Title: Amplitudes and the Riemann Zeta Function

Speakers: Grant Remmen

Series: Quantum Fields and Strings

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Abstract:

In this talk, I will connect physical properties of scattering amplitudes to the Riemann zeta function. Specifically, I will construct a closed-form amplitude, describing the tree-level exchange of a tower with masses m^2_n = \mu^2_n, where \zeta(\frac{1}{2}\pm i \mu_n) = 0. Requiring real masses corresponds to the Riemann hypothesis, locality of the amplitude to meromorphicity of the zeta function, and universal coupling between massive and massless states to simplicity of the zeros of \zeta. Unitarity bounds from dispersion relations for the forward amplitude translate to positivity of the odd moments of the sequence of 1/\mu^2_n.

Amplitudes and the Riemann Zeta Function

Grant Remmen

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Perimeter Institute, October 2021

UC SANTA BARBARA
Kavli Institute for Theoretical Physics

The Riemann zeta function

- Introduced by Bernhard Riemann in 1859, a particular function of a single complex variable:

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

for $\mathrm{Re}(z) > 1$. Extend to the rest of the complex plane by analytic continuation.

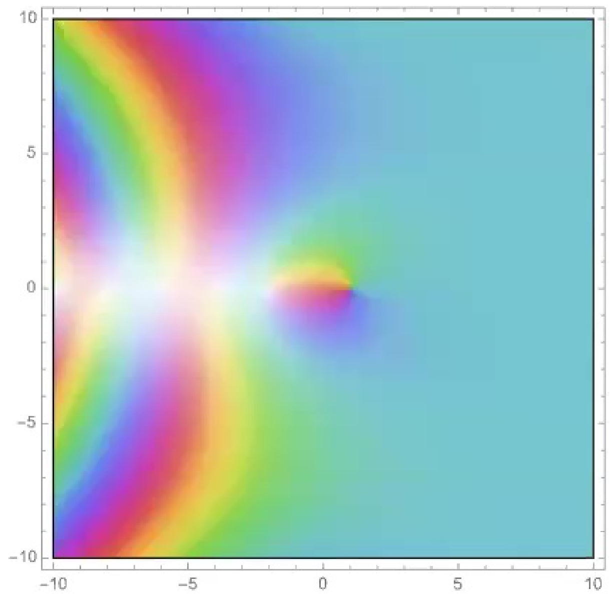
- Many interesting properties, with deep connections to the distribution of the primes:

$$\zeta(z) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-z}} \quad \text{(Euler)}$$

$$\log \zeta(z) = z \int_0^{\infty} \frac{\pi(x)}{x(x^z - 1)} \mathrm{d}x \quad \text{for} \quad \pi(x) = (\#\ \text{primes} \leq x)$$

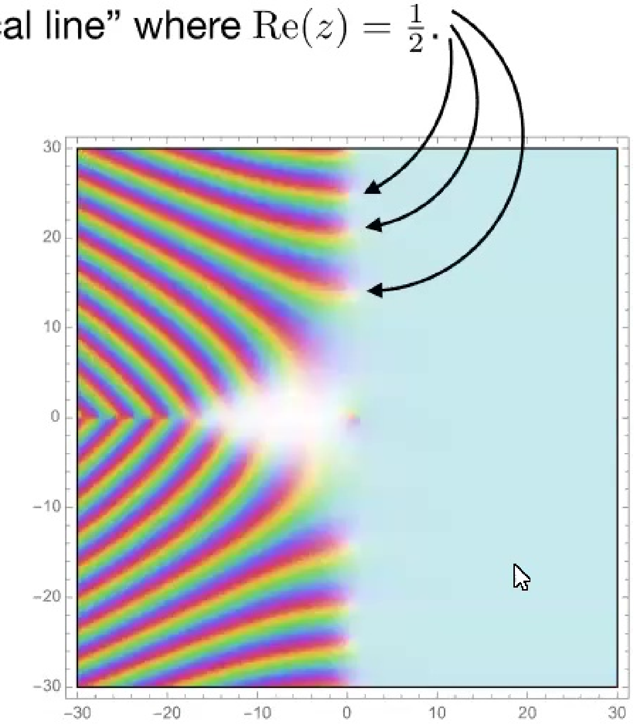
The Riemann zeta function

- The zeta function has been the subject of 150 years of mathematical interest, and its properties have been intensively investigated.



The Riemann zeta function

- Zooming out, we find a collection of additional zeros that all seem to lie on the “critical line” where $\mathrm{Re}(z) = \frac{1}{2}$.



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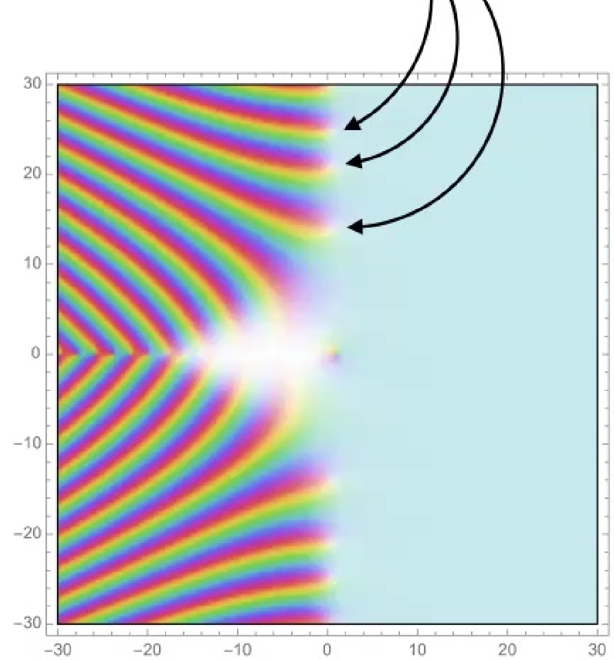


$$\zeta\left(\tfrac{1}{2} \pm i\mu_n\right) = 0$$

$$\mu_1 \simeq 14.135$$

$$\mu_2 \simeq 21.022$$

$$\vdots$$

(We take $\mathrm{Re}(\mu_n) > 0$ throughout.)

Riemann hypothesis

- The Riemann hypothesis asserts that all the nontrivial zeros do indeed lie on the critical line with $\mathrm{Re}(z) = \frac{1}{2}$.
- If true it would have various nice number theory consequences, e.g.,

$$\left| \pi(x) - \int_0^x \frac{\mathrm{d}t}{\log t} \right| < \frac{1}{8\pi} \sqrt{x} \log x \quad \text{for} \quad x \geq 2657$$

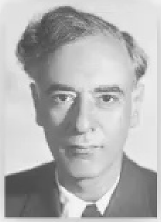
Schoenfeld (1976)

- One of Hilbert's 23 problems and a Millennium Problem
- Currently verified through the first 12 trillion zeros Platt, Trudgian [2004.09765]
- Other open questions:
  - Are all the zeros simple ones?
  - What can be be proven about the statistical properties of the zeros?
  - What is the asymptotic behavior of ζ on the critical line?

Connections to physics

- There is a long history of ideas connecting the Riemann zeta function to physics.

- Hilbert-Pólya conjecture (attributed to remark of Landau to Pólya in 1914):



Does there exist a quantum Hamiltonian whose eigenvalues ar correspond to the imaginary parts of the nontrivial zeros of zeta?

