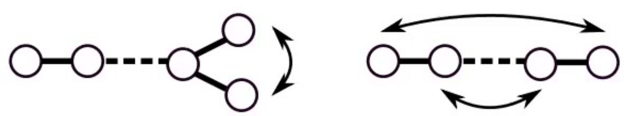
D-Structured Categorical Factorization Homology

Example: $\mathrm{Out}(G)$-action on representation categories of quantum groups



- $\mathrm{Out}(G)$: Outer automorphisms of G
- Action of $\kappa \in \mathrm{Out}(G)$ on $\mathrm{Rep}(G)$:
$$V \in \mathrm{Rep}(G) \mapsto (\kappa^{-1})^* V$$
$\rightsquigarrow$ Extend this action to representations of quantum groups
- Outer automorphisms are classified by Dynkin diagram automorphisms
  - I: set of nodes of Dynkin diagram
  - $(a_{ij})_{i,j \in I}$: Cartan matrix
  - $\kappa \colon I \to I$, such that $a_{ij} = a_{\kappa(i)\kappa(j)}$
- Examples: D_n $(n > 3)$ and A_n $(n > 1)$



D-Structured Categorical Factorization Homology

Example: $\mathrm{Out}(G)$-action on representation categories of quantum groups

Setup:

- $\mathfrak{g}$: finite-dimensional simple Lie algebra (type A_n, D_n or E_6)
- $\Pi = \{\alpha_1, \ldots, \alpha_n\}$: set of simple roots
- (a_{ij}): Cartan matrix

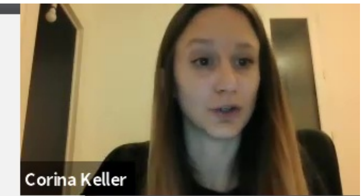
Formal quantum group $U_\hbar(\mathfrak{g})$:

- Hopf algebra deformation over $\mathbb{C}[[\hbar]]$ of $U(\mathfrak{g})$
- Generators: $\{H_{\alpha_i}, X^{\pm}_{\alpha_i}\}_{\alpha_i \in \Pi}$
- Action of $\kappa \in \mathrm{Aut}(\Pi)$ on $U_\hbar(\mathfrak{g})$:

$$H_{\alpha_i} \mapsto H_{\alpha_{\kappa(i)}}, \quad X^{\pm}_{\alpha_i} \mapsto X^{\pm}_{\alpha_{\kappa(i)}}$$

D-Structured Categorical Factorization Homology

Example: $\mathrm{Out}(G)$-action on representation categories of quantum groups



$\mathrm{Rep}_\hbar(G)$: Braided monoidal category with braiding coming from the universal R-matrix $\mathcal{R}_\hbar \in U_\hbar(\mathfrak{g})\hat{\otimes}U_\hbar(\mathfrak{g})$

Example: $U_\hbar(\mathfrak{sl}_3)$

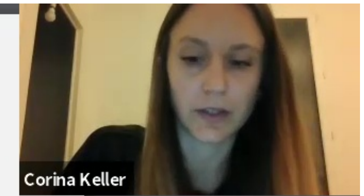
- Fix normal ordering on positive roots $\beta_1 = \alpha_1 \prec \beta_2 = \alpha_1 + \alpha_2 \prec \beta_3 = \alpha_2 \rightsquigarrow$ Root vectors $X^{\pm}_{\beta_i}$
- Universal R-matrix:

$$\mathcal{R}_\hbar = \prod_{\alpha_i \in \Pi} e^{\hbar a_{ij}^{-1} H_{\alpha_i} \otimes H_{\alpha_j}} \prod_{\beta_1 \prec \beta_2 \prec \beta_3} \exp_q((1-q^{-2})X^+_{\beta_i} \otimes X^-_{\beta_i}) = \Omega\widehat{\mathcal{R}}$$

- Dynkin diagram automorphism: 

