Title: Super Cartan geometry, loop quantum supergravity and applications

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Series: Quantum Gravity

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Abstract: This talk is devoted to the geometric approach to supergravity and applications in the framework of loop quantum gravity. Among other things, this approach leads to a reformulation of the theory in which (part of) supersymmetry manifests itself in terms of a gauge symmetry. Using the interpretation of supergravity in terms of a super Cartan geometry, we will derive the Holst variant of the MacDowell-Mansouri action for N=1 and N=2 AdS supergravity in D=4 for arbitrary Barbero-Immirzi parameters. We will show that these actions provide unique boundary terms that ensure local supersymmetry invariance at boundaries. The chiral case is special. The action is invariant under an enlarged gauge symmetry, and the boundary theory is a super Chern-Simons theory. The action also implies boundary conditions that link the super electric flux through, and the super curvature on, the boundary. Applications we have in mind are supersymmetric black holes and loop quantum cosmology. To this end, we will study a class of symmetry reduced models of chiral supergravity. The enlarged gauge symmetry of the chiral theory is essential as it allows for nontrivial fermionic degrees of freedom even if one imposes spatial isotropy. The quantization of the theory yields a natural state space and allows a consistent implementation of the constraint algebra.

Finally, we will give an outlook on applications towards a quantum description of supersymmetric black holes in the context of LQG and possible relations to superstring theory.

Super Cartan geometry, LQG and applications

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September 16, 2021

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Section 1

Introduction

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Supersymmetry

Coleman-Mandula: most general Lie algebra of symmetries of the S-matrix has form

$$\mathfrak{iso}(\mathbb{R}^{1,3}) \oplus \text{internal sym.} \qquad (1)$$

Only way around this seems to be through new form of symmetry:

Haag-Łopuszański-Sohnius theorem

Going away from (1) in an interacting QFT with mass gap requires **super Lie algebras**, i.e. $\mathbb{Z}_2$-graded algebras $(\mathfrak{g}, [\cdot,\cdot])$ of the form

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \quad \text{with} \quad [\cdot,\cdot] : \text{(anti) commutator on } \mathfrak{g}_0 \ (\mathfrak{g}_1)$$

such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ (+ graded Jacobi identity)

Supersymmetry

$\Rightarrow$ smallest possible superalgebra containing spacetime symmetries:

super Poincaré/super anti-de Sitter $\mathfrak{osp}(1|4)$

$$\mathfrak{iso}(\mathbb{R}^{1,3|4}) = \underbrace{\mathbb{R}^{1,3} \ltimes \mathfrak{so}(1,3)}_{\mathfrak{g}_0} \oplus \underbrace{S_{\mathbb{R}}}_{\mathfrak{g}_1}$$

generators: P_I, M_{IJ}, $Q_\alpha = (Q_A, Q^{A'})^T$ (Majorana spinor)

$$[P_I, Q_\alpha]_- = 0 - \frac{1}{2L} Q_\beta (\gamma_I)^\beta{}_\alpha$$

$$[M_{IJ}, Q_\alpha]_- = \frac{1}{2} Q_\beta (\gamma_{IJ})^\beta{}_\alpha$$

$$[Q_\alpha, Q_\beta]_+ = \frac{1}{2}(C\gamma_I)_{\alpha\beta} P^I + \frac{1}{4L}(C\gamma^{IJ})_{\alpha\beta} M_{IJ}$$

$$AdS_4 := \{x \in \mathbb{R}^5 | -(x^0)^2 + (x^1)^2 + \ldots + (x^3)^2 - (x^4)^2 = -L^2\}$$

Supersymmetry

Since new generators Q_α...

- transform as spinors, they relate particles of integer, half-integer spin,
- are anti-commuting, they relate bosons and fermions.

Can include further fermionic generators Q_α^r, $r = 1, \dots, \mathcal{N}$
→ $\mathcal{N}$**-extended SUSY**!