- Montgomery's pair correlation conjecture: Montgomery (1973)
  The correlation function for the normalized spacings of the nontrivial zeros is:

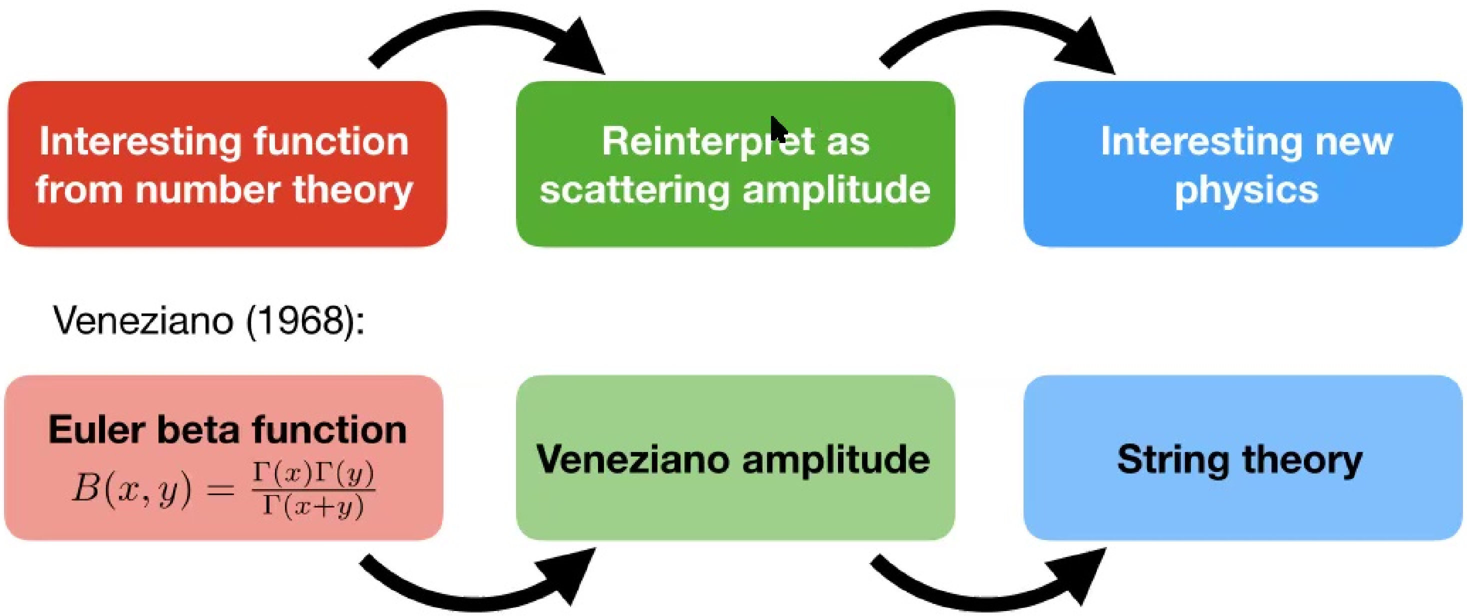
$$1 - \left(\frac{\sin \pi u}{\pi u} \right)^2 + \delta(u)$$

This is the same as the two-point function for a Gaussian unitary ensemble. Dyson

- Other work in quantum chaotic nonrelativistic scattering includes
  Gutzwiller (1983); Bhaduri, Khare, Law [chaodyn/9406006]; see also Srednicki [1105.2342]

What about amplitudes?

- Rather than try to prove the Riemann hypothesis, can we gain any insight if we somehow recast the zeta function as a relativistic scattering amplitude?
- General idea:



What about amplitudes?

- Rather than try to prove the Riemann hypothesis, can we gain any insight if we somehow recast the zeta function as a relativistic scattering amplitude?
- Indeed, the Veneziano amplitude itself can be written in terms of ζ: Freund, Witten (1987)

$$A_4(s,t,u) = B(-\alpha(s),-\alpha(t)) + B(-\alpha(t),-\alpha(u)) + B(-\alpha(s),-\alpha(u)) = \prod_{x=s,t,u} \frac{\zeta(1+\alpha(x))}{\zeta(-\alpha(x))}$$

However, this is somewhat illusory: the nontrivial zeros cancel out entirely.

He, Jejjala, Minic [1501.01975]

$$\frac{\zeta(1+z)}{\zeta(-z)} = \pi^{\frac{1}{2}+z} \frac{\Gamma\left(-\frac{z}{2}\right)}{\Gamma\left(\frac{1+z}{2}\right)}$$

Zeta/amplitudes correspondence

- Rather than try to prove the Riemann hypothesis, can we gain any insight if we somehow recast the zeta function as a relativistic scattering amplitude?

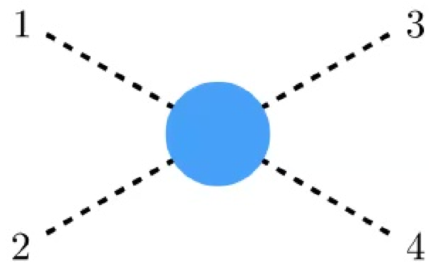
In this talk, we will construct a relativistic four-point scattering amplitude $\mathcal{M}(s,t)$ that truly captures the nontrivial properties of the zeta function.

$\mathcal{M}(s,t)$		$\zeta(z)$
Poles at $s, u = m_n^2$ for m_n real	$\longleftrightarrow$	Riemann hypothesis
Locality (simple poles)	$\longleftrightarrow$	Meromorphicity
Universal coupling	$\longleftrightarrow$	Simple zero conjecture
Dispersive bounds from analyticity/unitarity	$\longleftrightarrow$	Positive odd moments of μ_n^{-2} sequence
On-shell constructibility	$\longleftrightarrow$	Hadamard product expansion
CPT invariance	$\longleftrightarrow$	Reflection of zeros across critical line

Bottom-up approach

- Most important feature: ζ has nontrivial zeros that (appear to) all lie on a line

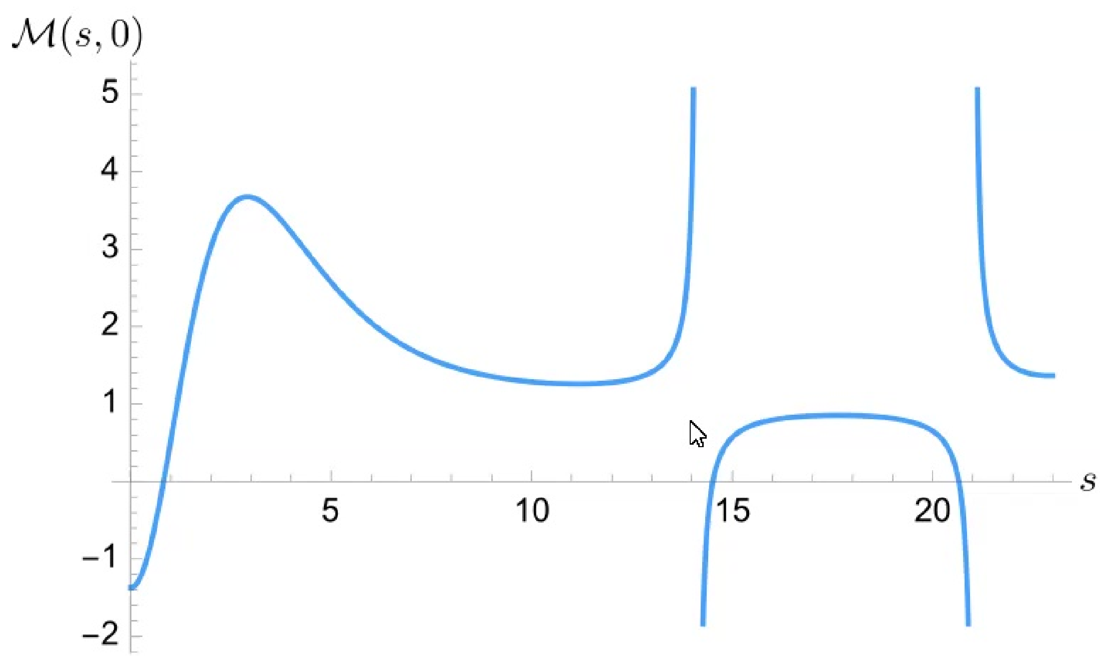
  Connection with amplitudes: poles all lie on lines corresponding to real kinematics, $s, t, u = m^2$



