Title: Sequential Discontinuities of Scattering Amplitudes

Speakers: Hofie Sigridar Hannesdottir

Series: Quantum Fields and Strings

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Abstract: In this talk, we explore new techniques to probe the analytic structure of scattering amplitudes in perturbative Quantum Field Theory. The goal of this approach is to leverage symmetries, limits, and analyticity in order to circumvent the explicit evaluation of multi-loop Feynman integrals. First, we generalize the cutting rules by relating sequential discontinuities (discontinuities of discontinuities) to multiple cuts through the corresponding Feynman diagram. Then, we determine the logarithmic branch-cut structure of massive scattering amplitudes by expanding around their branch points. As a corollary, we prove a conjectured bound on the transcendental weight of polylogarithmic Feynman integrals. These results pave a new path towards bootstrapping scattering amplitudes in perturbation theory.

Sequential Discontinuities of Scattering Amplitudes

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October 12, 2021

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arXiv: 2007.13747, 2109.09744

Outline

1. Introduction and motivation
2. Discontinuities and cuts
3. Two different cutting prescriptions
4. Branch-cut structure
5. Conclusions

Scattering Amplitudes $\mathcal{M}$

- $\mathcal{M} \sim \langle f|\, S\, |i\rangle$: Probability amplitude for measuring a final state $|f\rangle$ given $|i\rangle$.
- Properties extensively studied.
  - How to **encode their content**? *Spinors, twistors, amplituhedron?*
  - What are their **symmetries**? *Dual conformal invariance, Steinmann relations?*
  - What **functional forms** can they take? *Logarithms, polylogarithms?*
- Feynman-diagram calculations challenging at >1 loop.

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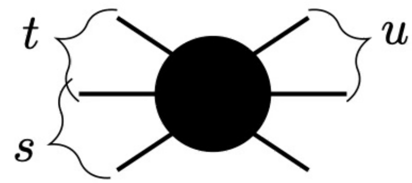
- Feynman-diagram calculations challenging at >1 loop.

Exploit constraints to calculate $\mathcal{M}$ more efficiently?

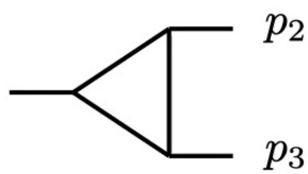
Analytic Structure of $\mathcal{M}$

- Leverage e.g. symmetries, limits and **analyticity**.



- $\mathcal{M}$ depends on Mandelstams $p_1^2, p_2^2, \ldots, s, t, u, \ldots$
- Results in logarithms, polylogarithms, ... with **branch cuts**.
- All propagators have $+i\varepsilon$'s.
- Here: Learn from discontinuities and **sequential discontinuities**.

Example of Analytic Structure of $\mathcal{M}$

$\mathcal{M} =$ 

$$\propto \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \frac{1}{2}\ln(z\bar{z})\ln\left(\frac{1-z}{1-\bar{z}}\right)$$

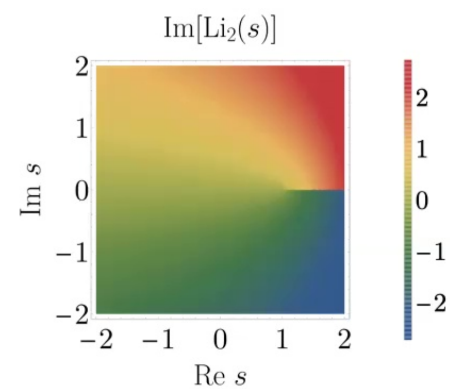
with $z\bar{z} = p_2^2/p_1^2, \quad (1-z)(1-\bar{z}) = p_3^2/p_1^2$

- Dilogarithm $\mathrm{Li}_2(z) = -\int_0^z \frac{\ln(1-s)}{s} ds$ has a branch point at $z = 1$:

$$\mathrm{Disc}_{z=1}\, \mathrm{Li}_2(z) = 2\pi i \int_1^z \frac{1}{s} ds = 2\pi i \ln(z)$$

- Logarithm $\ln(z) = \int_1^z \frac{1}{s} ds$ has a branch point at $z = 0$:

$$\mathrm{Disc}_{z=0}\, \ln(z) = 2\pi i$$



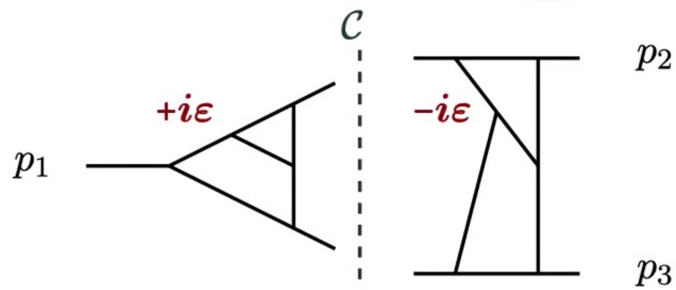
Example of Analytic Structure of $\mathcal{M}$

- Learn:
  - $\mathrm{Disc}_{z=1}\mathrm{Li}_2(z)$ exposes a new branch point at $z = 0$.
  - Useful information in $\mathrm{DiscDisc}\mathcal{M}$.
- Application:
  - Use analyticity information (e.g. transcendentality, Steinmann relations) to boostrap $\mathcal{M}$.
- Ultimate Goal:

  Apply analyticity constraints to avoid loop calculations in any theory.

Traditional Cutting Rules

$$\mathrm{Disc}\mathcal{M} = \mathcal{M}|_{+i\varepsilon} - \mathcal{M}|_{-i\varepsilon} = \sum \mathrm{Cut}\mathcal{M}$$



- L.h.s. of cut has $+i\varepsilon$, r.h.s. of cut has $-i\varepsilon$.
- Cuts put particles on shell with positive energy:

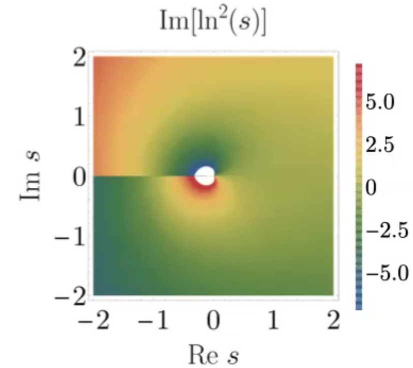
$$\frac{1}{k^2 - m^2 + i\varepsilon} \to -2\pi i \delta\left(k^2 - m^2\right)\Theta(k^0)$$

Extend to multiple discs & multiple cuts, how?