D-Structured Categorical Factorization Homology

Example: $\mathrm{Out}(G)$-action on representation categories of quantum groups



Example continued...

$$\mathcal{R}_\hbar = \prod_{\alpha_i \in \Pi} e^{\hbar a_{ij}^{-1} H_{\alpha_i} \otimes H_{\alpha_j}} \prod_{\beta_1 \prec \beta_2 \prec \beta_3} \exp_q((1-q^{-2}) X_{\beta_i}^+ \otimes X_{\beta_i}^-)$$

Action of automorphism on $\mathcal{R}_\hbar$:

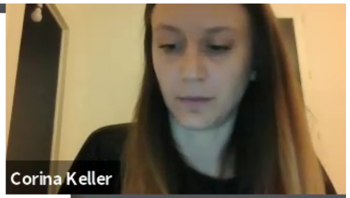
- Ω is invariant since $a_{ij} = a_{\kappa(i)\kappa(j)}$
- Action on $\widehat{\mathcal{R}}$ corresponds to change of normal order

$$\alpha_1 \prec \alpha_1 + \alpha_2 \prec \alpha_2 \Leftrightarrow \alpha_2 \prec \alpha_1 + \alpha_2 \prec \alpha_1$$

- $\mathcal{R}_\hbar$ is independent of chosen normal ordering of positive roots

Proposition

The balanced braided tensor category $\mathrm{Rep}_\hbar(\mathfrak{g})$ admits a left $\mathrm{Out}(G)$-action. Similarly, $\mathrm{Rep}_q(G)$ has a left $\mathrm{Out}(G)$-action.



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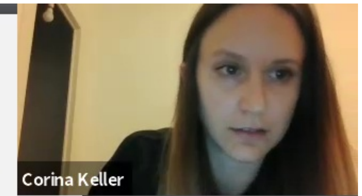


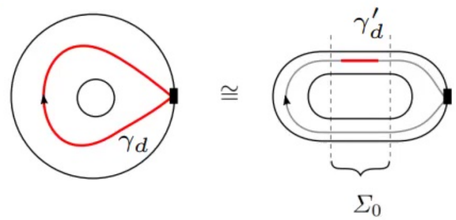
Computations on Surfaces with Boundary

Annulus with D-bundle

Excision:

- Undecorated case: $\int_{\mathbb{A}\mathrm{nn}} \mathcal{A} \cong \mathcal{A} \boxtimes_{\mathcal{A} \boxtimes \mathcal{A}} \mathcal{A}$
- D-Decorated case:

$\mathbb{A}\mathrm{nn}^d =$



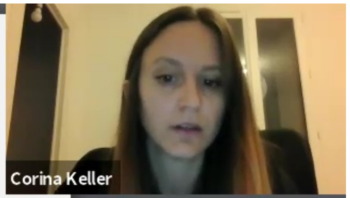
$$\phi \colon (\Sigma_0, \gamma'_d) \cong (N \times (-1,1), *) \text{ in } D\text{-}\mathrm{Man}^2$$

$\rightsquigarrow$ Compatibility of ϕ and excision leads to twisting of module structure:

$$\int_{\mathbb{A}\mathrm{nn}^d} \mathcal{A} \cong \mathcal{A}_{\langle d \rangle} \boxtimes_{\mathcal{A} \boxtimes \mathcal{A}} \mathcal{A}$$

Reconstruction for Module Categories

Ostrik, BZBJ



- $\mathcal{A}$: Rigid abelian tensor category
- $\mathcal{M}$: Abelian category with action $\mathcal{M} \boxtimes \mathcal{A} \xrightarrow{\text{act}} \mathcal{M}$ in Pr_c. For each $m \in \mathcal{M}$:

$$\mathcal{A} \xrightarrow{\text{act}_m} \mathcal{M}, \quad a \mapsto \text{act}(m \boxtimes a)$$