$$s = -(p_1 + p_2)^2$$
$$t = -(p_1 + p_3)^2$$
$$u = -(p_1 + p_4)^2 = -s - t$$

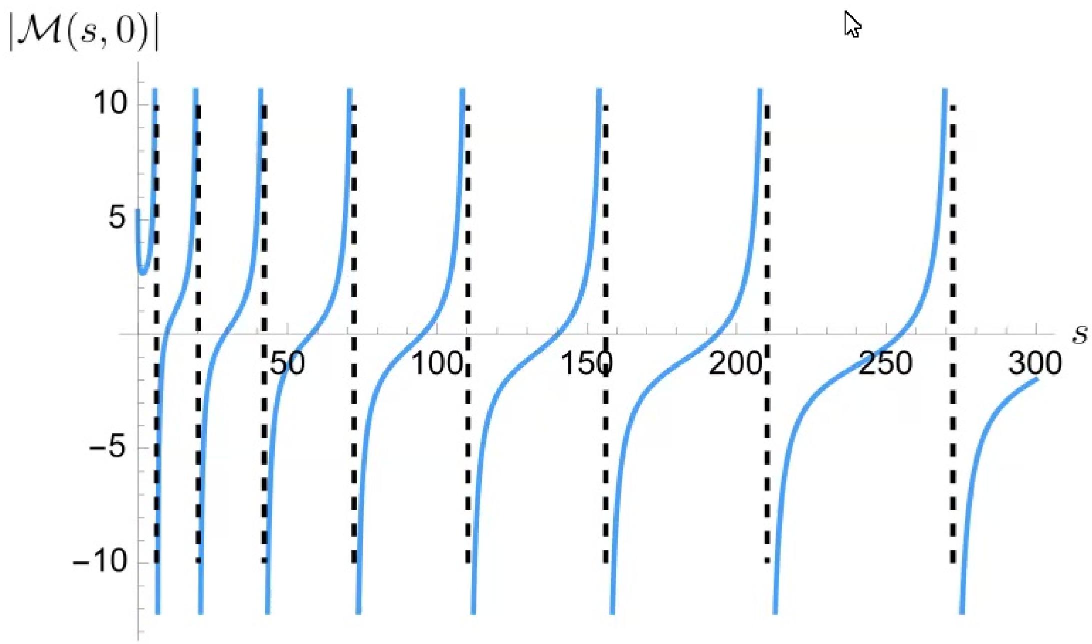
Bottom-up approach

- What about $\mathcal{A}(s) = 1/\zeta\left(\frac{1}{2} + is\right)$?
  - ✘ Poles with opposite-sign residues: tachyons



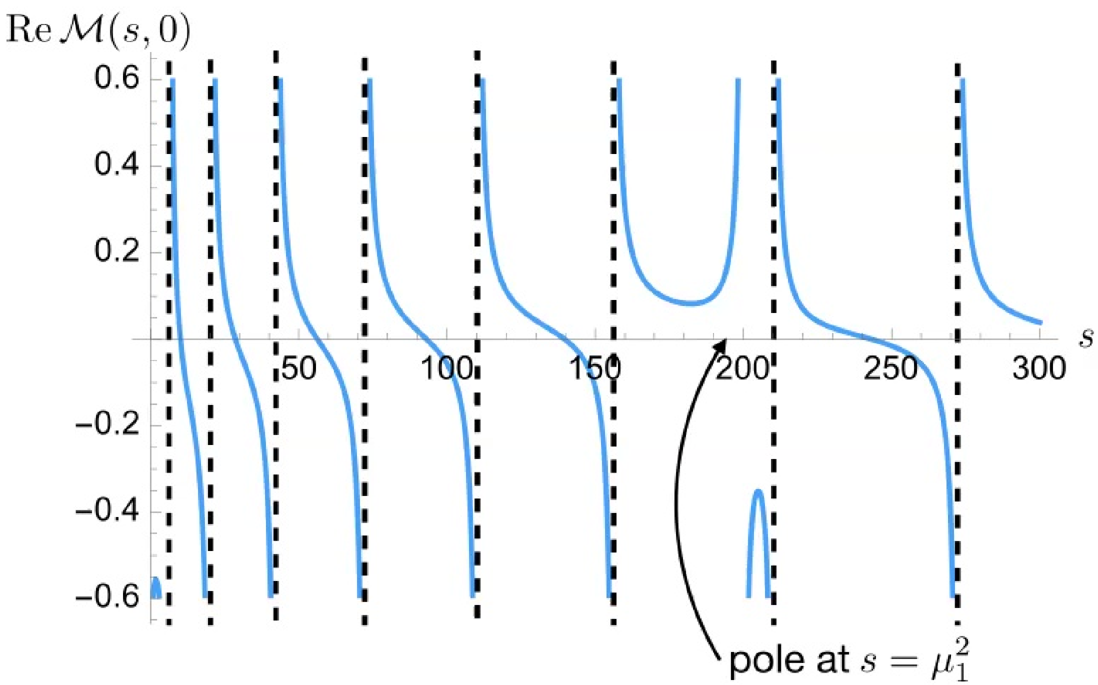
Bottom-up approach

- What about $\mathcal{A}(s) = \dfrac{\zeta'\left(\frac{1}{2}+i\sqrt{s}\right)}{\zeta\left(\frac{1}{2}+i\sqrt{s}\right)}$?
  - ✘ Only poles in the wrong places: $s = \frac{(4n+1)^2}{4}$



Bottom-up approach

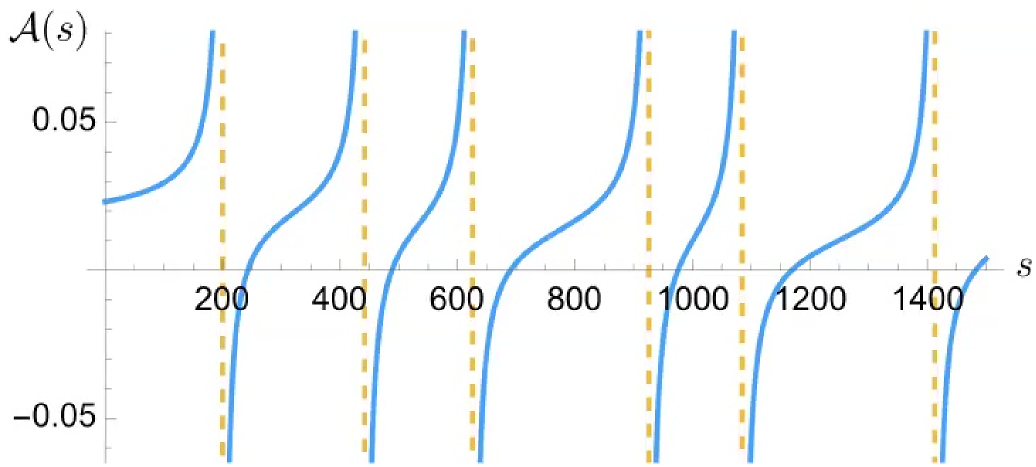
- What about $\mathcal{A}(s) = -\frac{i}{2\sqrt{s}} \frac{\zeta'\left(\frac{1}{2}+i\sqrt{s}\right)}{\zeta\left(\frac{1}{2}+i\sqrt{s}\right)}$?
  - ✘ Still have extra poles in the wrong places: $s = \frac{(4n+1)^2}{4}$



A Riemann zeta amplitude

- To cancel all the wrong poles, we compute their residues and add terms to remove them. Also adding a term to make the forward amplitude real, we find:

$$\mathcal{A}(s) = -\frac{i}{4\sqrt{s}}\left[\psi\left(\tfrac{1}{4}+\tfrac{i}{2}\sqrt{s}\right) + \frac{2\zeta'\left(\frac{1}{2}+i\sqrt{s}\right)}{\zeta\left(\frac{1}{2}+i\sqrt{s}\right)}\right] + \frac{i\log\pi}{4\sqrt{s}} - \frac{1}{s+\frac{1}{4}}$$



- Poles at $s = \mu_n^2$

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Digamma function: $\psi(z) = \Gamma'(z)/\Gamma(z)$
Poles at $\psi(-n)$ cancel trivial zeros at $\zeta(-2n)$ for integer $n > 0$
Pole at $\psi(0)$ canceled by $1/\left(s+\frac{1}{4}\right)$ term

No branch cuts: $\lim_{\epsilon\to 0}\mathcal{A}(s+i\epsilon)-\mathcal{A}(s-i\epsilon)=0$

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- In terms of the Landau-Riemann xi functions,

$$\Xi(z) = \xi\left(\tfrac{1}{2}+iz\right)$$
$$\xi(z) = \tfrac{1}{2}z(z-1)\pi^{-z/2}\Gamma\left(\tfrac{z}{2}\right)\zeta(z)$$

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$\mathcal{A}(s)$ can be written very compactly as:

$$\mathcal{A}(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\log\Xi(\sqrt{s})$$

$$\mathcal{M}(s,t) = \mathcal{A}(s) + \mathcal{A}(u)$$

A Riemann zeta amplitude

$\mathcal{M}$ is the simplest possible amplitude corresponding to the Riemann zeta function and satisfying three physical properties:

1. $\mathcal{M}$ is analytic everywhere except poles corresponding to the nontrivial zeros of the Riemann zeta function, and these poles are real if the Riemann hypothesis holds.