Problems with $i\varepsilon$ Definition, (1)

- $\text{Disc}\mathcal{M} = \mathcal{M}|_{+i\varepsilon} - \mathcal{M}|_{-i\varepsilon}$ **only** defined on the **branch cut**:

$$\text{Disc}_s \ln^2 s = \ln^2(s+i\varepsilon) - \ln^2(s-i\varepsilon) = 4\pi i\theta(-s)\ln|s$$



- What is the $i\varepsilon$ prescription of $\text{Disc}_s\mathcal{M}$?

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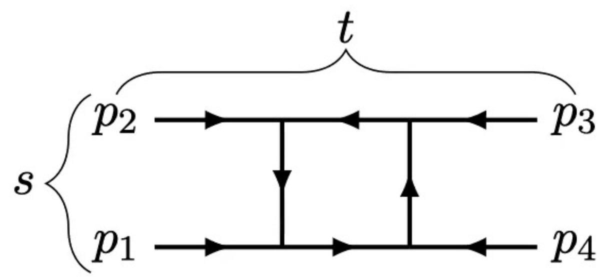
Problems with $i\varepsilon$ Definition, (2)

- Want to study Disc$\mathcal{M}$ in **each Mandelstam separately**



$$= \mathcal{M}\left(p_j^2, s, t, u\right)$$

- Intuitively: Define **discontinuity in a channel** s as

$$\text{Disc}_s\mathcal{M} = \mathcal{M}\left(p_j^2, s + i\varepsilon, t, u\right) - \mathcal{M}\left(p_j^2, s - i\varepsilon, t, u\right)$$

- **Problem:** Mandelstams are not all independent: $s + t + u = \sum p_j^2$.

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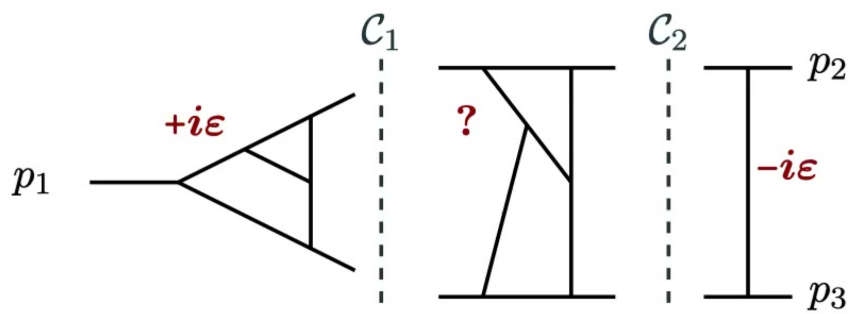
Problems with $i\varepsilon$ Definition, (3)

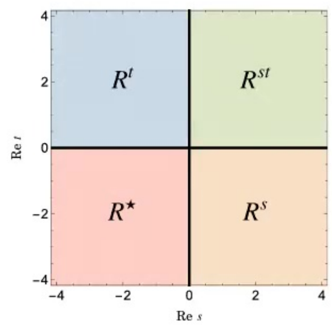
- For multiple cuts, how do we assign $\pm i\varepsilon$?

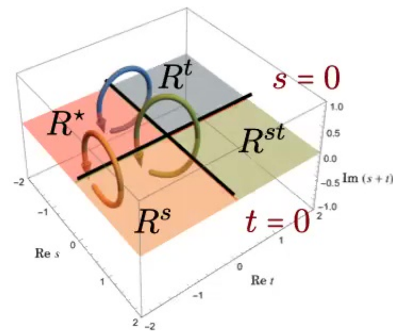


Definition of Discontinuity

Resolution: Abandon $\pm i\varepsilon$, take **monodromies**.

Definition: $\mathrm{Disc}_s\mathcal{M}$: **monodromy** of $\mathcal{M}$ around $s = 0$, starting in $\boldsymbol{R^s}$.

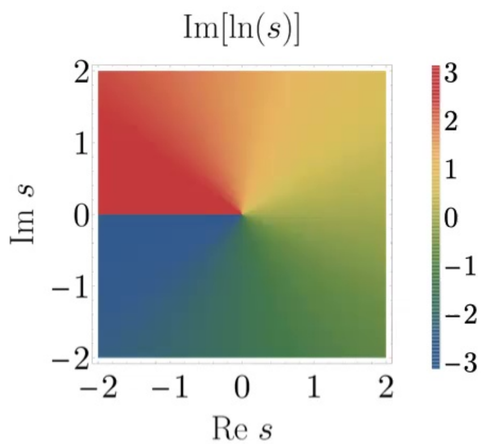




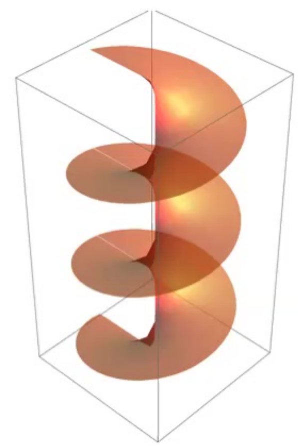
- **Monodromy:** Change when analytically continuing around a singularity.
- $\boldsymbol{R^s}$: Region in space of Mandelstams where $s > 0$, all other Mandelstams $s_{i,j,\ldots} < 0$.

DEFINITION OF DISCONTINUITY

Discontinuities with $\pm i\varepsilon$ only account for the **principal branch**:



Monodromies allow for maximal analytic continuation:



Definition of Discontinuity

$\text{Disc}_s\mathcal{M}$ is the **monodromy** of $\mathcal{M}$ around $s = 0$, starting in R^s.

- Agrees with the $i\varepsilon$ definition in R^s:

$$[\text{Disc}_s\mathcal{M}]_{R^s} = [\mathcal{M}|_{+i\varepsilon} - \mathcal{M}|_{-i\varepsilon}]_{R^s}$$

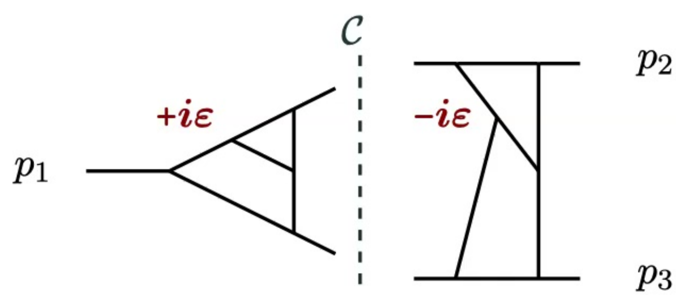
- Results in a **function on complex space**.