Application to gravity

Most ambitious use of this kind of symmetry:

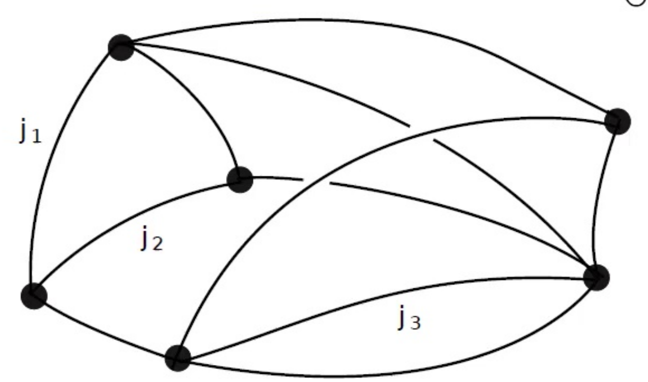
local supersymmetry ⇒ **supergravity** (SUGRA)

For $\mathcal{N} = 1, D = 4$ contains gravitational field and a spin $\frac{3}{2}$ fermion.

Loop quantum gravity

Framework for quantum gravity that

- attempts to retain as much as possible of general covariance through quantization
- starting point: reformulation of GR in terms of **Ashtekar-Barbero variables**
- gives canonical GR the kinematical structure of a **Yang-Mills theory**
- quantum states: string-like excitations of the gravitational field $\rightarrow$ **spin networks**, **spin foams**



LQG and supergravity

What has been done?

- Canonical SUGRA using Ashtekar-like chiral variables $\rightarrow$ additional SUSY constraints [Jacobson '88]
- Gauss and a SUSY constraint generate local $\mathfrak{osp}(1|2)$ symmetry $\rightarrow$ can construct $\mathfrak{osp}(1|2)$-valued connection [Fülöp '94, Gambini + Obregon + Pullin '96]
- Formal quantization [Gambini + Obregon + Pullin '96, Ling + Smolin '99]
- Canonical theory for higher D, quantization of RS fields, p-form fields [Bodendorfer+Thiemann+Thurn '11]

LQG and supergravity

My goals:

- Following [L+S] keep SUSY manifest as much as possible
- Where does the $\mathfrak{osp}(1|2)$ symmetry come from? $\rightarrow$ Understand the geometric origins with a view towards generalizations:
  - Immirzi parameters
  - higher $\mathcal{N} > 1$
  - boundary theory ($\rightarrow$ BPS states, **black holes**)
- mathematically rigorous formulation, both classically and in quantum theory:
  - structure and properties of graded holonomies
  - structure of Hilbert spaces $\leftrightarrow$ relation to standard quantization in LQG with fermions

In this talk

Classical theory:

- Supergravity via Cartan geometry
- Holst-MacDowell-Mansouri SUGRA action for any β
- extension to $\mathcal{N} > 1$!
- boundary theory ($\rightarrow$ black holes!)
- special properties of self-dual theory

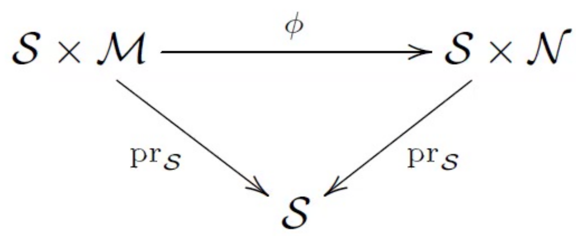
Quantum theory:

- graded connections, -holonomies, and -group integration
- applications: LQC, black holes (WIP)

In this talk

Mathematically clean formulation **studying enriched categories** (not part of this talk!) [KE '20+'21]

- enriched supermanifold as a tuple $\mathcal{M}_{/\mathcal{S}} := (\mathcal{S} \times \mathcal{M}, \mathrm{pr}_{\mathcal{S}})$ together with morphisms ϕ s.t.



- $\mathcal{S}$: parametrization supermanifold

Section 2

Gravity as Cartan geometry

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Klein geometry

F. Klein: "Classify geometry of space via group symmetries".

Example: Minkowski spacetime $\mathbb{M} = (\mathbb{R}^{1,3}, \eta)$

- isometry group $\mathrm{ISO}(\mathbb{R}^{1,3}) = \mathbb{R}^{1,3} \ltimes \mathrm{SO}_0(1,3)$
- event $p \in \mathbb{M}$: $G_p = \mathrm{SO}_0(1,3)$ (isotropy subgroup)

$$\mathrm{ISO}(\mathbb{R}^{1,3})/\mathrm{SO}_0(1,3) \cong \mathbb{M}$$

Definition

A *Klein geometry* is a pair (G, H) where G is a Lie group and $H \subseteq G$ a closed subgroup such that G/H is connected.