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1. $\mathcal{M}$ is analytic everywhere except poles corresponding to the nontrivial zeros of the Riemann zeta function, and these poles are real if the Riemann hypothesis holds.

2. Each pole has positive residue as required by unitarity.

3. The forward amplitude satisfies

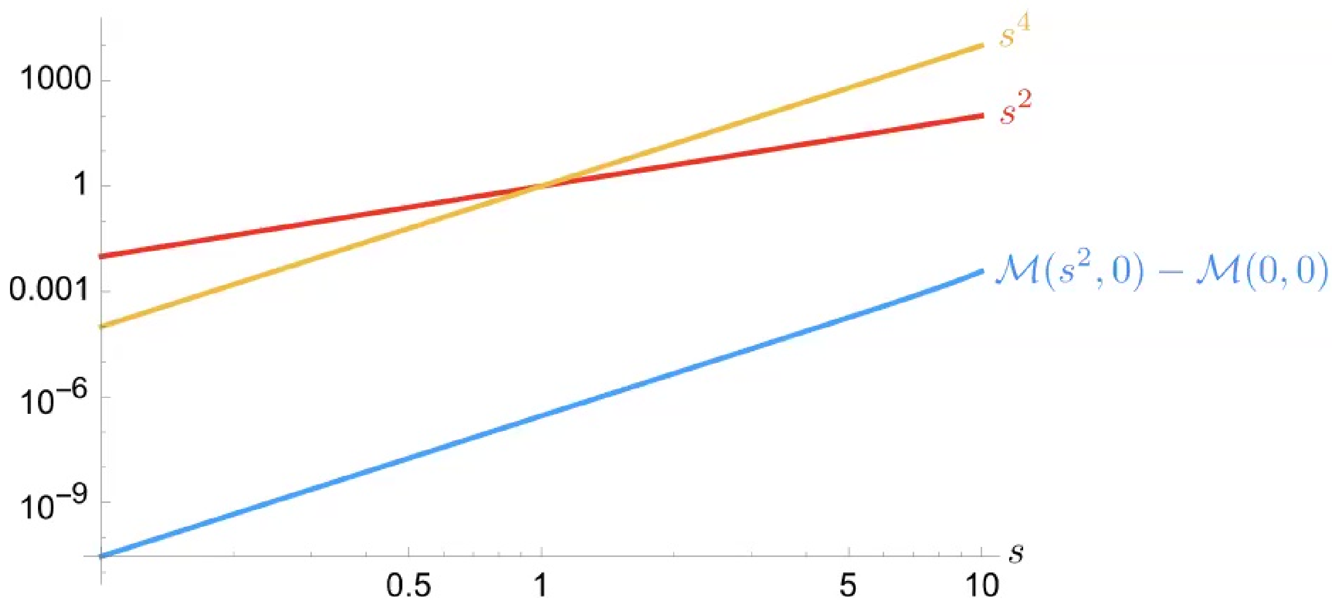
$$\lim_{s \to 0} \frac{\mathrm{d}^2}{\mathrm{d}s^2} \mathcal{M}(s,0) \neq 0$$

A Riemann zeta amplitude

- Were the square roots necessary?

  Yes: If we send $s \to s^2$ in $\mathcal{M}(s,0)$ to eliminate the square roots, then the forward amplitude scales with s^4 at small momentum.

  This violates the s^2 scaling required by dispersion relations. Adams et al. [hep-th/0602178]



Properties of $\mathcal{A}(s)$

- Connection between low-momentum behavior and the zeros of zeta:

$$\frac{c_0}{2} = \lim_{s\to 0} \mathcal{A}(s) = -4 + \frac{\pi^2}{8} + G + \frac{\zeta''\left(\frac{1}{2}\right)}{2\zeta\left(\frac{1}{2}\right)} - \frac{1}{8}\left(\gamma + \frac{\pi}{2} + \log 8\pi\right)^2$$

Catalan's constant $G = \sum_{k=0}^{\infty} (-1)^k/(2k+1)^2$

Properties of $\mathcal{A}(s)$

- Connection between low-momentum behavior and the zeros of zeta:

$$c_0 = \sum_{n=1}^{\infty} \frac{2}{\mu_n^2} \simeq 4.6210 \times 10^{-2}$$

using the Hadamard product form of the zeta function (more on this later).

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- Poles corresponding to the nontrivial zeros: $\zeta\left(\frac{1}{2} \pm i\mu_n\right) = 0$

If the Riemann hypothesis holds, these poles are all at real, positive masses.

$$m_n = \mu_n$$

The poles have the correct (positive) residue required by unitarity:

$$\oint_{s=\mu_n^2} i\mathcal{A}(s + i\epsilon)\mathrm{d}s > 0$$

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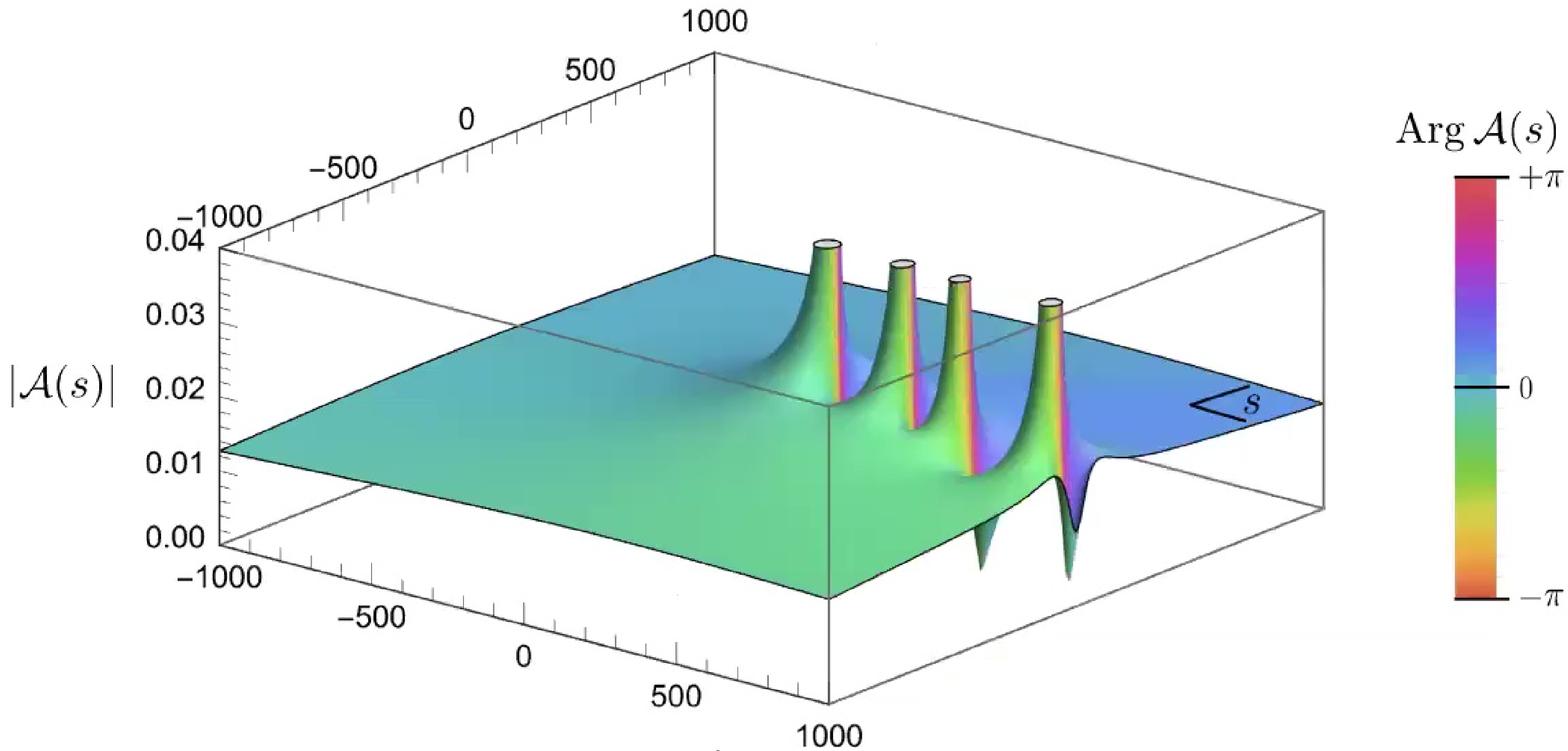
All simple zeros $\Longrightarrow$ Universal coupling of massive states