Get functor: $\mathcal{M} \xrightarrow{\widetilde{\text{act}_m^R}} \text{act}_m^R \circ \text{act}_m\text{-}\mathrm{Mod}_{\mathcal{A}}, \quad X \mapsto \text{act}_m^R(X)$

Theorem (BZBJ - based on Barr-Beck theorem)

Let $m \in \mathcal{M}$ be such that act_m^R is faithful and colimit preserving

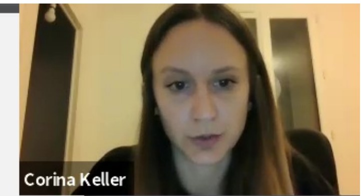
$$\mathcal{M} \xrightarrow{\widetilde{\text{act}_m^R}} \underline{\mathrm{End}}_{\mathcal{A}}(m)\text{-}\mathrm{Mod}_{\mathcal{A}}$$

is an equivalence of $\mathcal{A}$-module categories, where

$\underline{\mathrm{End}}_{\mathcal{A}}(m) := \text{act}_m^R \, \text{act}_m(1_{\mathcal{A}})$

Computations on Surfaces with Boundary

Annulus with D-bundle



Reconstruction:

Have $\mathcal{A}$-module functor

$$G\colon \mathcal{A} \to \mathcal{A}_{\langle d \rangle} \boxtimes_{\mathcal{A}\boxtimes\mathcal{A}} \mathcal{A}, \quad a \mapsto 1_{\mathcal{A}} \boxtimes a$$

with right adjoint

$$m \boxtimes a \xrightarrow{G^R} (\mathrm{reg}^d_{1_{\mathcal{A}}})^R(m) \triangleright a$$

Twisted regular action

$$\mathrm{reg}^d\colon \mathcal{A}_{\langle d \rangle} \boxtimes \mathcal{A} \boxtimes \mathcal{A} \xrightarrow{\mathrm{id}\boxtimes\mathrm{id}\boxtimes\vartheta(d)} \mathcal{A}_{\langle d \rangle} \boxtimes \mathcal{A} \boxtimes \mathcal{A} \xrightarrow{T} \mathcal{A}_{\langle d \rangle}$$

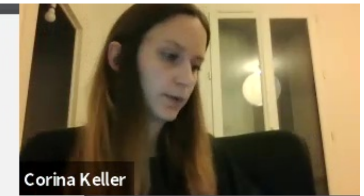
K-Müller (following BZBJ)

We have an equivalence

$$\int_{\mathbb{A}\mathrm{nn}^d} \mathcal{A} \cong T(\underline{\mathrm{End}}^d(1_{\mathcal{A}}))\text{-}\mathrm{Mod}_{\mathcal{A}}$$

Computations on Surfaces with Boundary

Annulus with D-bundle



Twisted coend algebra:

$$\mathfrak{F}_{\mathcal{A}}^{d} := T(\underline{\mathrm{End}}^{d}(1_{\mathcal{A}})) \cong \int^{\mathrm{comp}(\mathcal{A})} V^{\vee} \otimes \vartheta(d^{-1})(V)$$

Example:

- $\mathcal{A} = H\text{-}\mathrm{Mod}_{lf}$ for H a ribbon Hopf algebra with D-action
- $\mathfrak{F}_{\mathcal{A}}^{e}$ is REA
- $\mathfrak{F}_{\mathcal{A}}^{d} = \int^{V \in \mathrm{comp}(\mathcal{A})} V^{\vee} \otimes d^{*}V \cong (H^{\circ}, \mathrm{ad}_{d}^{*})$ with twisted multiplication:

$$m^{d}(\psi \otimes \phi) = \psi(\mathcal{R}_1(-) d.\mathcal{R}_1') \cdot \phi(S(\mathcal{R}_2')\mathcal{R}_2(-))$$

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Computations on Surfaces with Boundary

