Title: Effective field theories of topological crystalline insulators and topological crystals

Speakers: Sheng-Jie Huang

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Abstract: In this talk, I will present a general approach to obtain effective field theories for topological crystalline insulators whose low-energy theories are described by massive Dirac fermions. We show that these phases are characterized by the responses to spatially dependent mass parameters with interfaces. These mass interfaces implement the dimensional reduction procedure such that the state of interest is smoothly deformed into a network of defects (dubbed topological crystal), where each defect supports a short-ranged entangled state. Effective field theories are obtained by integrating out the massive Dirac fermions, and various quantized topological terms are uncovered. I will describe how to apply this strategy through a few simple examples and comment on the relation to the topological elasticity theory.

Effective field theories of topological crystalline insulators and topological crystals

Sheng-Jie Huang

JOINT QUANTUM INSTITUTE

CMTC

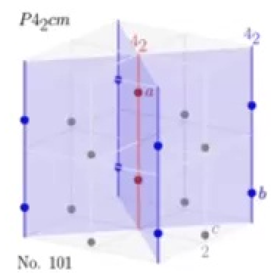
UNIVERSITY OF MARYLAND

Perimeter
Quantum Matter Seminar
09/28/2021

Main goal

Crystalline symmetry protected topological phases (cSPT phases)

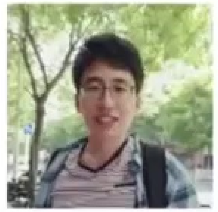
adiabatically connected to → Topological crystals



→ Effective field theories

Acknowledgement

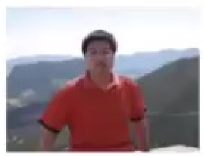
Chang-Tse Hsieh
(RIKEN -> NTU)



Jiabin Yu
(UMD)

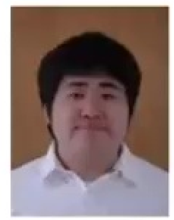
Michael
Hermele
(Boulder)

Yi-Ping Huang
(NTHU)



Hao Song
(McMaster)

Zhida Song
(Princeton)



Yang Qi
(Fudan)

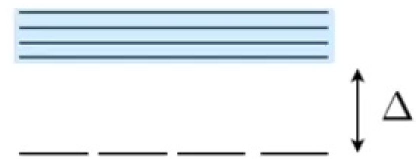
Chen Fang
(CAS)

Liang Fu
(MIT)

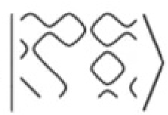
1. **SJH**, Chang-Tse Hsieh, Jiabin Yu, arXiv:2107.03409.
2. Zhida Song, **SJH**, Yang Qi, Chen Fang, Michael Hermele, Sci. Adv. 5, eaax2007 (2019)
3. **SJH***, Hao Song*, Yi-Ping Huang and Michael Hermele, Phys. Rev. B 96, 205106 (2017)
4. Hao Song, **SJH**, Liang Fu and Michael Hermele, Phys. Rev. X 7, 011020 (2017)

Introduction: Topological phases of matter

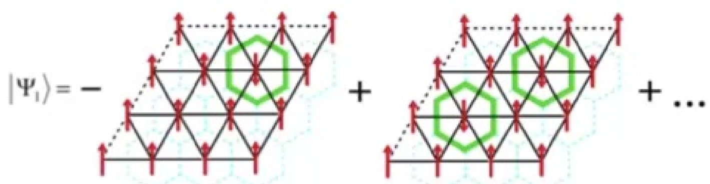
- Topological phases of matter ≡
  Phases with energy gap to bulk excitations



- Can be characterized by patterns of many-body entanglement in the ground state

$|\Phi_{\text{string}}\rangle = \sum_{\text{closed string pattern}}$ 

Quantum Information Meets Quantum Matter -- From Quantum Entanglement to Topological Phase in Many-Body Systems, Bei Zeng, Xie Chen, Duan-Lu Zhou, Xiao-Gang Wen



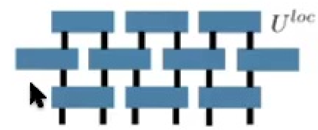
Levin and Gu (2012)

Haldane, Affleck, Kennedy, Lieb, Tasaki

Introduction: symmetry protected topological phases

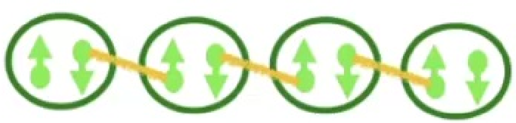
Symmetry protected topological (SPT) phases

1. Symmetry G not spontaneously broken
2. Unique ground state with only local excitation in the bulk
3. Ground state becomes trivial if all symmetries broken
4. Ground state has "Short-range entanglement"



Classic examples:

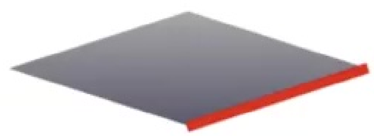
d=1



Haldane *S=1* chain

Symmetry: time reversal,
or SO(3) spin rotation,
or reflection

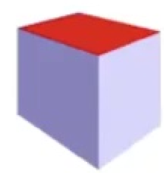
d=2



Quantum spin-hall insulator

Symmetry: charge conservation
+ time reversal

d=3



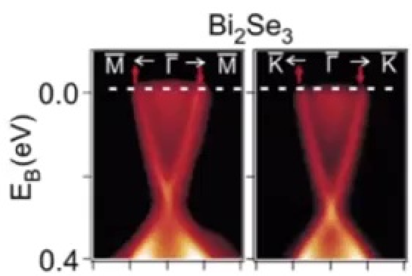
Topological band insulator

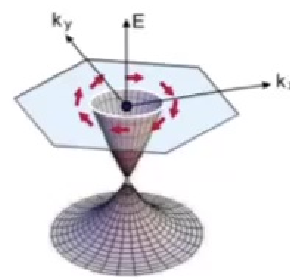
Symmetry: charge
conservation + time reversal

Introduction: boundaries of SPT phases

- Interesting boundary phenomena protected by the symmetry
- Free spin-1/2's on open ends of Haldane chain

- Surface Dirac fermions are hallmark signatures of 3d topological insulators

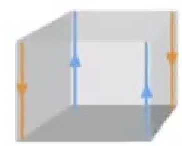


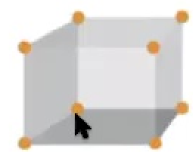


Adapted from Xia *et al.*, 2008, Hsieh, Xia, Qian, Wray, *et al.*, 2009a, and Xia, Qian, Hsieh, Wray, *et al.*, 2009.