Properties of $\mathcal{A}(s)$



Properties of $\mathcal{A}(s)$

- Locality: All poles are simple ones.

$$\mathcal{A}(s) \sim 1/(-s+\mu_n^2)$$

- Higher-degree poles would correspond to kinetic terms with too many derivatives: a failure of locality. For example,

$$(\Box - m^2)^k \phi \longrightarrow \frac{1}{(-s+m^2)^k}$$

- Nonlocality in $\mathcal{A}(s) \sim 1/(-s+\mu_n^2)^k$ for $k > 1$ would correspond to an essential singularity in the Riemann zeta function,

$$\zeta(z) \sim e^{\frac{\alpha}{(z-z_n)^{k-1}}}$$

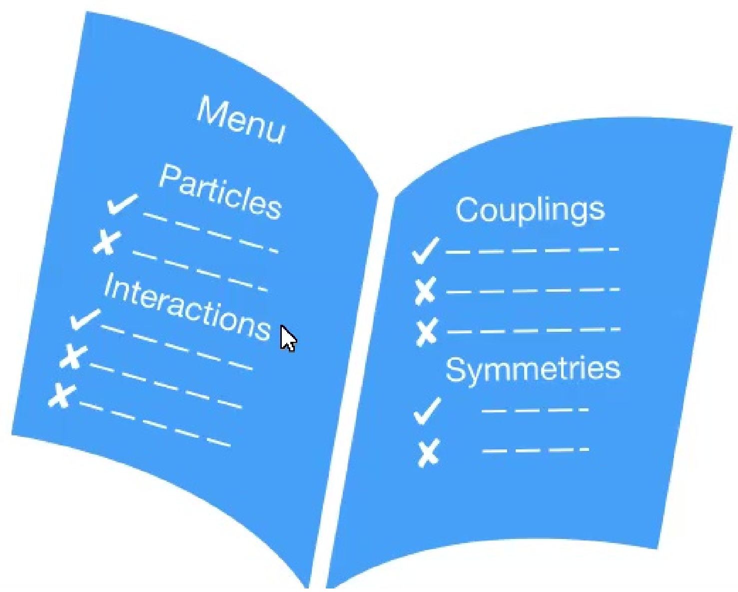
Locality in $\mathcal{A}$ $\longleftrightarrow$ Meromorphicity in ζ

Analytic dispersion relations

Which theories are possible?

Can any Lagrangian be a consistent EFT?

- Certain signs or magnitudes of couplings violate fundamental physics principles:
  - Unitarity
  - Causality
  - Analyticity
  - Thermodynamics



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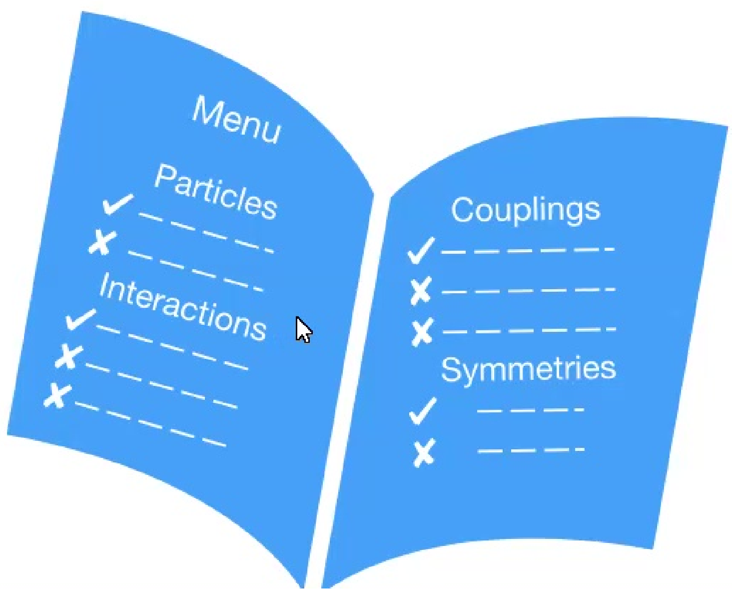
"infrared consistency"



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- Examples:
  - Standard Model EFT — **GR**, Rodd [1908.09845] & (2021, forthcoming)
  - Flavor physics — **GR**, Rodd [2004.02885, 2010.04723]
  - Higher-curvature terms — Bellazzini, Cheung, **GR** [1509.00851]; Cheung, **GR** [1608.02942]; Gruzinov, Kleban (2006)
  - Massive gravity — Cheung, **GR** [1601.04068]
  - Einstein-Maxwell theory — Cheung, **GR** [1407.7865]; Cheung, Liu, **GR** [1801.08546, 1903.09156]; Arkani-Hamed, Huang, Liu, **GR** [2109.13937]
  - Scalar theories — Adams et al. (2006); Chandrasekaran, **GR**, Shahbazi-Moghaddam [1804.03153]
  - a-theorem — Komargodski, Schwimmer (2011); Elvang et al. (2012)

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- Our $\mathcal{M}(s,t)$ built from the zeta function will by definition satisfy the requirements of analyticity and unitarity for scattering amplitudes.

- **Question:** What happens if we run $\mathcal{M}(s,t)$ through the mechanics of analytic dispersion relations?

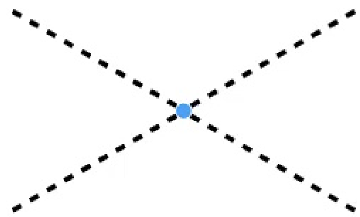
Example theory

We'll first briefly review how infrared consistency bounds the coefficients of an EFT, based on analyticity, unitarity, and causality. Adams et al. [hep-th/0602178]

Example EFT: massless scalar with shift symmetry

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + \frac{c}{M^4}(\partial\phi)^4$$