- Sequential discontinuities are **natural** and **algebraic**:

$$[\text{Disc}_s\text{Disc}_s\mathcal{M}]_{R^s} = \left[\left(\mathbb{1} - \mathscr{M}_{\circlearrowleft^s_0}\right)\left(\mathbb{1} - \mathscr{M}_{\circlearrowleft^s_0}\right)\mathcal{M}\right]_{R^s}$$

Traditional Cutting Rules

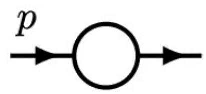
$$\text{Disc}\mathcal{M} = \mathcal{M}|_{+i\varepsilon} - \mathcal{M}|_{-i\varepsilon} = \sum \text{Cut}\mathcal{M}$$



Proofs:

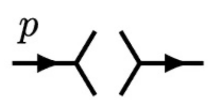
- Cutkosky, using the Landau equations.
- t'Hooft and Veltman, using the largest time equation.
- Time-ordered perturbation theory (TOPT).

Discontinuities, Monodromies and Cuts

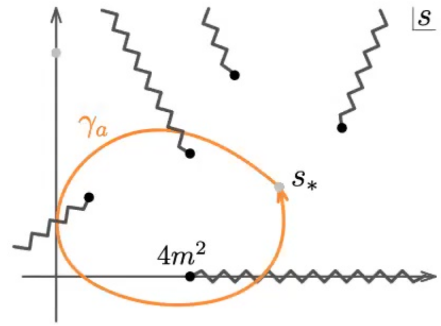
$\mathcal{M} =$  $\propto -\frac{i}{16\pi^2} \ln\left(-p^2 - i\varepsilon\right)$

$$\left(\mathbb{1} - \mathscr{M}_{\circlearrowleft_0^{p^2}}\right)\mathcal{M} \propto -\frac{i}{16\pi^2}(-2\pi i) = -\frac{1}{8\pi}$$

$$[\mathrm{Disc}\mathcal{M}]_{R^{p^2}} \propto -\frac{i}{16\pi^2}(-2\pi i)\,\Theta(p^2) = -\frac{1}{8\pi}\Theta(p^2)$$

$\mathrm{Cut}\mathcal{M} \propto$  $= -\frac{1}{8\pi}\Theta(p^2)$

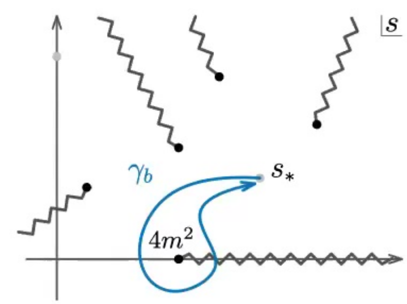
Relate DiscDisc$\mathcal{M}$ to cuts



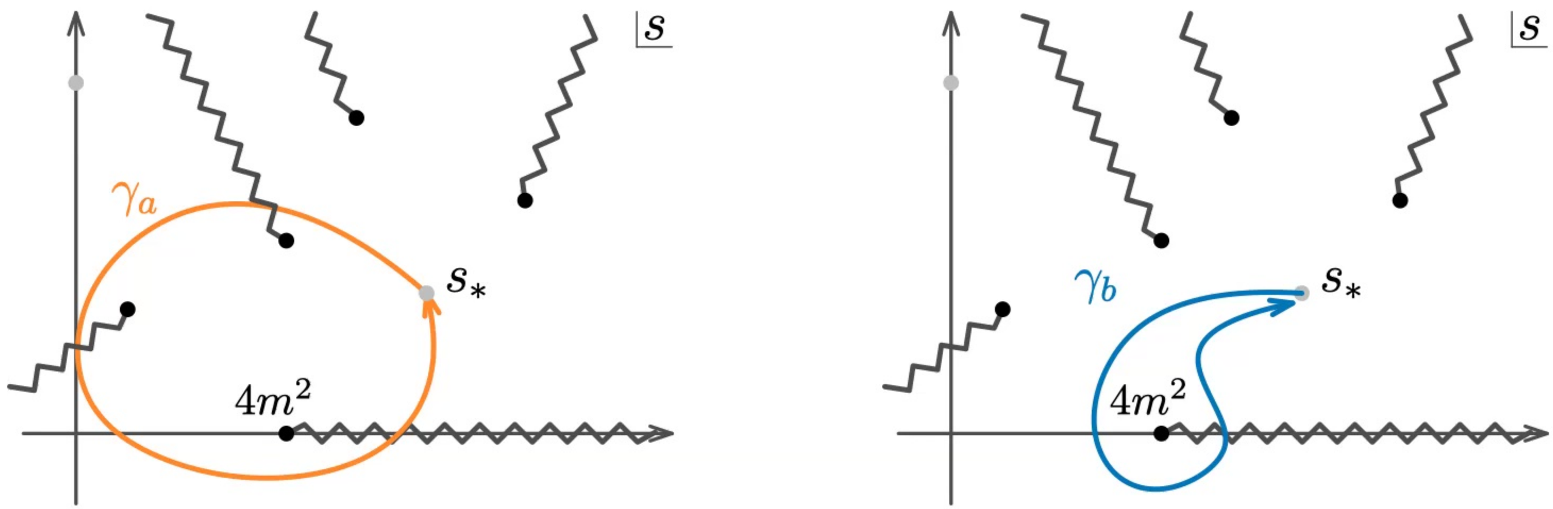
Two prescriptions:

a. Rotate all energies, encircle some branch points.
   - External particles massive, internal particles massive or massless.

b. Solve Landau equations which find branc points, encircle one at a time.
   - All particles massive.



RELATE DISCDISC$\mathcal{M}$ TO CUTS



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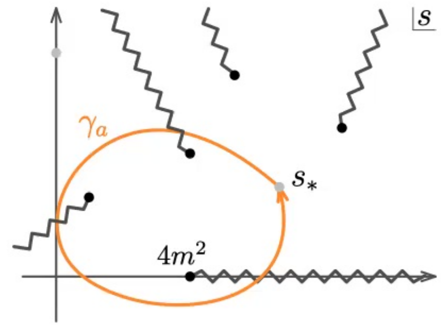

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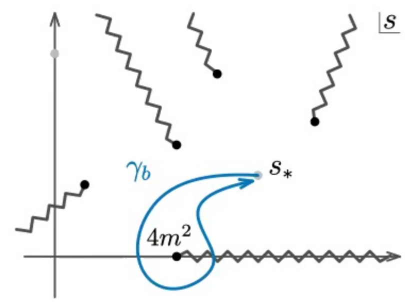

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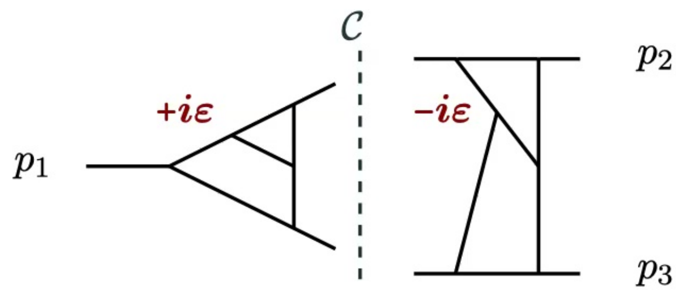

Proofs:

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- Time-ordered perturbation theory (TOPT).