Cartan geometry

- flat spacetime $\leftrightarrow$ Klein geometry
- $\Rightarrow$ Cartan geometry as deformed Klein geometry

Definition: Cartan geometry

A **Cartan geometry** modeled on a Klein geometry (G, H) is a principal H-bundle

$$\begin{array}{ccc} P & \xleftarrow{\;r\;} & H \\ \downarrow{\scriptstyle \pi} & & \\ M & & \end{array}$$

together with a **Cartan connection** $A \in \Omega^1(P, \mathfrak{g})$ s.t.

1. $\mathrm{pr}_{\mathfrak{h}} \circ A$ defines ordindary gauge field
2. $A: \; T_pP \to \mathfrak{g}$ isomorphism $\forall p \in P$

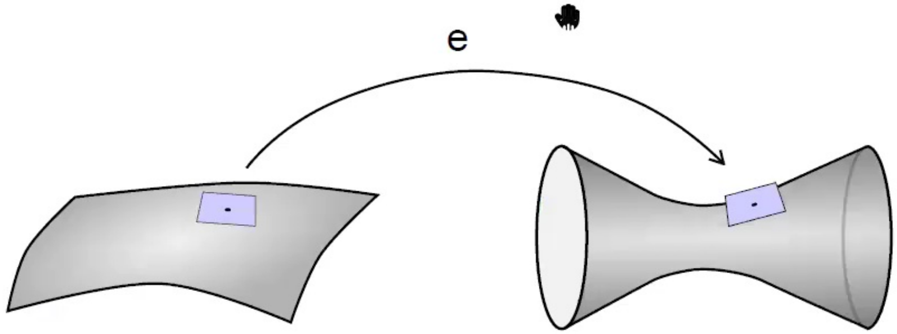
Gravity as Cartan geometry

Example: Cartan geometry modeled over **AdS$_4$** $(\mathrm{SO}(2,3), \mathrm{SO}_0(1,3))$

Cartan connection

$$A = \mathrm{pr}_{\mathbb{R}^{1,3}} \circ A + \mathrm{pr}_{\mathfrak{so}(1,3)} \circ A =: e + \omega$$

- ω: **Lorentz-connection**, e: **soldering form** (co-frame)



Holst as MacDowell-Mansouri

Holst action via Mac-Dowell-Mansouri [A. Randono '06, D. Wise '09]
via perturbed BF-Theory [Freidel+Starodubtsev '05]

Here: AdS_4: can use $\mathfrak{so}(2,3) \cong \mathfrak{sp}(4) \supset \mathfrak{so}(1,3)$, basis $M_{IJ} = \frac{1}{4}[\gamma_I, \gamma_J]$

Definition

$$\mathcal{P}_\beta := \frac{\mathbb{1} + i\beta\gamma_5}{2\beta} : \mathfrak{so}(1,3) \to \mathfrak{so}(1,3), \quad \beta : \textit{Immirzi}$$

$\to$ yields inner product on $\mathfrak{sp}(4)$:

$$\langle \cdot, \cdot \rangle_\beta := \mathrm{tr}(\cdot\, \mathbf{0} \oplus \mathcal{P}_\beta \cdot)$$

Holst-MacDowell-Mansouri action

$$S_{H-MM}[A] = \int_M \langle R[A] \wedge R[A] \rangle_\beta = S^\beta_{Holst} + \textit{boundary term}$$

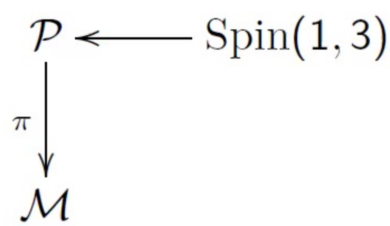
$R[A] = \mathrm{d}A + \frac{1}{2}[A \wedge A]$: Cartan curvature