Two-to-two scattering amplitude:



$$\mathcal{M}(s,t) = \frac{2c}{M^4}(s^2 + t^2 + u^2)$$

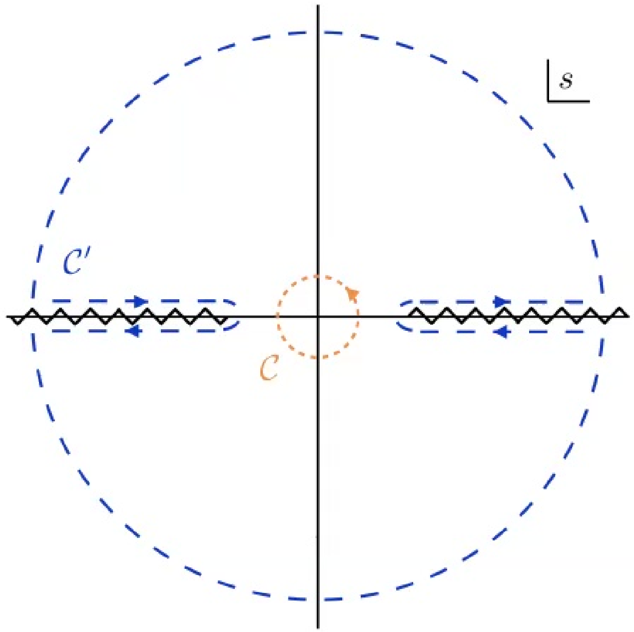
Forward amplitude (in state = out state):

$$\mathcal{M}(s,0) = \frac{4c}{M^4}s^2$$

$$s = -(p_1 + p_2)^2$$
$$t = -(p_1 + p_3)^2$$
$$u = -(p_1 + p_4)^2$$

Analyticity and unitarity

The Wilson coefficient of interest can be extracted via a contour integral of the forward amplitude:



$$\frac{4c}{M^4} = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{\mathrm{d}s}{s^3}\mathcal{M}(s,0)$$
$$= \frac{1}{2\pi i}\oint_{\mathcal{C}'} \frac{\mathrm{d}s}{s^3}\mathcal{M}(s,0)$$

use analyticity to deform the contour

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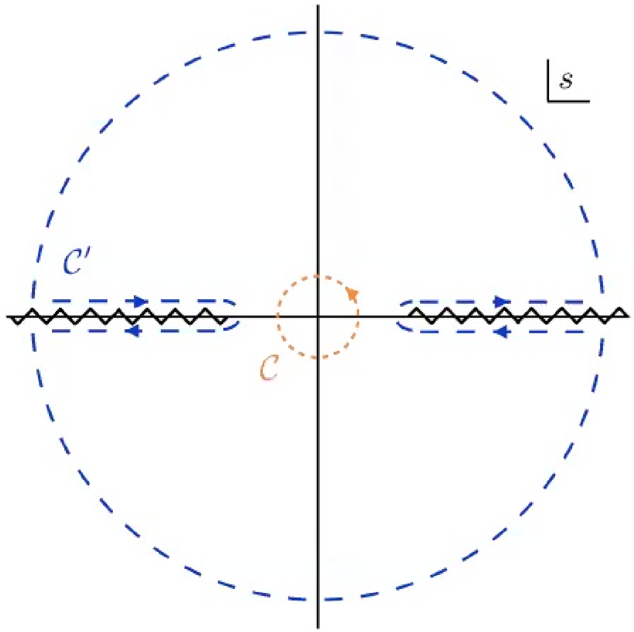

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$$= \frac{1}{2\pi i}\left(\int_{-\infty}^{0} + \int_{0}^{\infty}\right)\frac{\mathrm{d}s}{s^3}\mathrm{Disc}\,\mathcal{M}(s,0)$$

boundary term at infinity vanishes

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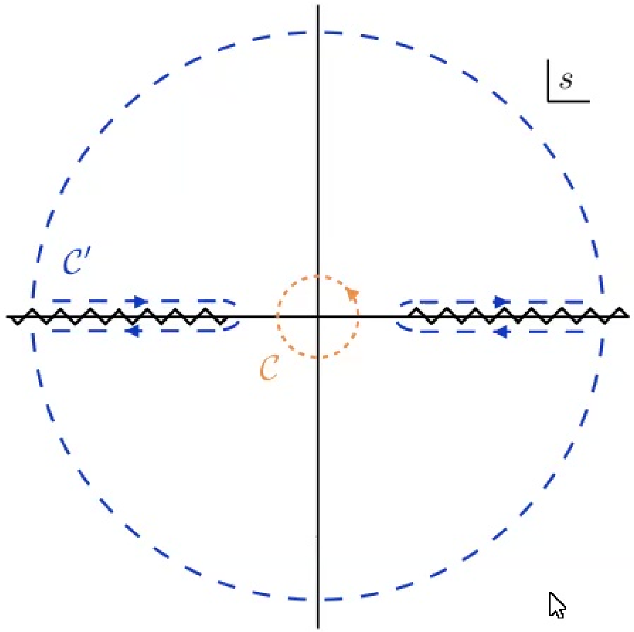

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$$= \frac{1}{2\pi i}\left(\int_{-\infty}^{0} + \int_{0}^{\infty}\right)\frac{\mathrm{d}s}{s^3}\mathrm{Disc}\,\mathcal{M}(s,0)$$
$$= \frac{1}{i\pi}\int_{0}^{\infty}\frac{\mathrm{d}s}{s^3}\mathrm{Disc}\,\mathcal{M}(s,0)$$

crossing symmetry: $\mathcal{M}(s,0) = \mathcal{M}(-s,0)$

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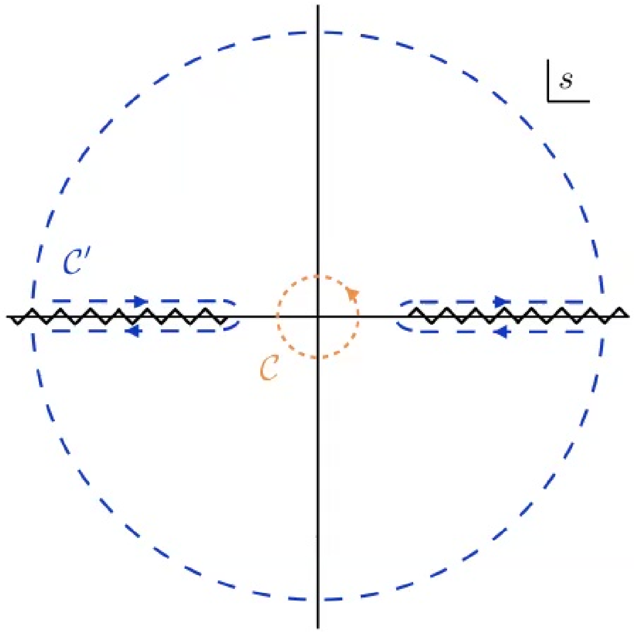

$$\frac{4c}{M^4} = \frac{1}{i\pi}\int_0^\infty \frac{\mathrm{d}s}{s^3}\,\mathrm{Disc}\,\mathcal{M}(s,0)$$
$$= \frac{1}{i\pi}\int_0^\infty \frac{\mathrm{d}s}{s^3}\lim_{\epsilon\to 0}[\mathcal{M}(s+i\epsilon,0)-\mathcal{M}(s-i\epsilon,0)]$$
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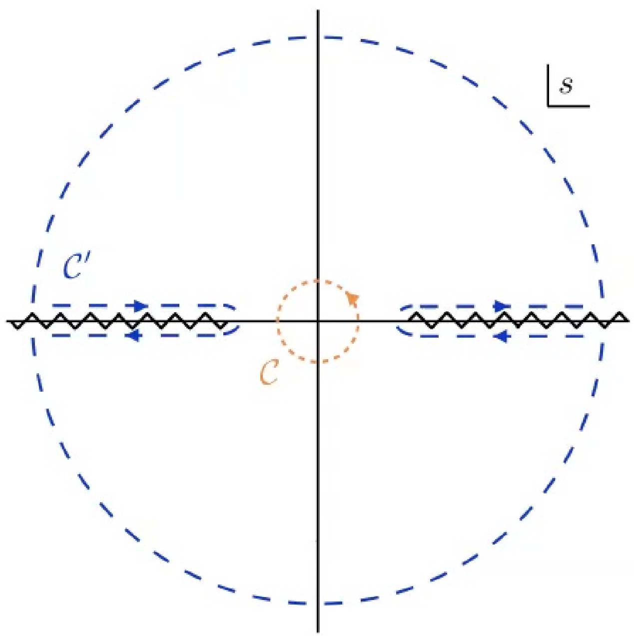
Schwarz reflection principle:
$$\mathcal{M}(s^*,0) = (\mathcal{M}(s,0))^*$$

Analyticity and unitarity

The Wilson coefficient of interest can be extracted via a contour integral of the forward amplitude:



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using the optical theorem (unitarity):

$$\mathrm{Im}\,\mathcal{M}(s,0) = s\,\sigma(s)$$

$$\implies c > 0$$

Adams et al. [hep-th/0602178]