3. Two different cutting prescriptions

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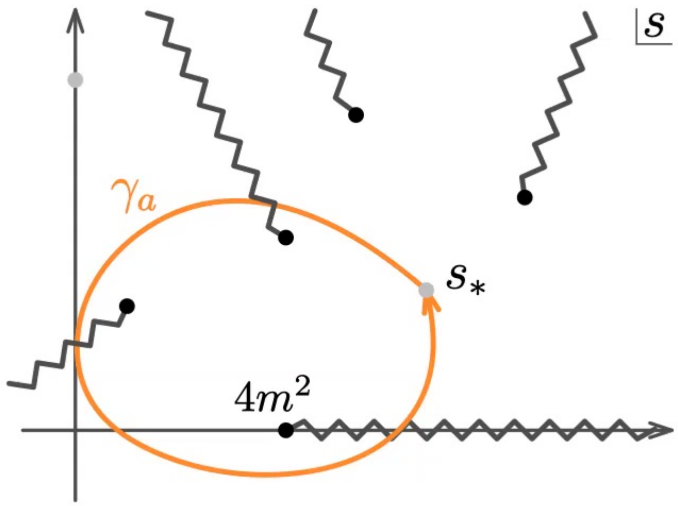
a. Rotate energies, encircle branch points



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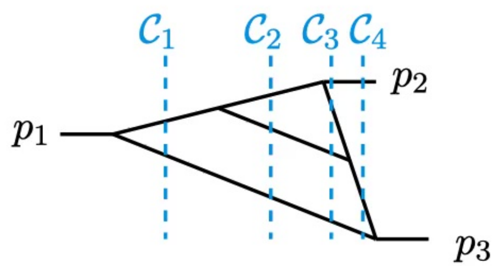
Cutting Rules in TOPT

- Energies are **independent**, Mandelstams are not.
- **One delta** function for **each cut**.
  - Various numbers of on-shell Feynman propagators for each cut.
  - Relate Disc$\mathcal{M}$ to cuts using $\frac{1}{E_i+i\varepsilon} - \frac{1}{E_i-i\varepsilon} = -2\pi i\delta\left(E_i\right)$



$$\mathcal{M}|_{+i\varepsilon} \propto \int \frac{1}{E_1-\omega_1+i\varepsilon}\frac{1}{E_1-\omega_2+i\varepsilon}\frac{1}{E_1-E_2-\omega_3+i\varepsilon}\frac{1}{E_1-E_2-\omega_4+i\varepsilon}$$

Results Relating Cuts and Discontinuities

$$C_{k,m} = \sum_{\ell=1}^{m} (-1)^{\ell} \binom{m}{\ell} (-\ell)^{k}$$

Same channel:

Sum of cut diagrams with a **combinatorial** factor.

$$[\mathrm{Disc}_s^m \mathcal{M}]_{R^s} = (\mathbb{1} - \mathscr{M}_{\leftrightarrows\bigcirc^{s}_{0}})^m \mathcal{M} = \sum_{k=m} C_{k,m} \left[\mathcal{M}_{k\text{-cuts}}\right]_{R^s_+}$$

$R_+^{\{s_i\}}$: $\mathcal{M}$ computed with all $+i\varepsilon$,

$\{s_i\} > 0$ while all other Mandelstams $\{s_j\} < 0$.

Results Relating Cuts and Discontinuities

$$C_{k,m} = \sum_{\ell=1}^{m} (-1)^{\ell} \binom{m}{\ell} (-\ell)^{k}$$

Same channel:

Sum of cut diagrams with a **combinatorial** factor.

$$\left[\mathrm{Disc}_s^m \mathcal{M}\right]_{R^s} = \left(\mathbb{1} - \mathscr{M}_{\leftrightarrows\bigcirc^{s}_{0}}\right)^m \mathcal{M} = \sum_{k=m} C_{k,m} \left[\mathcal{M}_{k\text{-cuts}}\right]_{R^s_+}$$

Different channels:

Sum of cut diagrams in a **region** $R^{\{s,t\}}$ where both cuts can be computed.

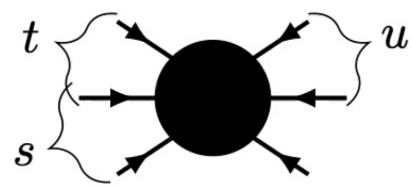
$$\left[\mathrm{Disc}_s \mathrm{Disc}_t \mathcal{M}\right]_{R^{\{s,t\}}} = \left(\mathbb{1} - \mathscr{M}_{\leftrightarrows\bigcirc^{t}_{0}}\right)\left(\mathbb{1} - \mathscr{M}_{\leftrightarrows\bigcirc^{s}_{0}}\right)\mathcal{M} = \sum_{k=1} \sum_{\ell=1} (-1)^{k+\ell} \left[\mathcal{M}_{\{k \text{ cuts in } s,\ \ell \text{ cuts in } t\}}\right]_{R_+^{\{s,t\}}}$$

$R_+^{\{s_i\}}$: $\mathcal{M}$ computed with all $+i\varepsilon$,

$\{s_i\} > 0$ while all other Mandelstams $\{s_j\} < 0$.

Steinmann Relations

$\mathcal{M}$ cannot have sequential discontinuities in partially overlapping channels



$\mathcal{M}$ cannot contain $\ln(s)\ln(t)$ but can contain $\ln(s)\ln(u)$.

STEINMANN RELATIONS

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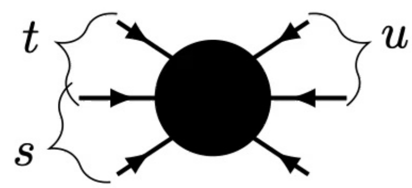
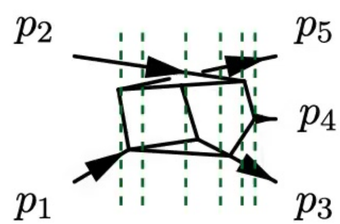


$\mathcal{M}$ cannot contain $\ln(s)\ln(t)$ but can contain $\ln(s)\ln(u)$.

- Important for **bootstrapping** amplitudes.
- Proof in S-matrix theory: *Non-perturbative, used unitarity.*
- Our new proof in TOPT: *Applies to* ***individual Feynman integrals***.

Proof of Steinmann Relations in TOPT

- TOPT denominators have a sequence of energies.