Section 3

Supergravity & boundary theory

Supergravity

- (AdS) Supergravity as **super Cartan geometry** modeled on super Klein geometry $(\mathrm{OSp}(1|4), \mathrm{Spin}(1,3))$ [D'Auria-Fré-Regge '80, D'Auria-Fré '80, Castellani-D'Auria-Fré '91, KE '20+'21]



- super Cartan connection $\mathcal{A} \in \Omega^1(\mathcal{P}, \mathfrak{osp}(1|4))$
- Decompose

$$\mathcal{A} = \underbrace{\mathrm{pr}_{\mathfrak{g}_1} \circ \mathcal{A}}_{\psi} + \underbrace{\mathrm{pr}_{\mathbb{R}^{1,3}} \circ \mathcal{A}}_{e} + \underbrace{\mathrm{pr}_{\mathfrak{spin}(1,3)} \circ \mathcal{A}}_{\omega}$$

- $\psi = \psi^\alpha Q_\alpha$ (spin-3/2) Rarita-Schwinger field

Supergravity and LQG

Holst action for (extended) Poincaré SUGRA [Tsuda '00, Kaul '07]
via MM ($\mathcal{N} = 1$, β as θ-ambiguity) [Obregon+Ortega-Cruz+Sabido '12]

Here: via Holst projection [KE+HS '21]

- $\mathfrak{osp}(1|4) \cong \mathfrak{sp}(4) \oplus S_{\mathbb{R}}$
- $S_{\mathbb{R}}$ Clifford module $\to$ can naturally extend $\mathcal{P}_\beta$

Definition

$$\mathbf{P}_\beta := \mathbf{0} \oplus \mathcal{P}_\beta \oplus \mathcal{P}_\beta : \mathfrak{osp}(1|4) \to \mathfrak{osp}(1|4)$$

$\to$ yields inner product on $\mathfrak{osp}(1|4)$:

$$\langle \cdot, \cdot \rangle_\beta := \mathrm{str}(\cdot\, \mathbf{P}_\beta \, \cdot)$$

Supergravity and LQG

super Holst-MacDowell-Mansouri action

$$S_{\text{sH-MM}}[\mathcal{A}] = \int_M \langle R[\mathcal{A}] \wedge R[\mathcal{A}] \rangle_\beta$$

Cartan curvature

$R[\mathcal{A}] = \mathrm{d}\mathcal{A} + \frac{1}{2}[\mathcal{A} \wedge \mathcal{A}]$

- $R[\mathcal{A}]^I = \Theta^{(\omega)I} - \frac{1}{4}\bar{\psi} \wedge \gamma^I \psi$
- $R[\mathcal{A}]^{IJ} = R[\omega]^{IJ} + \frac{1}{L^2}\Sigma^{IJ} - \frac{1}{4L}\bar{\psi} \wedge \gamma^{IJ}\psi$
- $R[\mathcal{A}]^\alpha = D^{(\omega)}\psi^\alpha - \frac{1}{2L} e^I (\gamma_I)^\alpha{}_\beta \wedge \psi^\beta$

- $\rightarrow$ yields Holst action of $D = 4$, $\mathcal{N} = 1$ AdS-SUGRA $+$ *bdy terms*

Boundary theory

Q: What is special about *bdy terms* contained in sH-MM-action?

Boundary action: Most general ansatz

$$\mathscr{L}_{\rm bdy} = C_1 F(\omega)^{IJ} \wedge F(\omega)^{KL} \epsilon_{IJKL} + C_2 F(\omega)^{IJ} \wedge F(\omega)_{IJ} + C_3 \mathrm{d}(\bar{\psi} \wedge \gamma_* D^{(\omega)} \psi) + C_4 \mathrm{d}(\bar{\psi} \wedge D^{(\omega)} \psi)$$

- C_1, C_2, C_3, C_4 arbitrary constant coefficients

- $\rightarrow$ require that SUSY of the full theory $\mathscr{L}_{\rm full} = \mathscr{L}_{\rm bulk} + \mathscr{L}_{\rm bdy}$ is preserved at boundary [Andrianopoli+D'Auria '14, A+D'A+Cerchiai +Trigiante '18]

Boundary theory

- in Castellani-D'Auria-Fré approach: **SUSY trafos** as inf. superdiffeomorphisms X such that $\iota_X e^I = 0$
- Since $L_X = \mathrm{d}\iota_X + \iota_X \mathrm{d} \rightarrow$ SUSY-invariance at boundary imposes

Boundary condition

$$\iota_X \mathscr{L}_{\text{full}}|_{\partial M} = 0$$

- $\rightarrow$ turns out that coefficients C_i **uniquely** fixed by this requirement!