Introduction: crystalline topological phases

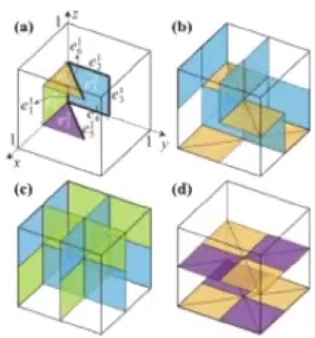
There has also been a great progress on understanding topological phases with crystalline symmetry





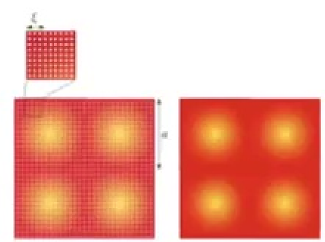
General approaches (real space)

Topological crystals (defect networks)

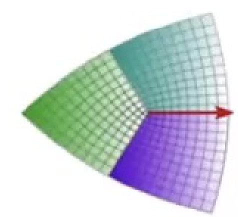


Song, **SJH**, Fu, Hermele, PRX 2017
SJH, Song, Huang, Hermele, PRB 2017
Song, **SJH**, Qi, Fang, Hermele, Sci. Adv. 2019

Smooth states



"Gauging" spatial symmetry



Thorngren and Else, PRX 2018
Else and Thorngren, PRB 2019

Works on classification of cSPT phases

General approaches (real space)

- Applies to any spatial dimension and any crystalline symmetry
- Works for bosons and fermions
- Classifications for free or interacting systems

- Formal mathematical structure of the topological crystal approach: Atiyah-Hirzebruch spectral sequence in equivariant generalized homology/cohomology

Ken Shiozaki, Charles Zhaoxi Xiong, Kiyonori Gomi; arXiv:1810.00801
Dominic V. Else, Ryan Thorngren; PRB 99, 115116 (2019)
Daniel S. Freed, Michael J. Hopkins; arXiv:1901.06419
Zhida Song, Chen Fang, Yang Qi; Nature Communications 11, 4197 (2020)

Works on classification of cSPT phases

General approaches (momentum space)

- K-theory (for free fermions)

  Freed and Moore; Annales Henri Poincaré 2013.
  T. Morimoto and A. Furusaki; PRB 2013.
  K. Shiozaki and M. Sato; PRB 2014.
  K. Shiozaki, M. Sato, and K. Gomi; PRB 2016, PRB 2017.
  J. Kruthoff, J. de Boer, J. van Wezel, C. L. Kane, and R.-J. Slager; PRX 2017
  Xue-Yang Song, Andreas P. Schnyder; PRB 2017
  K. Gomi; arXiv 2018
  K. Shiozaki, M. Sato, and K. Gomi; arXiv 2018.
  …

Topological crystal approach:

- An alternative approach to K-theory for classifying fermionic cSPT phases
- Provide physical pictures and build-boundary correspondence for the topological invariants in momentum space

SJH, YT Hsu, PRR 3, 013243 (2021)
Y Chen, **SJH**, YT Hsu, TC Wei, arXiv:2109.06959

Outline

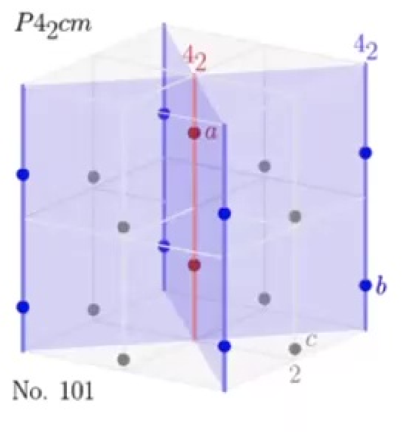
- Review: topological crystal approach
- Quantized topological responses of crystalline topological phases
- Examples:

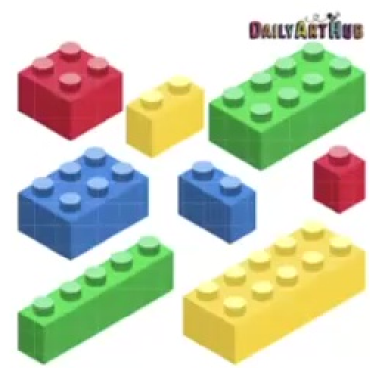
  1. Topological crystalline insulators
  2. Atomic insulators

- General form of the effective field theory
- Summary and outlook

General classification

cSPT phases —adiabatically connected to→ Topological crystals





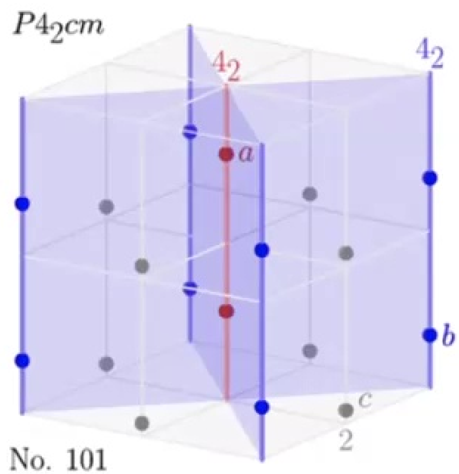
Hao Song, **SJH**, Liang Fu and Michael Hermele. *PRX* **7,** 011020 (2017)

SJH, Hao Song, Yi-Ping Huang and Michael Hermele. *PRB* **96**, 205106 (2017)

Zhida Song, **SJH**, Yang Qi, Chen Fang, Michael Hermele. Sci. Adv. 5, eaax2007 (2019)

Topological crystals

What are topological crystals ?

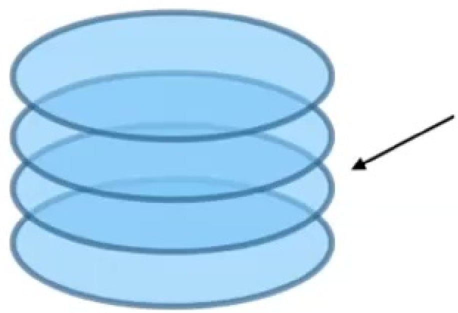


Topological crystal in *d* dimensions

- Crystalline pattern of "**building blocks**" in real space
- Each building block is a manifold of dimension $0 \leq d_b \leq d$, hosting a d_b-dimensional topological phase
- $d_b \equiv$ "block dimension"
- Blocks must be "glued together" at the intersections to eliminate any bulk gapless modes

Example 1: Weak topological insulators

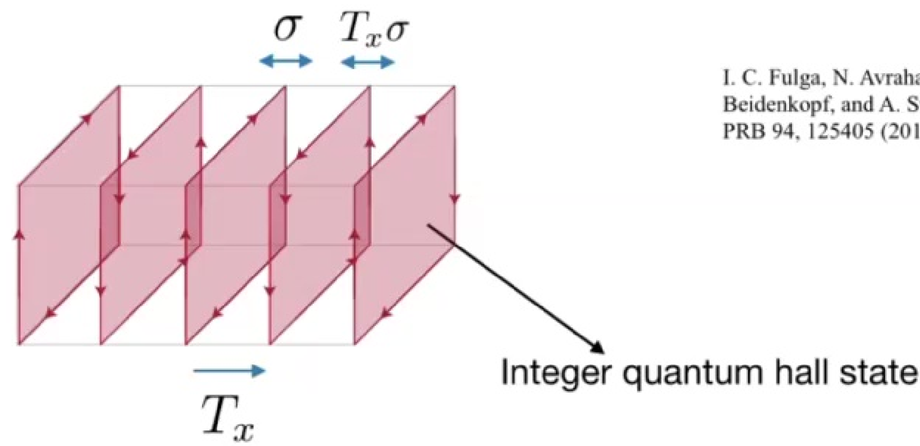
- Weak topological insulator of electrons in $d=3$ is a stack of d=2 topological insulators
- Relevant symmetries: <u>charge conservation</u>, <u>time reversal</u>, <u>lattice translation</u> normal to stacking direction



$d_b = 2$ blocks of 2d topological insulator

Example 2: Topological crystalline insulators

- 3D topological crystalline insulators with <u>charge conservation</u>, <u>reflection</u> and <u>translation</u> symmetry