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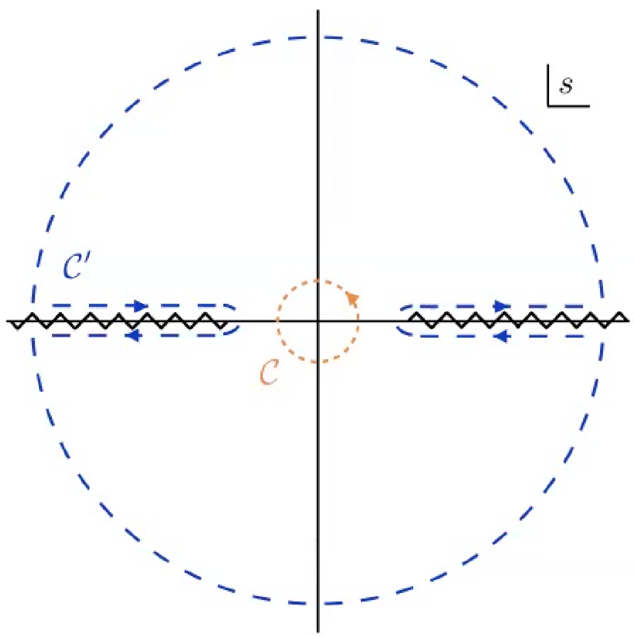

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using the optical theorem (unitarity):

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More generally,

$$\lim_{s\to 0}\frac{\mathrm{d}^{2k}}{\mathrm{d}s^{2k}}\mathcal{M}(s,0) > 0$$

Adams et al. [hep-th/0602178]

Wilson coefficients for the zeta amplitude

- Let's now apply the dispersion relation formalism to our zeta amplitude. Define a power series of the forward amplitude at small momentum:

$$\mathcal{M}(s,0) = \sum_{k=0}^{\infty} c_{2k} s^{2k}$$

- Extract the Wilson coefficient with a contour integral,

$$\begin{aligned} c_{2k} &= \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{\mathrm{d}s}{s^{2k+1}} \mathcal{M}(s,0) \\ &= \frac{2}{\pi} \int_0^{\infty} \frac{\mathrm{d}s}{s^{2k}} \sigma(s) + c_{\infty}^{(2k)} \end{aligned}$$

- Boundary term:

$$c_{\infty}^{(2k)} = \frac{1}{2\pi i} \oint_{|s|=\infty} \frac{\mathrm{d}s}{s^{2k+1}} \mathcal{M}(s,0)$$

Nonzero $c_{\infty}^{(2k)}$ would mean that $\Xi(z)$ grows at least as fast as $e^{\alpha z^{4k+2}}$ (i.e., growth order $4k+2$), contradicting known growth order 1. Titchmarsh (1951)

$$\implies c_{\infty}^{(2k)} = 0$$

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$$\mathcal{M}(s,0) = \sum_{k=0}^{\infty} c_{2k} s^{2k}$$

- The properties we have proven for $\mathcal{M}(s,t)$ give a beautiful relation between the Wilson coefficients and the nontrivial zeros:

$$c_{2k} = \sum_{n=1}^{\infty} \frac{2}{\mu_n^{2(2k+1)}}$$

e.g.,

$$c_0 = \sum_{n=1}^{\infty} \frac{2}{\mu_n^2}$$

$$c_2 = \sum_{n=1}^{\infty} \frac{2}{\mu_n^6}$$

$$c_4 = \sum_{n=1}^{\infty} \frac{2}{\mu_n^{10}}$$

$$\vdots$$

Riemann hypothesis $\implies c_{2k} > 0$

Wilson coefficients for the zeta amplitude

For example, the s^2 coefficient gives us the remarkable identity:

$$\begin{aligned}
c_2 &= \frac{1}{2}\lim_{s\to 0}\frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathcal{M}(s,0) \\
&= -128 + \frac{1}{7680}\psi^{(5)}\left(\tfrac{1}{4}\right) - \zeta_1^6\left(\tfrac{1}{2}\right) \\
&\quad + 3\zeta_1^4\left(\tfrac{1}{2}\right)\zeta_2\left(\tfrac{1}{2}\right) - \frac{9}{4}\zeta_1^2\left(\tfrac{1}{2}\right)\zeta_2^2\left(\tfrac{1}{2}\right) \\
&\quad + \frac{1}{4}\zeta_2^3\left(\tfrac{1}{2}\right) - \zeta_1^3\left(\tfrac{1}{2}\right)\zeta_3\left(\tfrac{1}{2}\right) \\
&\quad + \zeta_1\left(\tfrac{1}{2}\right)\zeta_2\left(\tfrac{1}{2}\right)\zeta_3\left(\tfrac{1}{2}\right) - \frac{1}{12}\zeta_3^2\left(\tfrac{1}{2}\right) \\
&\quad + \frac{1}{4}\zeta_1^2\left(\tfrac{1}{2}\right)\zeta_4\left(\tfrac{1}{2}\right) - \frac{1}{8}\zeta_2\left(\tfrac{1}{2}\right)\zeta_4\left(\tfrac{1}{2}\right) \\
&\quad - \frac{1}{20}\zeta_1\left(\tfrac{1}{2}\right)\zeta_5\left(\tfrac{1}{2}\right) + \frac{1}{120}\zeta_6\left(\tfrac{1}{2}\right) \\
&= \sum_{n=1}^{\infty}\frac{2}{\mu_n^6}
\end{aligned}$$

using the shorthand $\zeta_n(z) = \zeta^{(n)}(z)$

$$\zeta_n^k(z) = [\zeta_n(z)]^k$$

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Can prove (with great effort!) by computing analytic expressions for derivatives of $\zeta(z)$ at $z = \frac{1}{2}$ using polygamma identities and the product form of the zeta function,

$$\zeta(z) = \frac{1}{2(z-1)} (\pi e^{\gamma})^{z/2} \prod_{k=1}^{\infty} \left(1 + \frac{z}{2k}\right) e^{-z/2k} \prod_{z_n \text{ nontrivial zeros}} \left(1 - \frac{z}{z_n}\right)$$

which comes from the Hadamard expansion of the xi function,

$$\xi(z) = \xi(0) \prod_{z_n \text{ nontrivial zeros}} \left(1 - \frac{z}{z_n}\right)$$

What is remarkable is that our amplitude construction allows for much simpler, physical derivations of such identities!

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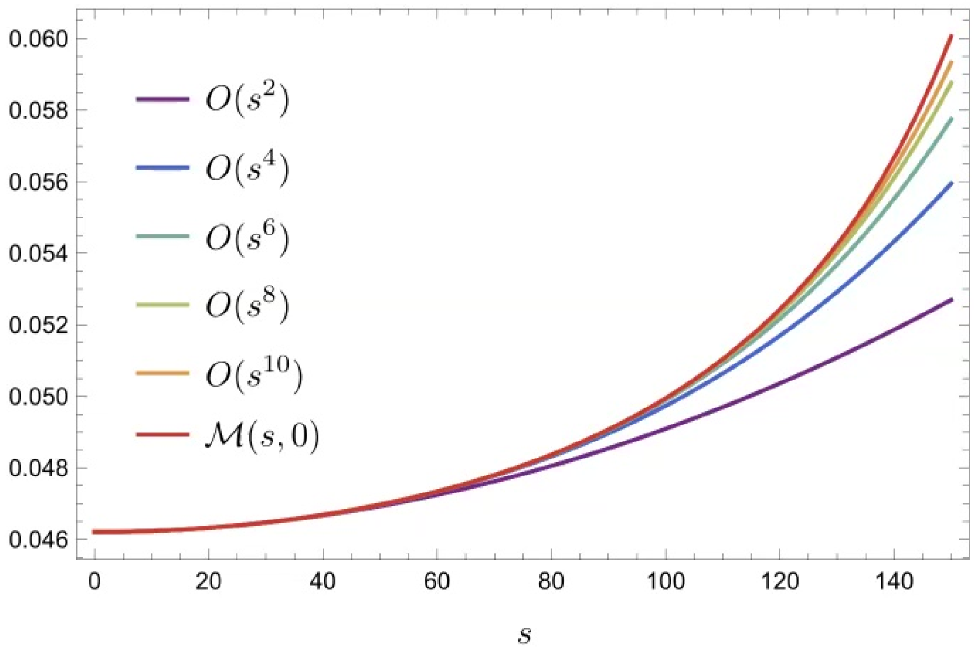
Wilson coefficients for the zeta amplitude



$c_0 \simeq 4.6210 \times 10^{-2}$

$c_2 \simeq 2.8835 \times 10^{-7}$

$c_4 \simeq 6.4273 \times 10^{-12}$

$c_6 \simeq 1.5807 \times 10^{-16}$

$c_8 \simeq 3.9465 \times 10^{-21}$

$c_{10} \simeq 9.8805 \times 10^{-26}$

Numerical tests of $c_{4,6,8,10}$ confirm prediction to within relative error of 10^{-30}.