Boundary action [KE+HS '21]

$$S_{\text{bdy}}(\mathcal{A}) = \frac{L^2}{\kappa} \int_{\partial M} \langle \omega \wedge \mathrm{d}\omega + \frac{1}{3}\omega \wedge [\omega \wedge \omega] \rangle_\beta + \langle \psi \wedge D^{(\omega)}\psi \rangle_\beta$$

- $\rightarrow$ **exactly** reproduces boundary term in sH-MM action!

Extended SUGRA and boundary theory

What about **extended** SUSY? → Consider $\mathcal{N} = 2$ [KE+HS '21]

Super Cartan connection

$$\mathcal{A} = e^I P_I + \frac{1}{2}\omega^{IJ} M_{IJ} + \Psi^\alpha_r Q^r_\alpha + \hat{A} T$$

- $r = 1, 2$: R-symmetry index (→ gauge group $\mathrm{SO}(2) \cong \mathrm{U}(1)$)
- $\hat{A}$: $\mathrm{U}(1)$ gauge field (*graviphoton*)

- make most general ansatz of boundary term compatible with local SUSY
- → full action $S_{\text{full}}(\mathcal{A})$ acquires MacDowell-Mansouri form! [Andrianopoli et al. '14+'21, KE+HS '21]

Extended SUGRA and boundary theory

super Holst-MacDowell-Mansouri action ($\mathcal{N}=2$) [KE+HS '21]

$$S^{\mathcal{N}=2}_{\text{sH-MM}}[\mathcal{A}] = \int_M \mathrm{str}(R[\mathcal{A}] \wedge \mathbf{P}_\beta R[\mathcal{A}])$$

Definition

$$\mathbf{P}_\beta : \Omega^2(M, \mathfrak{osp}(2|4)) \to \Omega^2(M, \mathfrak{osp}(2|4))$$

$$\mathbf{P}_\beta := \mathbf{0} \oplus \mathcal{P}_\beta \oplus \mathcal{P}_\beta \oplus \mathcal{P}_\beta \oplus \frac{1}{2\beta}(1 + \beta\star)$$

→ yields θ-topological term in $\mathrm{U}(1)$-sector

→ β literally has interpretation as θ-**ambiguity**!

Chiral Theory

- Since $\mathbf{P}_\beta$ for generic β not $\mathrm{OSp}(1|4)$-invariant, SUSY hidden, especially in canonical theory?
- (partial) resolution $\to$ **chiral theory** ($\beta = \mp i$)

Holst projection

$$\mathbf{P}_{-i} : \mathfrak{osp}(1|4)_\mathbb{C} \to \mathfrak{osp}(1|2)_\mathbb{C}$$

$M_{IJ} \mapsto T_i^+ = \frac{1}{2}(J_i + iK_i)$, $Q_\alpha \mapsto Q_A$

Proposition: (T_i^+, Q_A) generate subalgebra

$$[T_i^+, T_j^+] = \epsilon_{ij}{}^k T_k^+$$
$$[T_i^+, Q_A] = Q_B(\tau_i)^B{}_A$$
$$[Q_A, Q_B] = 0 + \frac{1}{L}(\epsilon\sigma^i)_{AB} T_i^+$$

generate D=2 super Poincaré algebra $\mathfrak{sl}(2,\mathbb{C}) \rtimes \mathbb{C}^2$ (orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2)_\mathbb{C}$)

Chiral Theory

Chiral Holst-MacDowell-Mansouri action

$$S_{\text{sH-MM}}[\mathcal{A}] = \int_M \mathrm{str}(R[\mathcal{A}] \wedge \mathbf{P}_{-i} R[\mathcal{A}])$$