I. C. Fulga, N. Avraham, H. Beidenkopf, and A. Stern, PRB 94, 125405 (2016)

- Mirror symmetry pins the integer quantum hall (IQH) states to different mirror planes

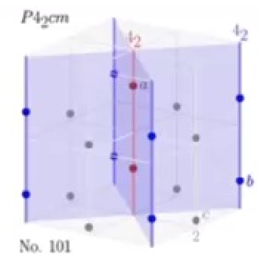
Topological crystals

Now we have a rough idea about topological crystals

cSPT phases —adiabatically connected to→ Topological crystals

↓

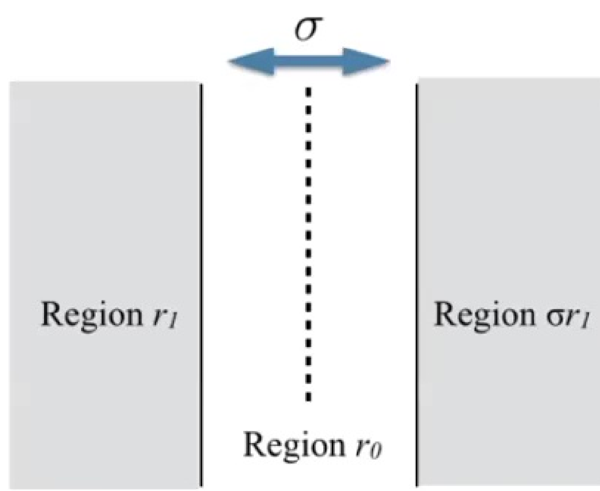
Dimensional reduction



Dimensional reduction

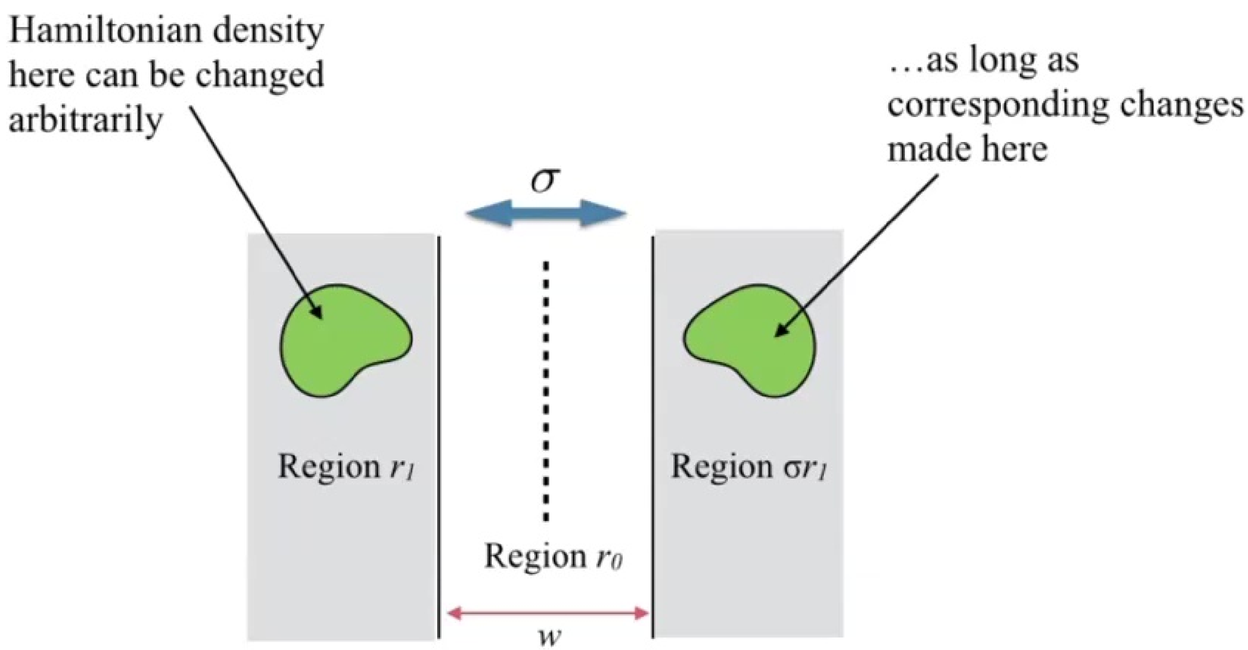
Example 1: 3D topological crystalline insulators
protected by reflection symmetry

Cross section through *bulk*



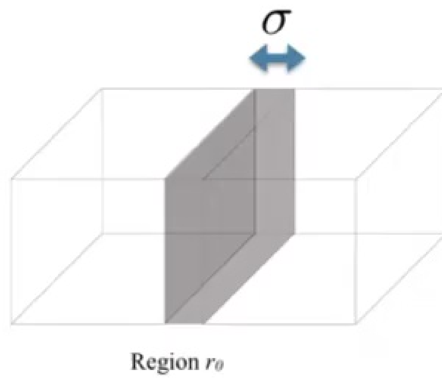
Dimensional reduction

- Quick argument: can locally trivialize any patch away from the mirror plane



Width $w \gg \xi$ for correlation length ξ.

Dimensional reduction

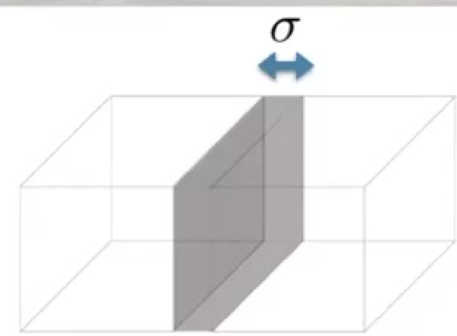


cSPT phases —adiabatically connected to→ Topological crystals

What is this statement good for?

- A general approach to obtain classifications of cSPT phases
- Works for bosonic and fermionic systems (with or without interaction)
- Simple physical picture for cSPT phases

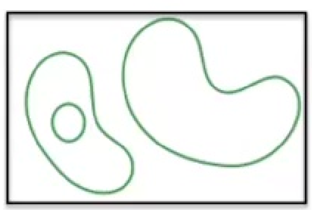
Dimensional reduction



- List all possible $d=2$ building block states

Ex: Topological crystalline insulators. $G=U(1) \times M$

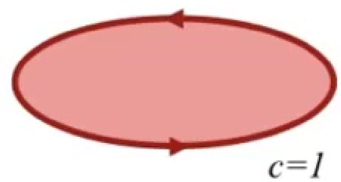
1. Fermionic Z_2 SPT state



(Isobe and Fu 2015)

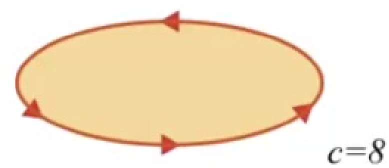
Z_4

2. $d=2$ IQH state



Z

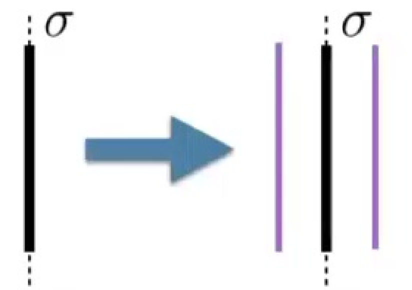
3. $d=2$ E_8 state (Kitaev 2006)



Z

Dimensional reduction

Block equivalence relations



$$|\psi\rangle_{r_0} \to |L\rangle \otimes |\psi\rangle_{r_0} \otimes |R\rangle$$