Annulus with D-bundle

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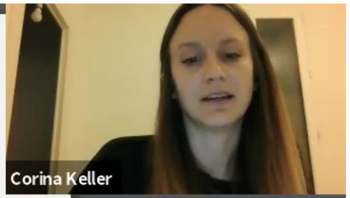
$$\int_{\mathbb{A}\mathrm{nn}^d} \mathcal{A} \cong \mathfrak{F}^d_{\mathcal{A}}\text{-}\mathrm{Mod}_{\mathcal{A}}$$

Corina Keller

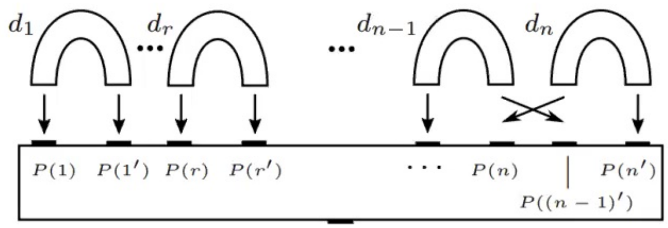


Computations on Surfaces with Boundary

General punctured surfaces with D-bundles

Combinatorial description of surface with D-bundle:

- Ribbon graph model for $\Sigma \Leftrightarrow$ Gluing pattern $P\colon \{1, 1', \ldots, n, n'\} \xrightarrow{\cong} \{1, 2, \ldots, 2n\}, P(i) < P(i')$
- Group elements $d_1, \ldots, d_n$, one for each edge of the graph



$\rightsquigarrow$ Attach algebra in $\mathcal{A}$ to this combinatorial data:

$$a_P^{d_1, \ldots, d_n} := \bigotimes_{i=1}^{n} \mathfrak{F}_{\mathcal{A}}^{d_i}$$

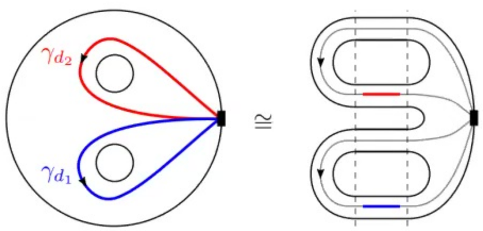
Computations on Surfaces with Boundary

General punctured surfaces with D-bundles

Algebra $a_P^{d_1,\dots,d_n}$

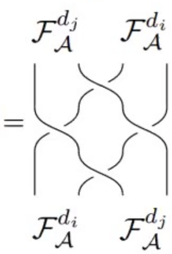
- $a_P^{d_1,\dots,d_n} := \bigotimes_{i=1}^n \mathfrak{F}_\mathcal{A}^{d_i}$ as object in $\mathcal{A}$
- Cross-relations determined by gluing pattern
- Example: $\Sigma = \Sigma_{0,3}$, $P(1,1',2,2') = (1,2,3,4)$

$a_P^{d_1,d_2} = \mathfrak{F}_\mathcal{A}^{d_1} \otimes \mathfrak{F}_\mathcal{A}^{d_2}$



Multiplication:

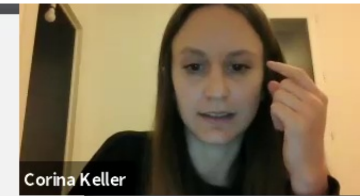
$C =$



$$\mathfrak{F}_\mathcal{A}^{d_1} \otimes \mathfrak{F}_\mathcal{A}^{d_2} \otimes \mathfrak{F}_\mathcal{A}^{d_1} \otimes \mathfrak{F}_\mathcal{A}^{d_2} \xrightarrow{\mathrm{id}\otimes C\otimes \mathrm{id}} (\mathfrak{F}_\mathcal{A}^{d_1})^{\otimes 2} \otimes (\mathfrak{F}_\mathcal{A}^{d_2})^{\otimes 2} \xrightarrow{m\otimes m} \mathfrak{F}_\mathcal{A}^{d_1} \otimes \mathfrak{F}_\mathcal{A}^{d_2}$$