- $\Rightarrow$ manifestly $\mathrm{OSp}(1|2)_{\mathbb{C}}$-gauge invariant (similar for $\mathcal{N} = 2$!)
- $\Rightarrow$ SUSY partially becomes true gauge symmetry! [Fülöp '94, Gambini + Obregon + Pullin '96, Ling + Smolin '99, KE '20+'21, KE+HS' 21]

Super Ashtekar connection

$$\mathcal{A}^+ := \mathbf{P}_{-i}\mathcal{A} = A^{+i} T_i^+ + \psi^A Q_A$$

Chiral Theory

Can show for $\mathcal{N} = 1, 2$ that action can be split in the form

Chiral action

$$S^{\beta=-i}_{\text{sH-MM}}(\mathcal{A}) = \frac{i}{\kappa}\int_M \langle \mathcal{E} \wedge F(\mathcal{A}^+)\rangle + \frac{1}{4L^2}\langle \mathcal{E} \wedge \mathcal{E}\rangle + S_{\text{bdy}}$$

$\mathcal{E}$: *super el. field* $\rightarrow (\mathcal{E}, \mathcal{A}^+)$ build up graded symplectic phase space

Boundary term

$$S_{\text{bdy}}(\mathcal{A}^+) = \frac{k}{4\pi}\int_{\partial M} \langle \mathcal{A}^+ \wedge \mathrm{d}\mathcal{A}^+ + \frac{1}{3}\mathcal{A}^+ \wedge [\mathcal{A}^+ \wedge \mathcal{A}^+]\rangle$$

$\rightarrow \mathrm{OSp}(\mathcal{N}|2)_{\mathbb{C}}$-Chern Simons theory, *level* $k = 4\pi i L^2/\kappa$

Boundary condition

$$\underleftarrow{F(\mathcal{A}^+)} = -\frac{1}{2L^2}\underleftarrow{\mathcal{E}}$$

Super Ashtekar connection

- $\rightarrow \mathcal{A}^+$ natural candidate to quantize SUGRA à la LQG
- even exists for extended SUSY
- contains both gravity and matter d.o.f. $\rightarrow$ unified description, more fundamental way of quantizing fermions
- substantially simplifies constraints (canonical form of Einstein equations): partial solution via gauge invariance
- **boundary theory** described by **super Chern-Simons theory** $\rightarrow$ natural candidate to study inner boundaries in LQG ($\rightarrow$ BPS states, black holes)
- $\leftrightarrow$ boundary theories in string theory [Mikhaylov + Witten '14]

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Quantum theory: Overview: Bosonic theory

- bosonic A^+ induces **holonomy** along path e

$$h_e[A^+] = \mathcal{P}\exp\left(-\int_e A^+\right)$$

- $\rightarrow$ for any finite graph $\gamma = \bigcup_i e_i$ yields element in

$$\mathcal{A}_\gamma := \mathrm{Hom}_{\mathbf{Cat}}(I(\gamma), \mathbf{G}),\ I(\gamma) \ni e \mapsto h_e[A^+]$$

$\mathcal{A}_\gamma$: *set of generalized connections*

- defines **projective family**:

$$\mathcal{A}_{\gamma'} \rightarrow \mathcal{A}_\gamma, \quad \forall \gamma \subseteq \gamma'$$

Quantum theory: Overview: Bosonic theory

- $\mathcal{A}_\gamma$ starting point for quantization à la Ashtekar-Lewandowski
- **inductive family** of pre-Hilbert spaces

$$\mathcal{H}_\gamma := \mathrm{Cyl}^\infty(\mathcal{A}_\gamma)$$

$\mathrm{Cyl}^\infty(\mathcal{A}_\gamma)$: *smooth functions on* $\mathcal{A}_\gamma$

- $\mathcal{H}_\gamma \cong C^\infty(G^{|E(\gamma)|}) \to$ Haar measure on G induced inner product on pre-Hilbert space $\mathcal{H}_\gamma$
- have to impose **cylindrical consistency** of inner product under

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Quantum theory: Chiral SUGRA

- Quantization: study $\mathcal{A}^+$ and associated holonomies
- turns out $\rightarrow$ need additional **parametrizing supermanifold** $\mathcal{S}$ to incorporate **anticommutative** nature of fermionic fields