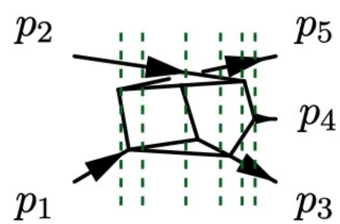


$$-E_5, \quad E_1–E_5, \quad E_1–E_5, \quad E_1–E_5, \quad E_1–E_5–E_3, \quad E_1–E_5–E_3+E_2$$
$$p_5^2, \quad (p_1-p_5)^2, \quad (p_1-p_5-p_3)^2, \quad (p_1-p_5-p_3+p_2)^2$$

- Each energy is a subset of the sequential ones.

Proof of Steinmann Relations in TOPT

- TOPT denominators have a sequence of energies.



$$-E_5, \quad E_1{-}E_5, \quad E_1{-}E_5, \quad E_1{-}E_5, \quad E_1{-}E_5{-}E_3, \quad E_1{-}E_5{-}E_3{+}E_2$$
$$p_5^2, \quad (p_1-p_5)^2, \quad (p_1-p_5-p_3)^2, \quad (p_1-p_5-p_3+p_2)^2$$

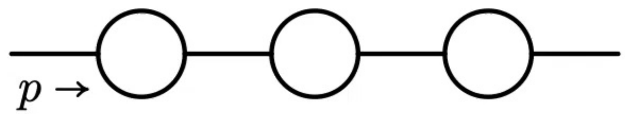
- Each energy is a subset of the sequential ones.

  No sequential discontinuities in partially overlapping channels

  - Regions may not exist when some particles are massless.
  - Cannot fix external masses to zero.

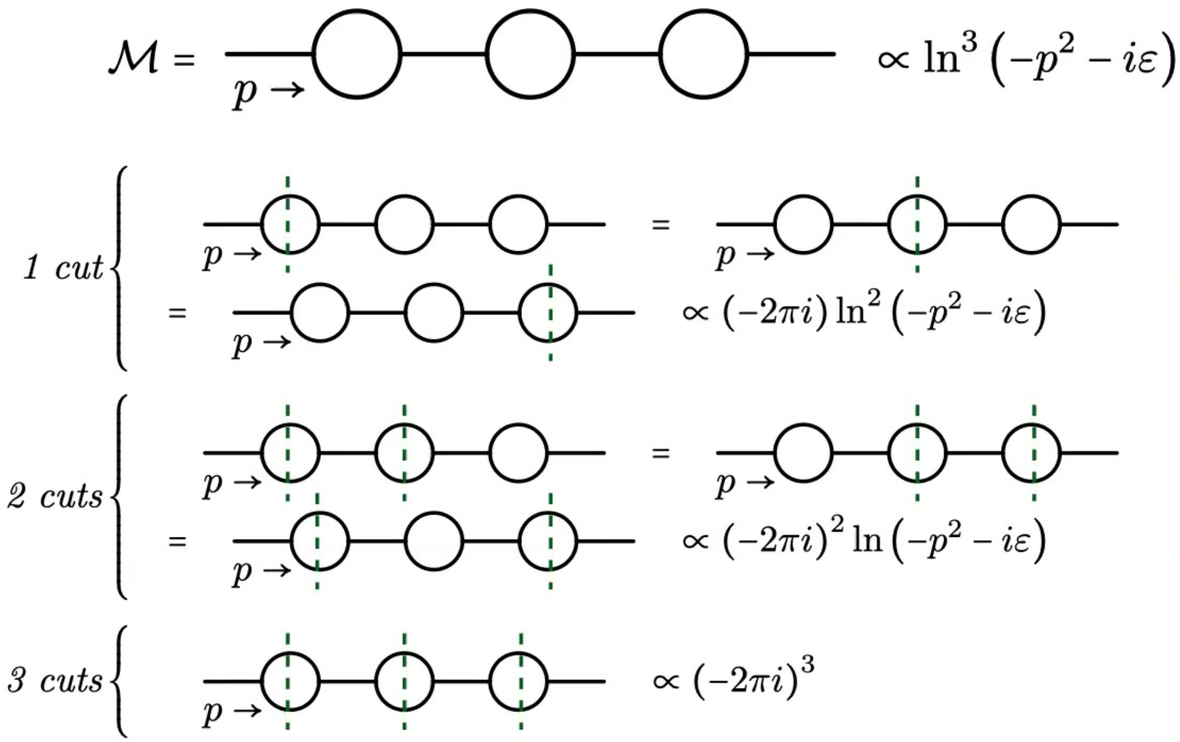
Example: Chain of Bubbles

$\mathcal{M} =$



$$\propto \ln^3\left(-p^2 - i\varepsilon\right)$$

EXAMPLE: CHAIN OF BUBBLES



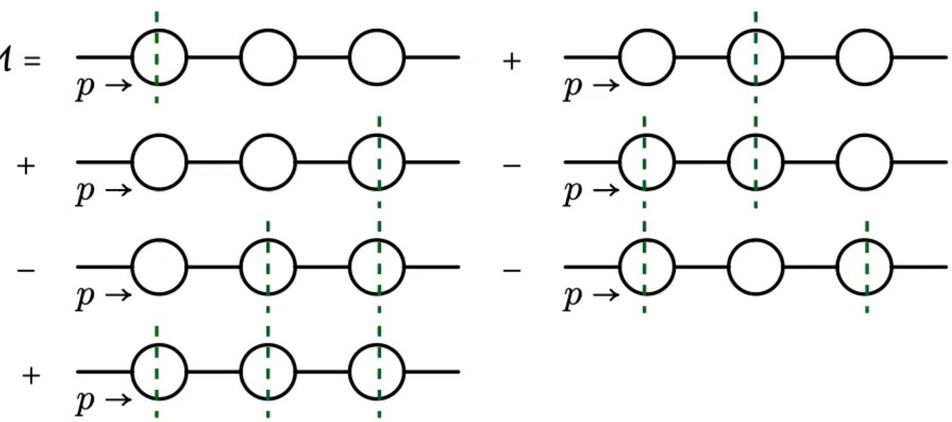
EXAMPLE: DISC$\mathcal{M}$ FOR CHAIN OF BUBBLES

- **Discontinuity** calculated using monodromy matrices:

$$\left[\mathrm{Disc}_{p^2}\mathcal{M}\right]_{R^{p^2}} \propto (-2\pi i)\ln^2\left(-p^2 - i\varepsilon\right) - 3(-2\pi i)^2 \ln\left(-p^2 - i\varepsilon\right) + (-2\pi i)^3$$

- **Cuts** calculated by putting particles on shell, using $+i\varepsilon$:

$\sum \mathrm{Cut}\mathcal{M} =$



- Agreement with formula:

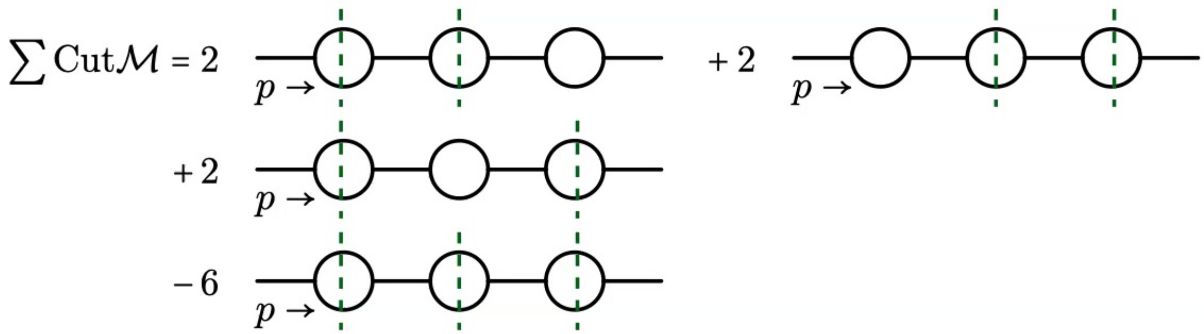
$$\left[\mathrm{Disc}_{p^2}\mathcal{M}\right]_{R^{p^2}} = \mathcal{M}^{(\text{1-cuts})} - \mathcal{M}^{(\text{2-cuts})} + \mathcal{M}^{(\text{3-cuts})}$$

Example: $\mathrm{Disc}^2\mathcal{M}$ for Chain of Bubbles

- **Discontinuity:**

$$\left[\mathrm{Disc}^2_{p^2}\mathcal{M}\right]_{R^{p^2}} \propto 6\,(-2\pi i)^2 \ln\left(-p^2 - i\varepsilon\right) - 6(-2\pi i)^3$$

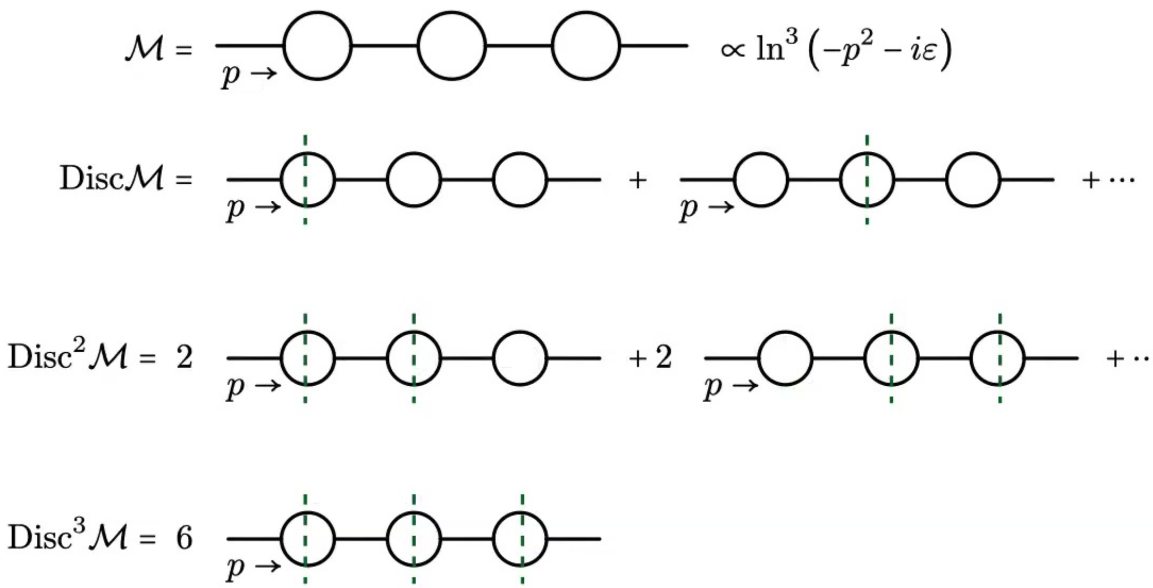
- **Cuts:**



- Agreement with formula:

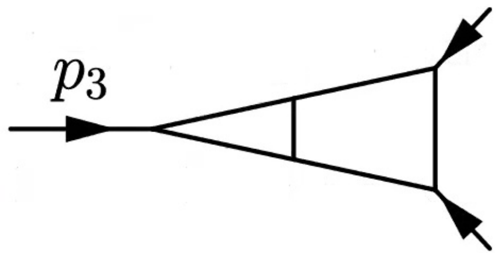
$$\left[\mathrm{Disc}^2_{p^2}\mathcal{M}\right]_{R^{p^2}} = 2\mathcal{M}^{(2\text{-cuts})} - 6\mathcal{M}^{(3\text{-cuts})}$$

Example: Chain of Bubbles, Summary



Discontinuities $= \sum$ **multiple cut** diagrams

Example: Two-loop Triangle



$$\mathcal{M} \propto 6[\mathrm{Li}_4(z) - \mathrm{Li}_4(\bar{z})] - 3\ln(z\bar{z})[\mathrm{Li}_3(z) - \mathrm{Li}_3(\bar{z})] + \frac{1}{2}\ln^2(z\bar{z})[\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z})]$$

Compare the following **discontinuities** and **cuts**:

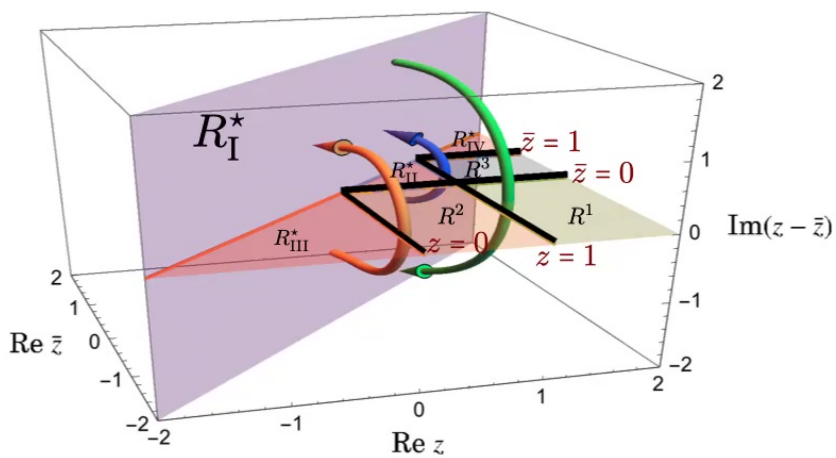
$$\mathrm{Disc}_{p_2^2}\mathrm{Disc}_{p_2^2}\mathcal{M} \qquad \mathrm{Disc}_{p_1^2}\mathrm{Disc}_{p_2^2}\mathcal{M}$$