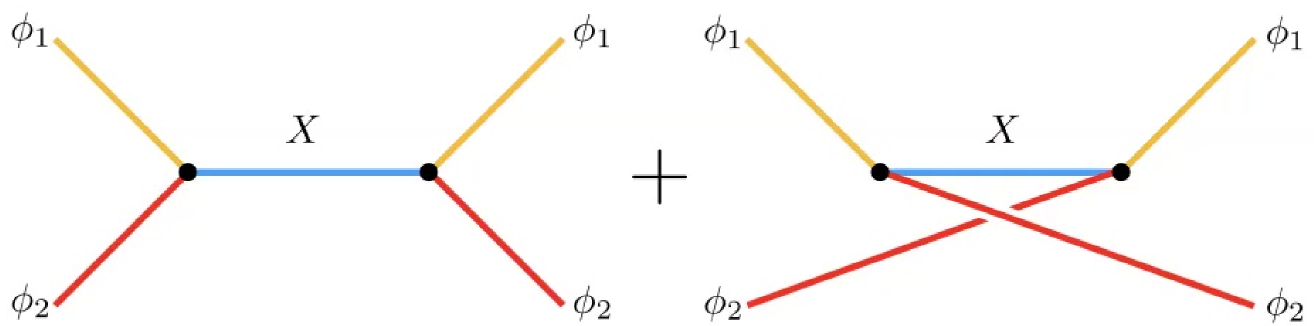
Other properties

What sort of theory is this?

- Our amplitude $\mathcal{M}(s,t)$ describes a theory of two types of massless scalars, ϕ_1 and ϕ_2, exchanging a tower of massive states X in the s and u channels for the process:

$$\phi_1\phi_2 \longrightarrow \phi_1\phi_2$$

- We alternatively could have defined $\mathcal{M}(s,t)$ as $\mathcal{A}(s)+\mathcal{A}(t)+\mathcal{A}(u)$ to have full Bose symmetry, in which case our amplitude would describe single-scalar scattering $\phi\phi \rightarrow \phi\phi$



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On-shell constructibility

- Based on the properties of $\mathcal{A}(s)$, our Riemann zeta amplitude is on-shell constructible from the UV amplitudes $\phi_1\phi_2 \to X$:

$$\mathcal{M}(s,t) = \sum_X \mathcal{M}_{\phi_1\phi_2\to X}(p_1,p_2)\frac{1}{-s+\mu_n^2}\mathcal{M}_{\phi_1\phi_2\to X}(p_3,p_4) + \sum_X \mathcal{M}_{\phi_1\phi_2\to X}(p_1,p_4)\frac{1}{-u+\mu_n^2}\mathcal{M}_{\phi_1\phi_2\to X}(p_3,p_2)$$

- Universal coupling: $\mathcal{M}_{\phi_1\phi_2\to X}(p_1,p_2) = \text{constant}$ for all X ($= 1$ in our units)

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$$\mathcal{M}(s,t) = \sum_n \left(\frac{1}{-s+\mu_n^2-i\epsilon} + \frac{1}{-u+\mu_n^2-i\epsilon}\right)$$

On-shell constructibility

- Proof:

Let $\Delta(s) = \mathcal{A}(s) - \sum_n \frac{1}{-s+\mu_n^2}$, where $\mathcal{A}(s) = -\frac{\mathrm{d}\,\log \Xi(\sqrt{s})}{\mathrm{d}s}$.

We have previously shown that $\mathcal{A}(s)$ has zeros only at $s = \mu_n^2$ and (if we allow the possibility of degenerate μ_n) all of the residues are 1.

$$\implies \Delta(s) \text{ is entire.}$$

Expand $\Delta(s)$ in a Laurent series about $s = \infty$.
The definition of $\Delta(s)$ and the absence of a pole at infinity in $\mathcal{A}(s)$ imply:

$$\Delta(s) \text{ is bounded.}$$

Liouville's theorem $\implies \Delta(s)$ is constant.

Hadamard product

- Integrating our result

$$\mathcal{A}(s) = -\frac{\mathrm{d}\,\log\Xi(\sqrt{s})}{\mathrm{d}s} = \sum_n \frac{1}{-s+\mu_n^2}$$

gives the product form for the Riemann-Landau xi function:

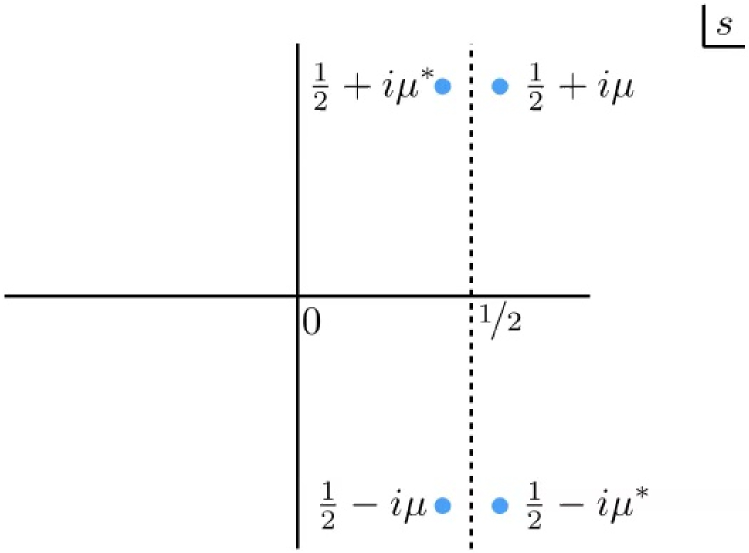
$$\xi(z) = \xi(0) \prod_{z_n \text{ nontrivial zeros}} \left(1 - \frac{z}{z_n}\right) \qquad z_n = \frac{1}{2} \pm i\mu_n$$

$$\zeta(z) = \frac{2\xi(0)}{z(z-1)} \frac{\pi^{z/2}}{\Gamma\left(\frac{z}{2}\right)} \prod_{z_n} \left(1 - \frac{z}{z_n}\right)$$

- Using the Weierstrass product $\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}$ along with $\zeta(0) = -\frac{1}{2}$, we have…

Reflection symmetry and CPT

- Functional equation $\zeta(z) = 2^z \pi^{z-1} \sin(\pi z/2)\Gamma(1-z)\zeta(1-z)$
  and Schwarz reflection $\zeta(z^*) = [\zeta(z)]^*$
  together imply that the nontrivial zeros enjoy a four-fold symmetry:



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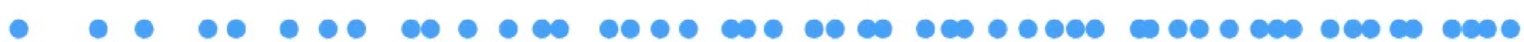
- Zero-counting: $N(T) = \Big|\{z|\zeta(z) = 0 \;\&\; 0 < \mathrm{Im}(z) \leq T\}\Big|$

$$= \frac{1}{\pi}\int_0^{\sqrt{T}} \sigma(s)\,\mathrm{d}s$$

Outlook

We have constructed an amplitude whose physical attributes correspond to the known or conjectured properties of the nontrivial zeros of zeta.

- **Open question:** What dynamics gives rise to $\mathcal{M}(s,t)$?



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- **Other future directions:**

    - Spin for intermediate states
    - Zeta function universality
    - Dirichlet *L*-functions


Questions