Computations on Surfaces with Boundary

General punctured surfaces with D-bundles



Theorem (K-Müller)

Let Σ be a surface with at least one boundary component. Fix a principal D-bundle $\varphi\colon \Sigma \to BD$ on Σ and a corresponding decorated ribbon graph $(P, d_1, \ldots, d_n)$. We have an equivalence of categories

$$\int_{\varphi\colon \Sigma \to BD} \mathcal{A} \cong a_P^{d_1,\ldots,d_n}\text{-}\mathrm{Mod}_{\mathcal{A}} \quad .$$

1. Introduction and Background on Factorization Homology

2. Computation of D-Structured Factorization Homology

2.1 Annuli with D-bundles

2.2 Punctured Surfaces with D-bundles

3. Quantization of $\mathrm{Out}(G)$-Twisted Character Varieties

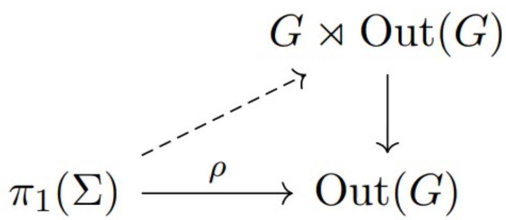
$\mathrm{Out}(G)$-Twisted Representation Variety

Meinrenken, Müller-Szabo-Szegedy

Setup

- $\Sigma = \Sigma_{g,r}$, $r > 0$
- $\rho\colon \pi_1(\Sigma) \to \mathrm{Out}(G) \Leftrightarrow (\kappa_1, \ldots, \kappa_n) \in \mathrm{Out}(G)^{2g+r-1}$

ρ-twisted G-representation variety:



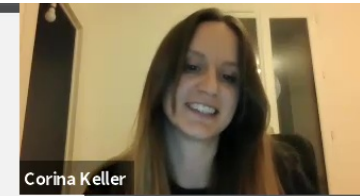
Definition

An element in the ρ-twisted representation variety is a map of sets $\varphi\colon \pi_1(\Sigma) \to G$ such that $\varphi(\gamma_1 \circ \gamma_2) = \varphi(\gamma_1)\rho(\gamma_1).\varphi(\gamma_2)$

Twisted conjugation action: $\varphi(\gamma) \mapsto g\varphi(\gamma)\rho(\gamma).g^{-1}$

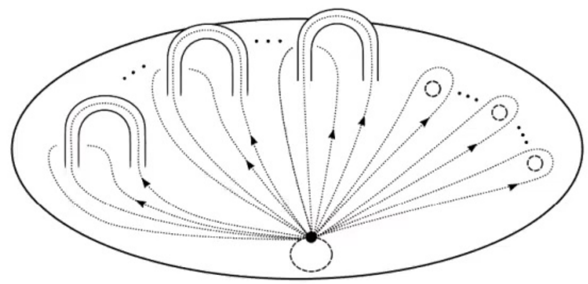
Poisson Structure on G-Representation Variety

Lattice gauge theory à la Fock and Rosly



Input:

- Graph model (v, E) for Σ
- Graph connection
  $A \colon E \to G$
- Set of graph connections
  $\mathcal{A} \cong G^{2g+r-1}$
- Lattice gauge group $\mathcal{G} = G$
- Classical r-matrix



Get: Fock-Rosly Poisson structure on G-representation variety, such that $\mathcal{G}$-action is Poisson-Lie

$\rightsquigarrow$ FR-Poisson structure on $\mathcal{A}/\mathcal{G}$

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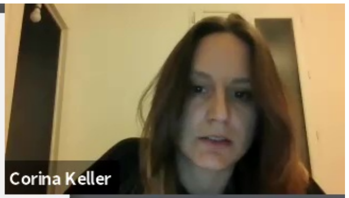
Out(G)-Twisted Representation Variety