$$z\bar{z} = \frac{p_2^2}{p_1^2}$$

$$(1-z)(1-\bar{z}) = \frac{p_3^2}{p_1^2}$$

Energy Rotations in z, $\bar{z}$ Plane

$$\mathcal{M} \propto 6[\mathrm{Li}_4(z) - \mathrm{Li}_4(\bar{z})] - 3\ln(z\bar{z})[\mathrm{Li}_3(z) - \mathrm{Li}_3(\bar{z})] + \frac{1}{2}\ln^2(z\bar{z})[\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z})]$$



$$z\bar{z} = \frac{p_2^2}{p_1^2}$$

$$(1 - z)(1 - \bar{z}) = \frac{p_3^2}{p_1^2}$$

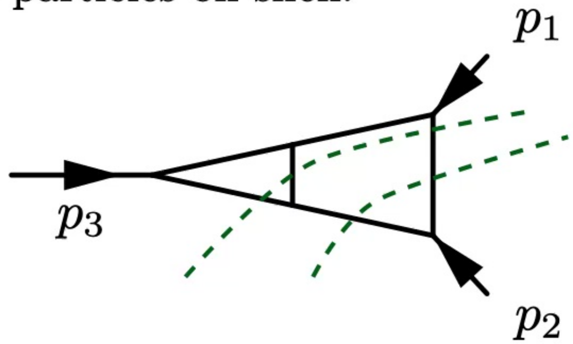
2-loop Triangle: Same Channel

- **Discontinuity** calculated using monodromy matrices:

$$\left[\mathrm{Disc}_{p_2^2}\mathrm{Disc}_{p_2^2}\mathcal{M}(z,\bar{z})\right]_{R^2} \propto \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z})$$

- **Cut** calculated by putting particles on shell:

$$\left[\mathcal{M}^{(2\text{-cuts})}\right]_{R^2} =$$



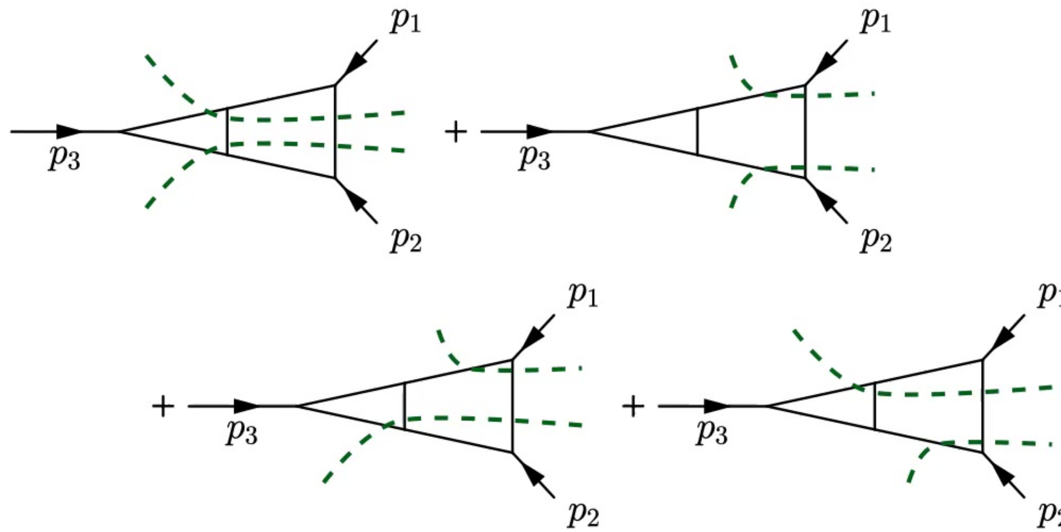
$$\propto \frac{\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z})}{2}$$

- As predicted, agree up to the **combinatorial factor**:

$$\boxed{\left[\mathrm{Disc}_{p_2^2}\mathrm{Disc}_{p_2^2}\mathcal{M}(z,\bar{z})\right]_{R^2} = 2\left[\mathcal{M}^{(2\text{-cuts})}\right]_{R^2}}$$

2-loop Triangle: Different Channel Cuts

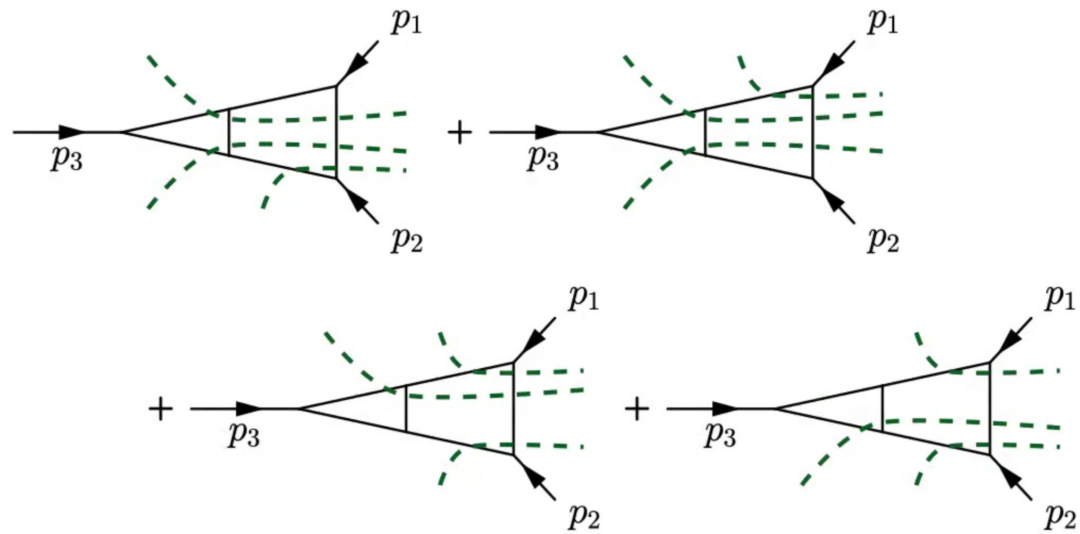
$\left[\mathcal{M}^{(2\text{-cuts})}\right]_{R^{12}} =$



$$\propto (2\pi i)^2 \left\{ \mathrm{Li}_2(\bar{z}) - \mathrm{Li}_2(z - i\varepsilon) - \frac{1}{2}\ln^2 z + \ln z \ln \bar{z} + i\pi \ln z - 2\pi i \ln \bar{z} \right\}$$