- Corresponds to making region surrounding the mirror plane wider
- Adjoining layer operation is an equivalence relation

Classification collapses: $Z_4 \times Z \times Z \to Z_8 \times Z_2$

Quantized topological responses of crystalline topological phases

SPTs and quantized responses

- We can understand SPT phases by considering their *quantized* responses
- For example, if we couple a (2+1)-D system with a U(1) symmetry to a background gauge field A, we can have a topological term

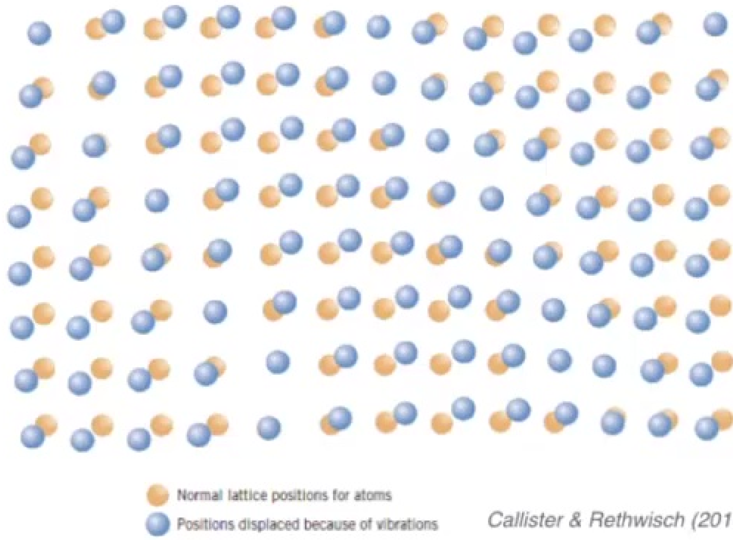
$$S = \frac{K}{4\pi} \int A \wedge dA$$

- Hall conductance is given by the coefficient K.
- For crystalline symmetries, need a notion of *gauging spatial symmetries*

Thorngren and Else, PRX 2018
Manjunath and Barkeshli, PRR 2021; arXiv:2012.11603
Xue-Yang Song, Yin-Chen He, Ashvin Vishwanath, and Chong Wang, PRR 2021
L. Gioia, Chong Wang, and A.A. Burkov, arXiv:2103.09841

Elastic responses

- An alternative approach to characterize crystalline SPT phases is given by quantized topological responses in the elasticity theory —topological terms for phonons



Callister & Rethwisch (2010)

V. Cvetkovic, Z. Nussinov, and J. Zaanen, Philos. Mag. 86, 2995 (2006),

J. Nissinen and G. E. Volovik, J. Exp. Theor. Phys. 127, 948 (2018),

J. Nissinen and G. E. Volovik, Phys. Rev. Research 1, 023007 (2019), arXiv:1812.03175.

Jaakko Nissinen, arXiv:2009.14184.

Dominic V. Else, **SJH**, Abhinav Prem, and Andrey Gromov, arXiv:2103.13393

SJH, Chang-Tse Hsieh, Jiabin Yu, arXiv:2107.03409

Elastic responses

- An alternative approach to characterize crystalline SPT phases is given by quantized topological responses in the elasticity theory —topological terms for phonons

- For systems with translation and U(1) symmetry:

$$\mathcal{L} = \mathcal{L}_0 + \frac{\nu}{2\pi}\epsilon^{\mu\nu} A_\mu \partial_\nu \theta^1 \quad (d=1),$$

$$\mathcal{L} = \mathcal{L}_0 + \frac{\nu}{8\pi^2}\epsilon^{\mu\nu\lambda}\epsilon_{IJ} A_\mu \partial_\mu \theta^I \partial_\nu \theta^J, \quad (d=2)$$

$$\mathcal{L} = \mathcal{L}_0 + \frac{\nu}{48\pi^3}\epsilon^{\mu\nu\lambda\sigma}\epsilon_{IJK} A_\mu \partial_\nu \theta^I \partial_\lambda \theta^J \partial_\sigma \theta^K \quad (d=3).$$

V. Cvetkovic, Z. Nussinov, and J. Zaanen, Philos. Mag. 86, 2995 (2006),

J. Nissinen and G. E. Volovik, J. Exp. Theor. Phys. 127, 948 (2018),

J. Nissinen and G. E. Volovik, Phys. Rev. Research 1, 023007 (2019), arXiv:1812.03175.

Jaakko Nissinen, arXiv:2009.14184.

Dominic V. Else, **SJH**, Abhinav Prem, and Andrey Gromov, arXiv:2103.13393

SJH, Chang-Tse Hsieh, Jiabin Yu, arXiv:2107.03409

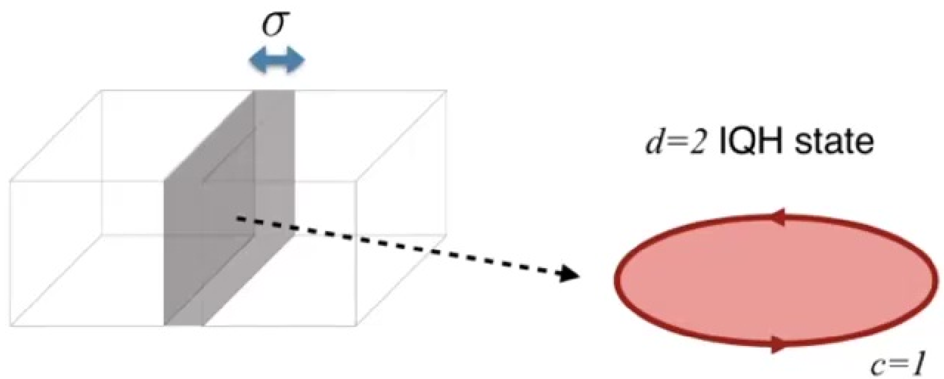
- How to relate the responses to the general classification?

- How to derive these topological terms?

Some simple examples

Effective field theory for TCI

Topological crystalline insulators: $G = U(1) \times \mathbb{Z}_2^P$



Assuming the bulk is described by the following massive Dirac theory:

$$\mathcal{L} = -i\bar{\psi}\gamma^\mu\partial_\mu\psi - im_0\bar{\psi}\psi,$$

$$g_r^y : \psi(\boldsymbol{r}) \to i\sigma^2\tau^3\psi(g_r^y\boldsymbol{r}),$$

Effective field theory for TCI

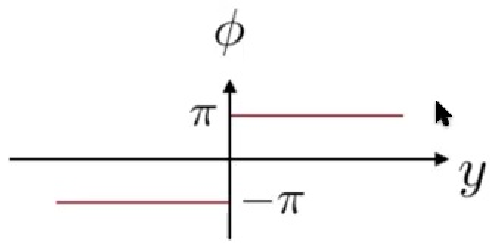
$$\mathcal{L} = -i\bar{\psi}\gamma^\mu\partial_\mu\psi - im_0\bar{\psi}\psi,$$

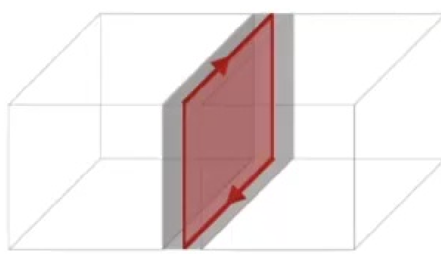
$$g_r^y : \psi(\boldsymbol{r}) \to i\sigma^2\tau^3\psi(g_r^y\boldsymbol{r}),$$