Super holonomies [KE '20+'21]

$$h_e[\mathcal{A}^+] = h_e[A^+] \cdot \mathcal{P}\exp\left(-\oint_e \mathrm{Ad}_{h_e[A^+]^{-1}}\psi\right) : \mathcal{S} \to \mathcal{G}$$

$\mathcal{A}^+ = A^+ + \psi$ and $e: [0,1] \to M \subset \mathcal{M}$

- for $\mathcal{S} = \{*\}$: $h_e[\mathcal{A}^+]$ reduces to bosonic holonomy $h_e[A^+]$

Quantum theory: Chiral SUGRA

- $\mathcal{A}_{\mathcal{S},\gamma}$: *set of generalized super connections*

$$\begin{array}{ccc} \mathcal{A}_{\mathcal{S}',\gamma'} & \longrightarrow & \mathcal{A}_{\mathcal{S}',\gamma} \\ \downarrow & & \downarrow \\ \mathcal{A}_{\mathcal{S},\gamma'} & \longrightarrow & \mathcal{A}_{\mathcal{S},\gamma} \end{array}$$

- for fixed graph γ induces functor $\mathcal{A}_\gamma : \mathcal{S} \to \mathcal{A}_{\mathcal{S},\gamma}$
  $\to$ **Molotkov-Sachse-type smf.**
- $\to$ can study cylindrical functions and invariant measures on $\mathcal{A}_\gamma$
- covariance under reparametrization requires **Berezin-type integral** for fermionic d.o.f.
- $\to$ induces **Krein structure** on state space $\leftrightarrow$ strong similarities to quantization of fermions in standard LQG

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Comparison: standard LQG with fermions

- pre-Hilbert space at a vertex

$$\mathcal{H}_v = \mathcal{O}(\mathcal{G}),\ \mathcal{G} = \mathrm{SU}(2)^{|E(v)|} \times \mathbb{C}_v^{0|4}$$

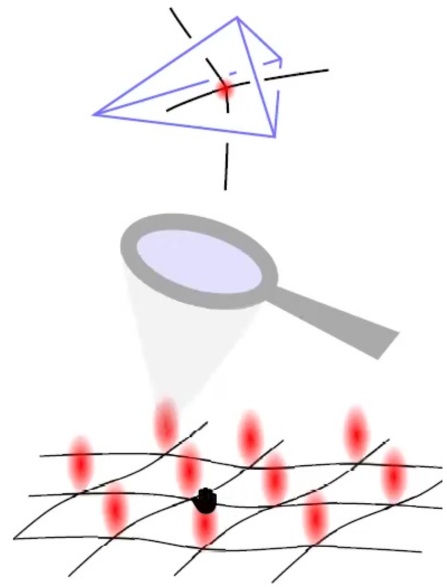
- local gauge transformation

$$g \triangleright f = f(\{g \cdot h_e\}_e, g(v) \cdot \theta_v)$$

- invariant measure:

$$\int_{\mathcal{G}} = \int_{\mathrm{SU}(2)^{|E(v)|}} \int_B$$

yields **Krein space** $(\mathcal{H}_v, \int_{\mathcal{G}}) \overset{Krein\ compl.}{\rightarrow}$ [Thiemann '98]



Quantum theory: Chiral SUGRA

Issues:

- work with complex variables $\rightarrow$ gauge group generically non-compact
- have to deal with indefinite inner product spaces and cylindrical consistency
- still need to recover ordinary real SUGRA $\rightarrow$ have to solve **reality conditions**
- possible resolution: unitary orthosmplectic group $\mathrm{UOSp}(1|2)$ (WIP)
- exact solution in symmetry-reduced model [KE+HS '20]

Symmetry reduction

- supersymmetric minisuperspace models [D'Eath + Hughes '88+'92, D'Eath + Hawking + Obregon '93]
- hybrid homogeneous isotropic ansatz for bosonic and fermionic d.o.f.
- in general: fermions not compatible with isotropy
- **But**: in (chiral) LQC can exploit enlarged gauge symmetry!
- $\rightarrow$ natural interpretation in terms of homogeneous isotropic super connection [KE '20, KE+HS '20]