2-loop Triangle: Different Channel Cuts

$$\left[\mathcal{M}^{(3\text{-cuts})}\right]_{R^{12}} =$$



$$\propto (2\pi i)^3 \left\{\ln z - \ln \bar{z}\right\}$$

2-loop Triangle: Different Channels

- Discontinuity:

$$\left[\text{Disc}_{p_2^2}\text{Disc}_{p_1^2}\Phi_2\right]_{R^{12}} \propto (2\pi i)^2\Big\{\text{Li}_2(\bar{z}) - \text{Li}_2(z - i\varepsilon) - \frac{1}{2}\ln^2 z + \ln z \ln \bar{z} - i\pi \ln z\Big\}$$

- Cuts:

$$\mathcal{M}^{(2\text{-cuts})} \propto (2\pi i)^2\Big\{\text{Li}_2(\bar{z}) - \text{Li}_2(z - i\varepsilon) - \frac{1}{2}\ln^2 z + \ln z \ln \bar{z} + i\pi \ln z - 2\pi i \ln \bar{z}\Big\}$$

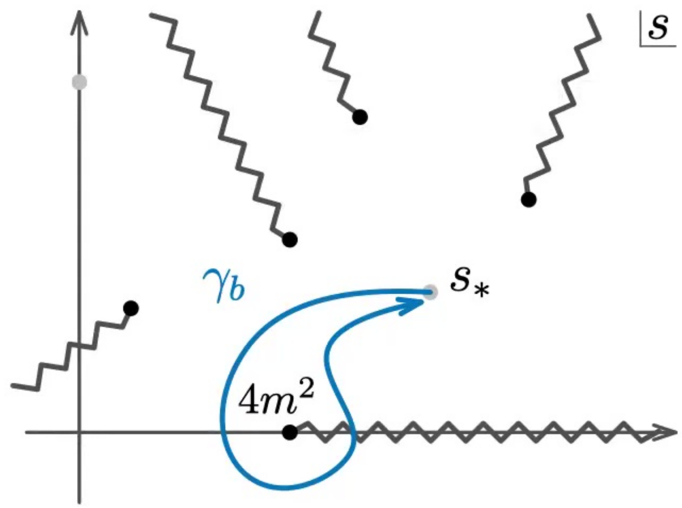
$$\mathcal{M}^{(3\text{-cuts})} \propto (2\pi i)^3 \{\ln z - \ln \bar{z}\}$$

- Agreement with formula:

$$\boxed{\left[\text{Disc}_{p_2^2}\text{Disc}_{p_1^2}\mathcal{M}_2\right]_{R^{12}} = \mathcal{M}^{(2\text{-cuts})} - \mathcal{M}^{(3\text{-cuts})}}$$

4. Branch-cut structure

b. Encircle individual branch points



b. Encircle individual branch points, procedure

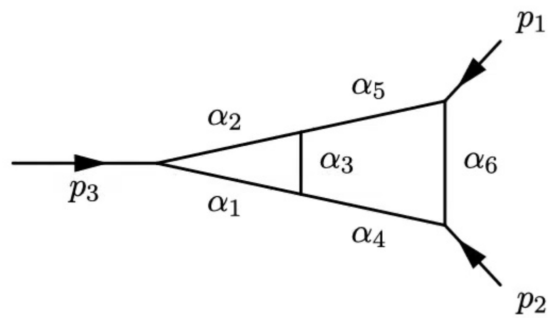
i. Find branch points using Landau equations.

ii. Compute monodromy using Cutkosky's formula.

iii. **What Riemann sheet is the branch point on?**

LANDAU EQUATIONS

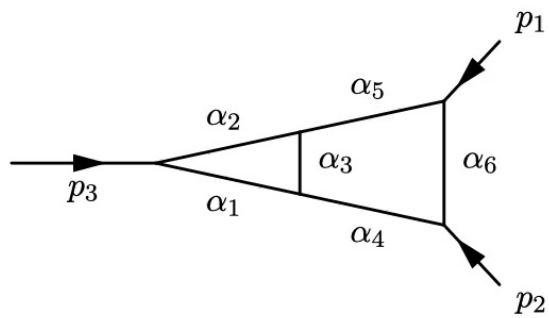
- Determine potential branch points of $\mathcal{M}$, at $\varphi_0 = 0$:



$$\mathcal{M} = \int d^{DL} k_i \frac{1}{\ell_1^2 - m_1^2 + i\varepsilon} \cdots \frac{1}{\ell_E^2 - m_E^2 + i\varepsilon}$$

Either $\alpha_i = 0$ or $\ell_i^2 = m_i^2$, & $\sum \pm\alpha_i \ell_i = 0$ around every loop.

Monodromy where cut can be computed



Cutkosky's formula gives monodromy around the branch point φ_0:

$$(1 - \mathscr{M}_{\varphi_0})\,(\qquad) = \int d^{DL}k_i \prod_{q \text{ with } \alpha_q \neq 0} (-2\pi i)\,\delta\left(\ell_q^2 - m_q^2\right)\Theta(\ell_q^0)$$

$$\times \prod_{p \text{ with } \alpha_p = 0} \frac{1}{\ell_q^2 - m_q^2 + i\varepsilon}$$

WHAT SHEET IS THE BRANCH POINT ON?

- Feynman parametrize $\mathcal{M}$ and expand around φ_0:

$$\mathcal{M}(\varphi_0) \sim \begin{cases} C\varphi_0^\gamma \log\varphi_0 & \text{if } \gamma \in \mathbb{Z}, \gamma \geq 0 \\ C\varphi_0^\gamma & \text{otherwise.} \end{cases}$$

with

$$\gamma = \frac{LD - n - 1}{2}$$

(L : # loops, D : dimensions, n : # of non-zero α_i's)

- Similar to $\text{Li}_{\gamma+1}(1-\varphi_0) \sim \gamma_0^\gamma \log(\varphi_0)$, get:

φ_0 is on the $\gamma + 1$-st sheet

Corollary on transcendental weight

- In symbol of the form $b_1 \otimes \cdots \otimes b_p \otimes \varphi_0 \otimes c_1 \otimes \cdots \otimes c_q, \qquad \boldsymbol{\gamma = q}$.
- Since $\gamma = \frac{LD-n-1}{2}$, γ takes its largest value when $n = 1$,

Maximum transcendentality of polylogarithmic Feynman integrals:

$$\left\lfloor \frac{LD}{2} \right\rfloor$$

Conclusions & Future Directions

- Defined discontinuities as monodromies around branch points.
- Two different cutting prescriptions:
  a. Use energy rotations and TOPT to encircle some branch points.
  b. Use Cutkosky & Landau equations to encircle one branch point at a time.
- Future directions:
  - Extend to massless particles.
  - Use relations to bootstrap amplitudes.
  - Explore non-physical region.