Add a spatially dependent mass term:

$$\mathcal{L}_m = -im\bar{\psi}e^{i\phi(\boldsymbol{r})\gamma^0\tau^2}\psi.$$

The mass interface supports an IQH state with Chern number +1

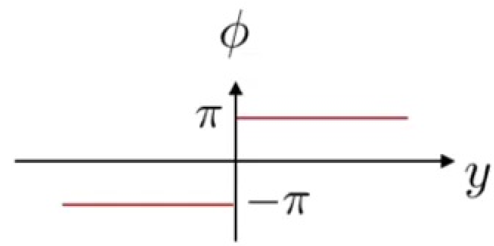




Effective field theory for TCI

Couple to the U(1) gauge field and integrate out the massive fermions,

$$S = \frac{1}{4\pi}\int AdA \wedge dP, \qquad P = \phi/2\pi.$$



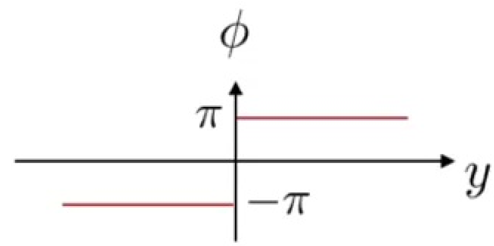
Can check that there is a c=1 IQH state at the interface
after integrating along y-direction

$$S_{\text{Interface}} = \frac{1}{4\pi}\int AdA,$$

Effective field theory for TCI

Couple to the U(1) gauge field and integrate out the massive fermions,

$$S = \frac{1}{4\pi} \int AdA \wedge dP, \qquad P = \phi/2\pi.$$

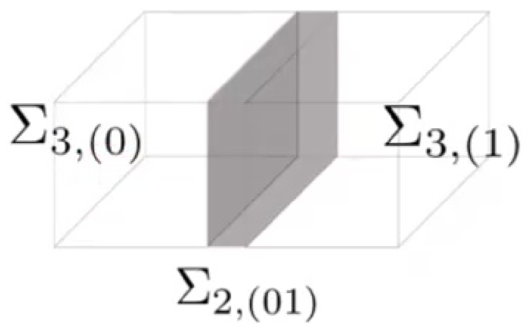


Can check that there is a c=1 IQH state at the interface
after integrating along y-direction

$$S_{\text{Interface}} = \frac{1}{4\pi} \int AdA,$$

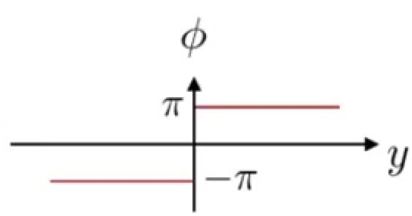
Mass interfaces

Classification of mass interfaces



$$\phi(\Sigma_{3,(1)}) = \phi(\Sigma_{3,(0)}) + 2\pi r(\Sigma_{2,(01)}), \qquad r(\Sigma_{2,(01)}) \in \mathbb{Z}$$

$$g_r^y \cdot r = -r.$$



Mass interfaces

There is a redundancy

$$\phi(\Sigma_{3,(i)}) \to \phi(\Sigma_{3,(i)}) + 2\pi h(\Sigma_{3,(i)}),$$
$$r(\Sigma_{2,(01)}) \to r(\Sigma_{2,(01)}) + h(\Sigma_{3,(1)}) - h(\Sigma_{3,(0)})$$

➡ r is a Z-valued cocycle in $H^1(M, \mathbb{Z}^r)$

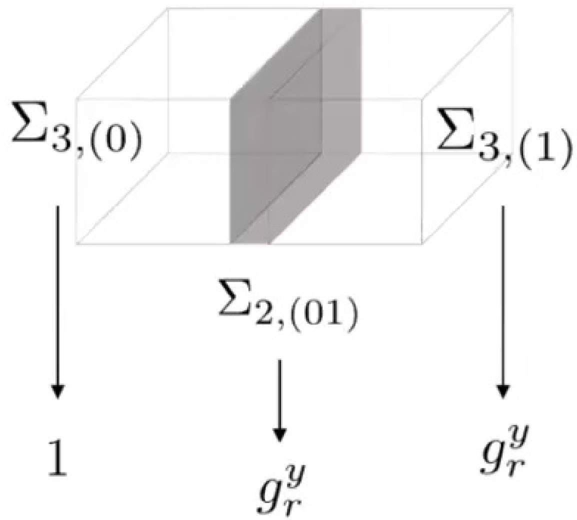
However, the mass interface is classified by $H^1(B\mathbb{Z}_2^P, \mathbb{Z}^r)$

Mass interfaces

Classification of mass interfaces

$$f : M \to B\mathbb{Z}_2^P$$

Cell-decomposition given by the
fundamental domain or asymmetric unit

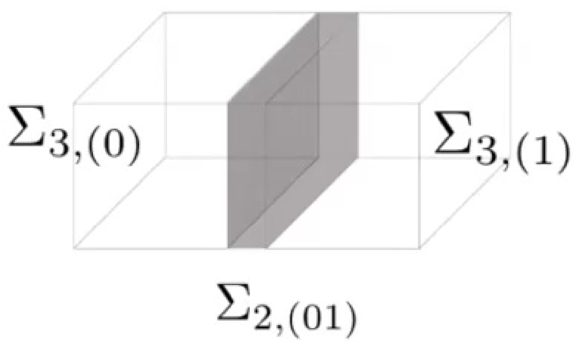


Mass interfaces

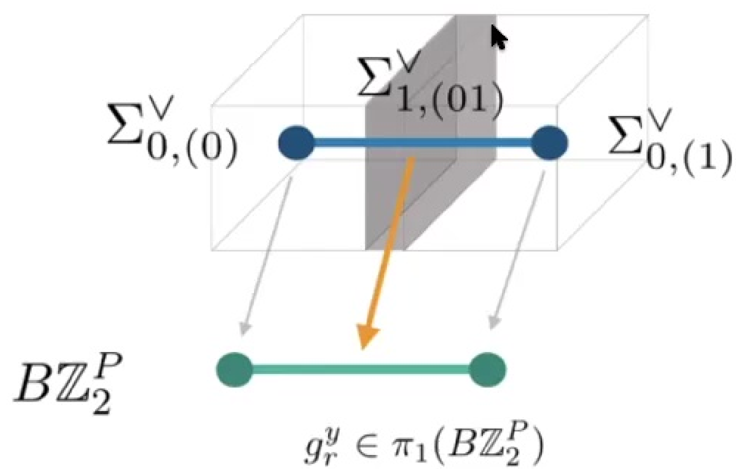
Classification of mass interfaces

$$f : M \to B\mathbb{Z}_2^P$$

Cell-decomposition given by the fundamental domain or asymmetric unit



Dual Cell-decomposition



(Dark green gots are identified)

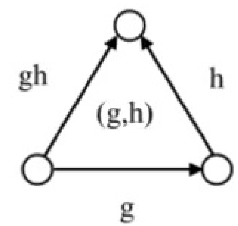
Review: classifying space BG

Assuming G is a discrete group

The **classifying space BG** is a simplicial complex whose n-simplices are labelled by n-tuples of group elements.

There is a unique 0-simplex given by the empty tuple.