Symmetry reduced connection

$$\mathcal{A}^{+} = c\,\mathring{e}^{i}\,T_{i}^{+} + \psi_{A}\mathring{e}^{AA'}\,Q_{A'}$$

$\mathring{e}^{i}$: fiducial co-triad

- **Also**: can show that this is the most general ansatz consistent with reality conditions (contorsion tensor isotropic)

Loop quantum cosmology

- Construction of state space via super holonomies $h_e[\mathcal{A}^+]$ along straight edges of a fiducial cell
- $\Rightarrow$ motivates state space of quantum theory

Hilbert space

$$\mathcal{H} = \overline{H_{\mathrm{AP}}(\mathbb{C})}^{\|\cdot\|} \otimes \Lambda[\psi_{A'}]$$

- reality condition in quantum theory can be solved exactly! [Wilson-Ewing '15, KE+HS '20]

Quantum right SUSY constraint

$$\widehat{S}^R_{A'} = \frac{3g^{\frac{1}{2}}}{2\lambda_m}|v|^{\frac{1}{4}}\left((\mathcal{N}_- - \mathcal{N}_+) - \frac{\kappa\lambda_m}{6g|v|}\widehat{\Theta}\right)|v|^{\frac{1}{4}}\widehat{\phi}_{A'}$$

- λ_m: quantum area gap (full theory)

Loop quantum cosmology

- Quantum algebra between left and right SUSY constraint closes and reproduces Poisson algebra exactly!

Quantum algebra

$$[\widehat{S}^{LA'}, \widehat{S}^{R}_{A'}] = 2i\hbar\kappa\widehat{H} - \frac{\hbar\kappa}{6g^{\frac{1}{2}}|v|^{\frac{1}{2}}}\widehat{\pi}^{A'}_{\psi}\widehat{S}^{R}_{A'}$$

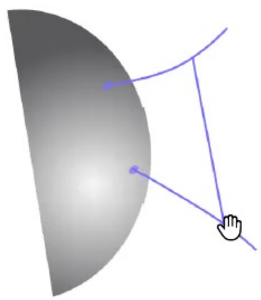
- $\rightarrow$ fixes some of the quantization ambiguities
- semiclassical limit: $\lambda_m \rightarrow 0$ i.e. corrections from quantum geometry negligible
- $\rightarrow$ **Hartle-Hawking state** as solution of constraints $\Leftrightarrow$[D'Eath '98]

Hartle-Hawking state

$$\Psi(a) = \exp\left(\frac{3a^2}{\hbar}\right)$$

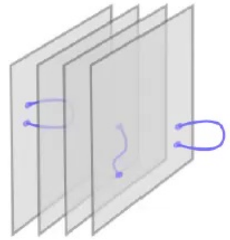
Boundary theory in LQG (WIP)

Holst-MacDowell-Mansouri in the presence of inner boundary:[Ling+Smolin '99, Ling '03, KE+HS '21]



- geometric theory induces super-CS theory on inner boundary
- for $\mathcal{N} = 2$: $\quad G = \mathrm{OSp}(2|2)$
- state counting feasible?

Open strings on coincident D-brane system: interesting similarities [Mikhaylov + Witten '14]



- super-CS theory in the location of the branes in the low energy limit
- for $G = \mathrm{OSp}(m|n)$

Summary & Outlook

"Classical" theory:

- SUSY: Symmetry involving bosons/fermions, internal/spacetime symmetry
- (Cartan) geometric description of $\mathcal{N} = 1, 2$, $D = 4$ Holst-SUGRA including **unique** boundary terms compatible with SUSY
- Chiral SUGRA the has structure of a **YM theory** $\rightarrow$ super Ashtekar connection
- boundary theory: **Super Chern-Simons theory**

Summary & Outlook

Outlook:

- Hilbert space, reality conditions, observables
- (charged) supersymmetric BHs $\leftrightarrow$ entropy calculations in string theory [Strominger+Vafa '96, Cardoso et al. '96]
- generalization to $\mathcal{N} \geq 3$ ($\leftrightarrow$ compare enriched Cartan approach with other approaches [Castellani+Grassi et al '14])
- limit $L \to \infty$ (vanishing cosmological constant) [Concha+Ravera+Rodríguez '19]
- etc. etc.

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Thank you!