Twisted Fock-Rosly Poisson structure



Proposition (K-Müller)

Fix an Out(G)-bundle on Σ, described by $\underline{\kappa} = (\kappa_1, \ldots, \kappa_n)$. The bivector field

$$\pi = \sum_{\alpha < \beta \in \bar{E}} r^{ij} x_i(\alpha) \wedge x_j(\beta) + \frac{1}{2} \sum_{\alpha \in \bar{E}} r^{ij} x_i(\alpha) \wedge x_j(\beta)$$

where

$$x(\alpha) = \begin{cases} -x^R(\alpha) \text{ if } \alpha \text{ incoming} \\ (\kappa_\alpha)_* x^L(\alpha) \text{ if } \alpha \text{ outgoing} \end{cases}$$

is a Poisson structure on the $\underline{\kappa}$-twisted representation variety and the twisted conjugation action is Poisson-Lie

Quantization

Algebra defined via $\underline{\kappa}$-decorated gluing pattern

$$a_\hbar^{\kappa_1,\dots,\kappa_n} = \otimes_{i=1}^n \mathcal{O}_\hbar^{\kappa_i}(G) \in \mathrm{Rep}_\hbar(G),$$

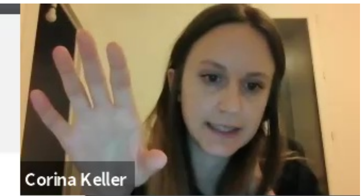
- Want to show:

$$\frac{[f_\hbar^{\kappa_i}, g_\hbar^{\kappa_j}]}{\hbar} \mathrm{mod}(\hbar) = \{f^{\kappa_i}, g^{\kappa_j}\}$$

  for $f_\hbar^{\kappa_i} \in \mathcal{O}_\hbar^{\kappa_i}(G)$ and $g_\hbar^{\kappa_j} \in \mathcal{O}_\hbar^{\kappa_j}(G)$

- $\mathcal{R} = 1 + \hbar r + \mathcal{O}(\hbar^2)$
- $r = \omega + t \in \mathfrak{g} \otimes \mathfrak{g}$: classical r-matrix

Quantization



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- $\mathcal{R} = 1 + \hbar r + \mathcal{O}(\hbar^2)$
- $r = \omega + t \in \mathfrak{g} \otimes \mathfrak{g}$: classical r-matrix

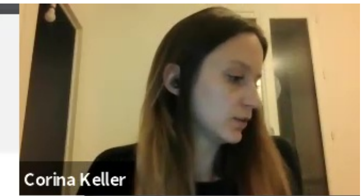
Example: $\mathbb{A}\mathrm{nn}^\kappa$

Twisted STS-Poisson structure

$\pi_{STS}^\kappa = \omega^{\mathrm{ad}(\kappa),\mathrm{ad}(\kappa)} + t^{R,L(\kappa)} - t^{L(\kappa),R}$

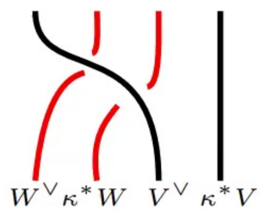
with $x^{\mathrm{ad}(\kappa)} = x^R - \kappa_* x^L$

Quantization



Example continued...

$\pi_{Drin} = \omega^{L,L} - \omega^{R,R}$



$$\pi_{Drin} \rightsquigarrow \pi_{Drin} + r^{R,R} - r^{L(\kappa),R}$$

$$\rightsquigarrow \underbrace{\pi_{Drin} + r^{R,R} - r^{L(\kappa),R} - r_{2,1}^{R,R} + r_{2,1}^{R,L(\kappa)}}_{=\pi^{\kappa}_{STS}}$$

Theorem (K-Müller)

The algebra $a_{\hbar}^{\kappa_1,\ldots,\kappa_n}$ is a deformation quantization of the twisted Fock-Rosly Poisson structure

Questions?