1-simplices are labeled by group elements



2-simplices are labeled by 2-tuples with boundary rules given by the group multiplication

......

$$\pi_1(BG) = G \qquad \pi_{>1}(BG) = 0$$

Mass interfaces

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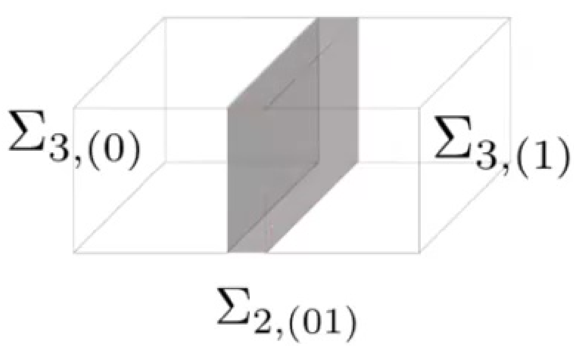

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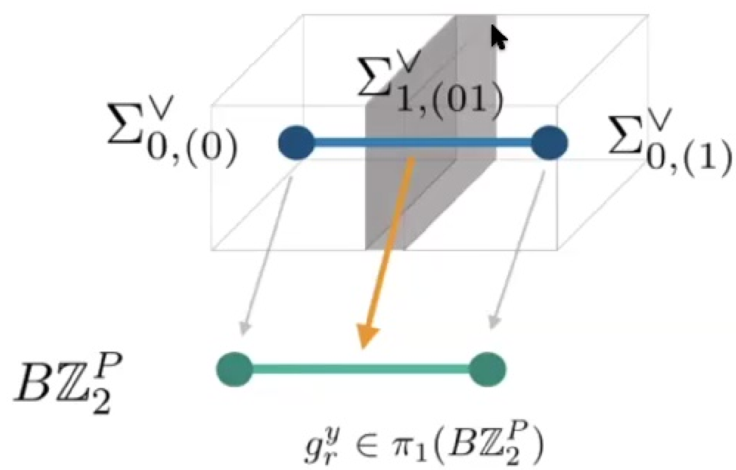


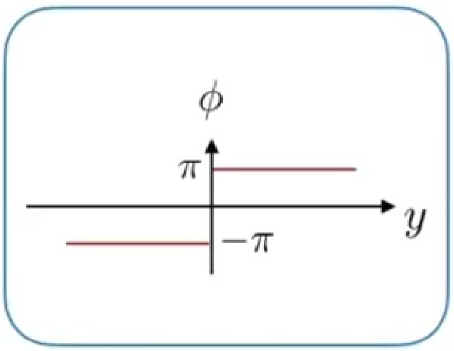
(Dark green gots are identified)

Mass interfaces

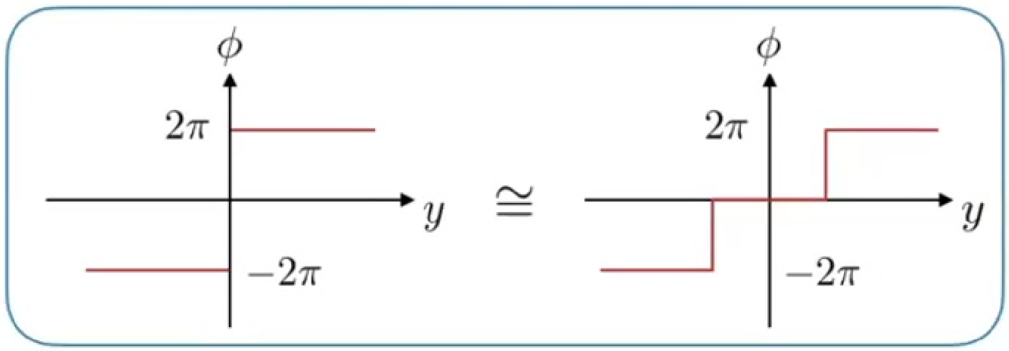
The correct statement is that

$$r = f^* c$$

$$c \in H^1(B\mathbb{Z}_2^P, \mathbb{Z}^r) = \mathbb{Z}_2$$



Non-trivial



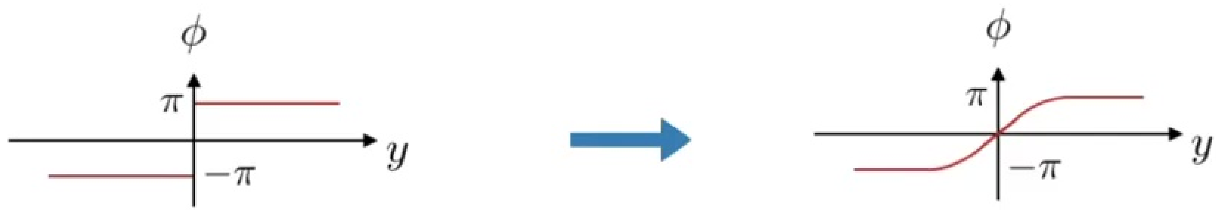
Trivial

Effective field theory for TCI

Effective field theory for TCI:

$$S = \frac{1}{4\pi}\int AdA \wedge dP, \qquad P \;=\; \phi/2\pi.$$

Can take the smooth limit:



$$\int_{\boldsymbol{r}_0}^{g_r^y \boldsymbol{r}_0} d\tilde{P} = r,$$

$d\tilde{P}$ is a smooth 1-form

$|\boldsymbol{r}_o| >> \xi$

$r = f^* c$

$c \in H^1(B\mathbb{Z}_2^P, \mathbb{Z}^r) = \mathbb{Z}_2$

Effective field theory for TCI

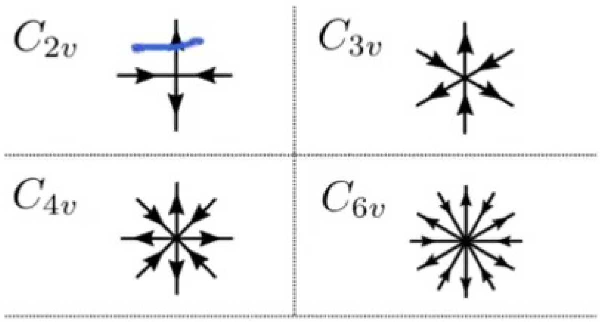
Generalize to other point group symmetries

Effective field theory for TCIs: $G = U(1) \times C_{nv}$

$$S = \frac{1}{4\pi}\int AdA \wedge d\tilde{P}^{(n)}.$$

$$\int_{\boldsymbol{r}_0}^{g_r^y \boldsymbol{r}_0} d\tilde{P}^{(n)} = r$$

$$\int_{\boldsymbol{r}_0}^{R\boldsymbol{r}_0} d\tilde{P}^{(n)} = 0.$$



Electromagnetic responses for TCIs

Anomalous Hall effect

By the gauge invariance, the following current is conserved:

$$J^{\mu} = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda\delta}\partial_{\nu}\tilde{P}^{(n)}\partial_{\lambda}A_{\delta}.$$

The 3d crystalline Hall conductivity is defined as

$$J^{i} = \sigma^{ij}E_{j},$$

$$\sigma^{ij} = \frac{1}{2\pi}\epsilon^{ijk}\partial_{k}\tilde{P}^{(n)}.$$

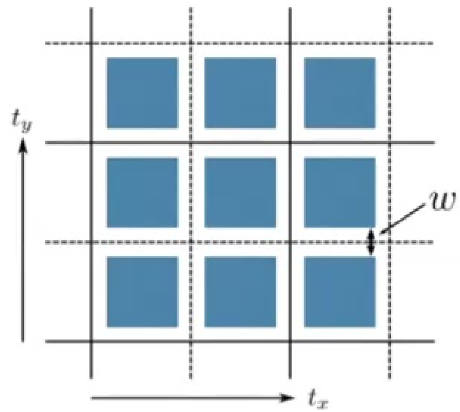
The Hall conductivity tensor is space-time dependent and is not quantized by itself.

An example of the quantized quantity is

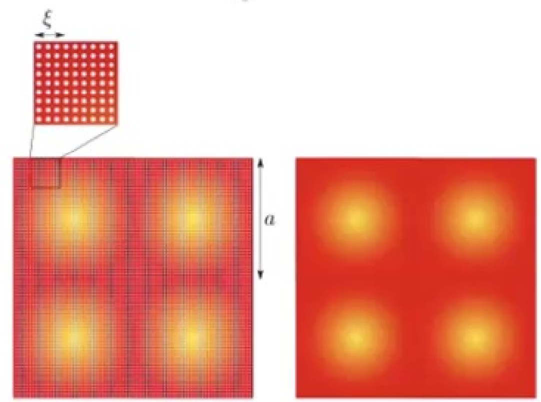
$$\frac{1}{2}\int_{\boldsymbol{r}_0}^{g_r^y \boldsymbol{r}_0}\epsilon_{ijk}\sigma^{jk}dx^{i} = \frac{1}{2\pi}\int_{\boldsymbol{r}_0}^{g_r^y \boldsymbol{r}_0}\partial_{i}\tilde{P}^{(n)}dx^{i} = \frac{1}{2\pi}r,$$

Space group symmetry

- Can't run dimensional reduction for **wallpaper/space groups** because correlation length can be larger than lattice constant



(**SJH**, Song, Huang and Hermele 2017)



(Thorngren and Else 2018)

1. Add a fine mesh of ancillas to obtain a much finer lattice.
2. **Assumption:** can make correlation length arbitrarily small.
3. If true, can reduce any space group SPT state down to a topological crystal state

1d Atomic insulators

$$G = U(1) \times \Gamma$$

$$H = -t \sum_{x}^{L} (c_x c_{x+l}^\dagger + h.c.) - \mu \sum_{x}^{L} c_x^\dagger c_x + \cdots .$$

$a \gg \xi$

Assume there is an unit charge per unit cell

Translation acts on the fermion by

$$t : c_x \to c_{x+a},$$

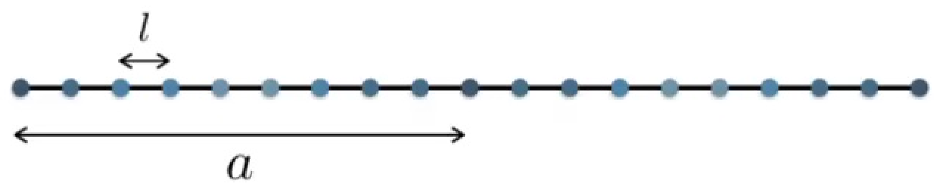
Go to the IR limit by expanding the fermion in terms of slowly varying fields

$$c_x \sim \psi_R(x) e^{k_F x} + \psi_L(x) e^{-k_F x}, \qquad k_F = \pi / l.$$

1d Atomic insulators

The low-energy theory is described by a massive Dirac fermion:

$$\mathcal{L} = -i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi + im\bar{\psi}\psi. \qquad \psi(x) = (\psi_R, \psi_L)^T$$



$a \gg \xi$

Translation acts on the low-energy fields by

$$t : \psi(x) \to e^{ik_F a \sigma_z}\psi(x+a) \qquad k_F = \pi/l$$

Translation is NOT a pure internal symmetry as in the usual IR theory.

$$\psi(x) \sim \psi(x+l)$$

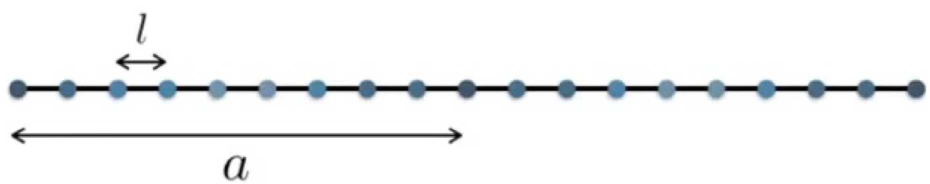
$$\psi(x) \not\approx \psi(x+a)$$

Could have spatial variation on the scale $R >> l, R < a$.

1d Atomic insulators

The low-energy theory is described by a massive Dirac fermion:

$$\mathcal{L} = -i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi + im\bar{\psi}\psi. \qquad \psi(x) = (\psi_R, \psi_L)^T$$



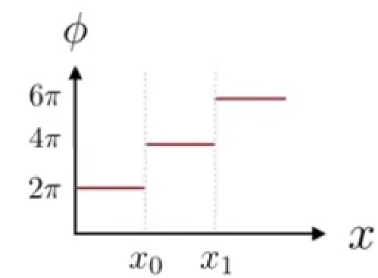
$a \gg \xi$

How to recover the picture that there is a unit charge per unit cell?

Add the following spatially dependent mass term:

$$\mathcal{L}_m = im_0\bar{\psi}e^{i\phi(x)\gamma^{01}}\psi,$$

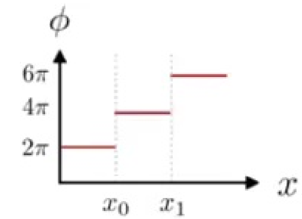
Translation requires that $\phi(x+a) = \phi(x) + 2\pi\nu$



Effective field theory for 1d Atomic insulators

Couple to the U(1) gauge field and integrate out the massive fermions,

$$S = \nu \int A \wedge (\frac{d\phi}{2\pi})$$

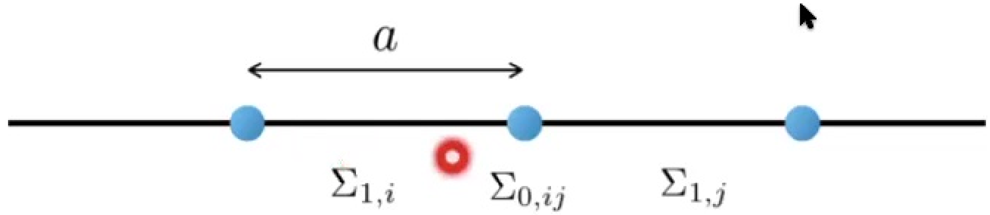


The effective action at the mass interfaces

$$\begin{aligned} S_{\text{eff}} &= \frac{\nu}{2\pi} \int \epsilon^{\mu\nu} A_\mu \partial_\nu \phi d^2 x \\ &\sim \nu \sum_i \int A_0 \delta(x - x_i) d^2 x \\ &= \sum_i \nu \int A_0 dt. \end{aligned}$$

Mass interfaces

Classification of mass interfaces



$$\phi(\Sigma_{1,j}) = \phi(\Sigma_{1,i}) + 2\pi\tau(\Sigma_{0,ij}),\ \tau(\Sigma_{0,ij}) \in \mathbb{Z}.$$

Can show that

$$\tau(\Sigma_{0,ij}) + \tau(\Sigma_{0,jk}) = \tau(\Sigma_{0,ik}).$$

There is a redundancy

$$\phi(\Sigma_{1,i}) \to \phi(\Sigma_{1,i}) + 2\pi h(\Sigma_{1,i}),\ h(\Sigma_{1,i}) \in \mathbb{Z}$$
$$\tau(\Sigma_{0,ij}) \to \tau(\Sigma_{0,ij}) + h(\Sigma_{1,j}) - h(\Sigma_{1,i})$$

Mass interfaces

Cocycle condition

$$\tau(\Sigma_{0,ij}) + \tau(\Sigma_{0,jk}) = \tau(\Sigma_{0,ik}).$$

Coboundary

$$\tau(\Sigma_{0,ij}) \to \tau(\Sigma_{0,ij}) + h(\Sigma_{1,j}) - h(\Sigma_{1,i})$$

→ τ is a Z-valued cocycle in $H^1(M, \mathbb{Z})$

Similar to the previous example, the mass interfaces are classified by

$$\alpha \in H^1(B\Gamma, \mathbb{Z})$$

$$\tau = f^*\alpha$$

$$f : M \to B\Gamma$$

f realized by the cell-decomposition of fundamental domains

Effective field theory for 1d Atomic insulators

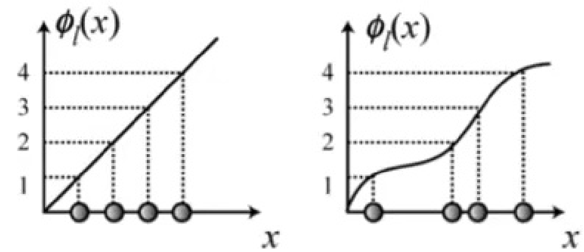
Take the smooth limit:

$$\int_{x_0}^{x_0+a} d\phi = (\phi(x_0+a) - \phi(x_0)) = \frac{2\pi}{a}\tau, \qquad |x_0| \gg \xi$$

Simplest configuration:

$$\phi(x) = \frac{2\pi}{a}\tau x = b_1 \tau x, \qquad b_1 = 2\pi/a$$

Reciprocal lattice vector



(From Giamarchi, 2004)

This is essentially the "labeling field" introduced by Haldane in 1981.

(Haldane, Phys. Rev. Lett. 47, 1840 (1981))

Define a 1-form $E = d\phi/2\pi$

This is essentially the elastic tetrad or vielbein in the topological elasticity theory

Higher dimensional atomic insulators

d=1:

$$S_{\text{eff}} = \nu \int A \wedge E$$

d=2:

$$S_{\text{eff}} = \frac{\nu}{2} \int \epsilon_{IJ} A \wedge E^I \wedge E^J$$

d=3:

$$S_{\text{eff}} = \frac{\nu}{6} \int \epsilon_{IJK} A \wedge E^I \wedge E^J \wedge E^K$$

The 1-form $E^I = d\phi^I / 2\pi$

classified by $H^1(B\Gamma, \mathbb{Z}) = H^1(\mathbb{T}^d, \mathbb{Z})$

General form of the effective field theory

$$S = \int \mathcal{L}_{\text{CS}}^{d-k+1}[A] \wedge \Omega_k, \qquad \mathcal{L}_{\text{CS}}^{2s+1}[A] = \frac{1}{(s+1)!} A \wedge (\frac{dA}{2\pi})^s,$$

$$\int_{C_{\{g\}}} \Omega_k = N_{\{g\}}$$

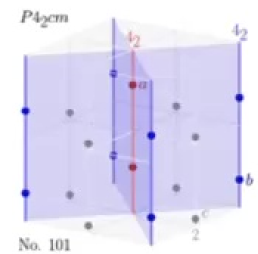
, where $C_{\{g\}}$ is a k-cycle intersecting with some Wyckoff position and

$$N_{\{g\}} = \int_{C_{\{g\}}} f^* \alpha \qquad \begin{matrix} \alpha \in H^k(BG_c, \mathbb{Z}) \\ f : M \to BG_c \end{matrix}$$

Spacetime dimensions	Symmetry group G	Ω_k in Eq. (3)	Integral conditions
$1+1$D	$U(1) \times \Gamma$	$\Omega_1 = E$	$\int_{C_{t_1}} E = \int_{C_{t_1}} f^* \tau \in \mathbb{Z}, \tau \in H^1(B\Gamma, \mathbb{Z})$
$1+1$D	$U(1) \times D_1$	$\Omega_1 = dP$	$\int_{C_{g_r}} dP = \int_{C_{g_r}} f^* r \in \mathbb{Z}_2, r \in H^1(BD_1, \mathbb{Z}^r)$
$2+1$D	$U(1) \times \Gamma$	$\Omega_2 = \frac{1}{2}\epsilon_{IJ} E^I \wedge E^J$	$\int_{C_{t_I}} E^I = \int_{C_{t_I}} f^* \tau^I \in \mathbb{Z}, \tau^I \in H^1(BT_I, \mathbb{Z})$
$2+1$D	$U(1) \times C_N$	$\Omega_2 = d\omega_1$	$\int_{D_u} d\omega_1 = \int_{D_u} f^* b \in \mathbb{Z}_N, b \in H^2(BC_N, \mathbb{Z})$
$3+1$D	$U(1) \times \Gamma$	$\Omega_3 = \frac{1}{6}\epsilon_{IJK} E^I \wedge E^J \wedge E^K$	$\int_{C_{t_I}} E^I = \int_{C_{t_I}} f^* \tau^I \in \mathbb{Z}, \tau^I \in H^1(BT_I, \mathbb{Z})$
$3+1$D	$U(1) \times C_{nv}$	$\Omega_1 = dP^{(n)}$	$\int_{C_{g_r}} dP^{(n)} = \int_{C_{g_r}} f^* r \in \mathbb{Z}_2, r \in H^1(BD_1, \mathbb{Z}^r)$ $\int_{C_u} dP^{(n)} = \int_{C_u} f^* a = 0, a \in H^1(BC_N, \mathbb{Z})$

Summary and outlook

cSPT phases —adiabatically connected to→ Topological crystals

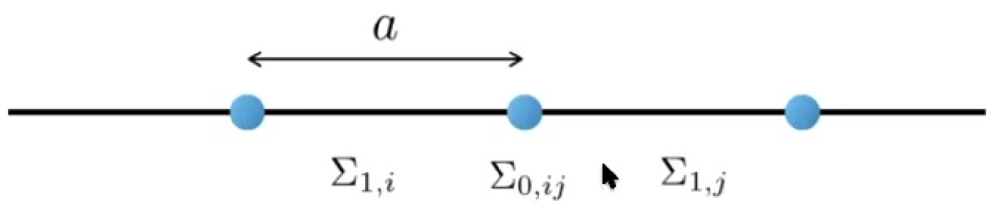


Effective field theories

- Studying other topological terms that are sensitive to the point group "charges" of the building blocks
- Generalizing to full space group symmetries
- Applying it to topologically ordered states, interacting gauge theories, topological semimetals….
- Effective field theories for non-invertible defect networks

Mass interfaces

Classification of mass interfaces



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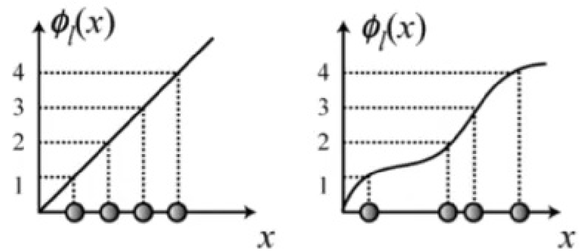

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