Title: q- deformed LQG with a cosmological constant

Speakers: Qiaoyin Pan

Series: Quantum Gravity

Date: September 02, 2021 - 2:30 PM

URL: https://pirsa.org/21090001

Abstract: The inclusion of the cosmological constant is one of the main questions faced by quantum gravity. In three dimensions, non-perturbative approaches to quantum gravity including loop quantum gravity (LQG), combinatorial quantization and spinfoam path integrals encode the cosmological constant as a deformation parameter in a quantum group structure. In this talk, I will focus on the LQG approach: I will explain the Poisson-Lie structure of the classical phase space and how its quantization naturally leads to the emergence of quantum groups. I will use the holonomy-flux algebra and its spinorial presentation introduced in the series of work by Bonzom, Dupuis, Girelli, Livine and myself. This allows to construct the Hamiltonian constraint, understand its matrix elements as Turaev-Viro amplitudes. This connects LQG to the other approaches in a unified mathematical setting.

q-deformed LQG with a cosmological constant

Qiaoyin Pan

Quantum Gravity Seminar @ PI, Sept. 2nd, 2021

In collaboration with V. Bonzom, M. Dupuis, F. Girelli, E. Livine

Motivation

- **3D gravity** Adding a cosmological constant as a coupling constant in different approaches of quantum gravity realize the quantum group structure:

↪ Spinfoam:

↪ Chern-Simons formulation:

↪ Loop quantum gravity (LQG):

2:35 PM Thu Sep 2

Motivation

- **3D gravity** Adding a cosmological constant as a **coupling constant** in different approaches of quantum gravity realize the quantum group structure:

↪ Spinfoam: Ponzano-Regge model $\xrightarrow{\text{regularization}}$ Tureav-Viro model

✔ Semiclassical limit → Regge action with Λ

✘ Quantum group put by hand, origin not clear

↪ Chern-Simons formulation:

↪ Loop quantum gravity (LQG):

3 / 72

Motivation

- **3D gravity** Adding a cosmological constant as a coupling constant in different approaches of quantum gravity realize the quantum group structure:

↪ Spinfoam: Ponzano-Regge model —regularization→ Tureav-Viro model

✓ Semiclassical limit→ Regge action with Λ

✗ Quantum group put by hand, origin not clear

↪ Chern-Simons formulation: Fock-Rosly phase space ⟶ combinatorial quantization

✓ Quantum groups appears naturally, different signatures and different signs of Λ in a unified way

✗ Not a good guide for 4D

↪ Loop quantum gravity (LQG):

q-deformed spin network states
quantum group structure

[R. Borissov, S. Major, L. Smolin, etc., 1995-, K. Noui, A. Perez, D. Pranzetti, 2010-2012]

2:39 PM Thu Sep 2

Motivation

- **3D gravity** Adding a cosmological constant as a coupling constant in different approaches of quantum gravity realize the quantum group structure:

↪ Spinfoam: Ponzano-Regge model $\xrightarrow{\text{regularization}}$ Tureav-Viro model

✔ Semiclassical limit ⟶ Regge action with Λ

✘ Quantum group put by hand, origin not clear

↪ Chern-Simons formulation: Fock-Rosly phase space ⟶ combinatorial quantization

✔ Quantum groups appears naturally, different signatures and different signs of Λ in a unified way

✘ Not a good guide for 4D

↪ Loop quantum gravity (LQG):

q-deformed spin network states quantum group structure $\xleftarrow{\text{quantization}}$ Poisson-Lie structure $\xleftarrow{\text{discretization}}$ gravity action with Λ Lie bialgebra structure

[R. Borissov, S. Major, L. Smolin, etc., 1995-, K. Noui, A. Perez, D. Pranzetti, 2010-2012]

7 / 72

Motivation

- **3D gravity** Adding a cosmological constant as a coupling constant in different approaches of quantum gravity realize the quantum group structure:

↪ Spinfoam: Ponzano-Regge model —regularization→ Tureav-Viro model

✓ Semiclassical limit→ Regge action with Λ

✗ Quantum group put by hand, origin not clear

↪ Chern-Simons formulation: Fock-Rosly phase space ⟶ combinatorial quantization

✓ Quantum groups appears naturally, different signatures and different signs of Λ in a unified way

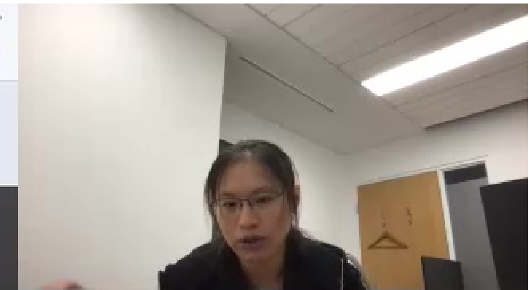
✗ Not a good guide for 4D

This talk

↪ Loop quantum gravity (LQG):

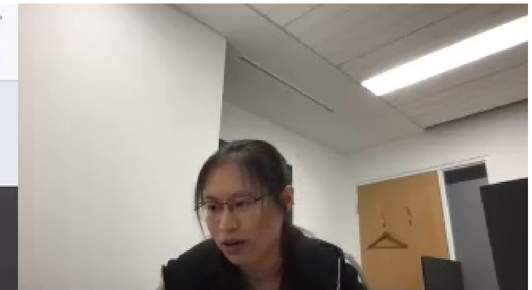
q-deformed spin network states, quantum group structure ◀····quantization···· Poisson-Lie structure ◀····discretization···· gravity action with Λ, Lie bialgebra structure

[R. Borissov, S. Major, L. Smolin, etc., 1995-, K. Noui, A. Perez, D. Pranzetti, 2010-2012]



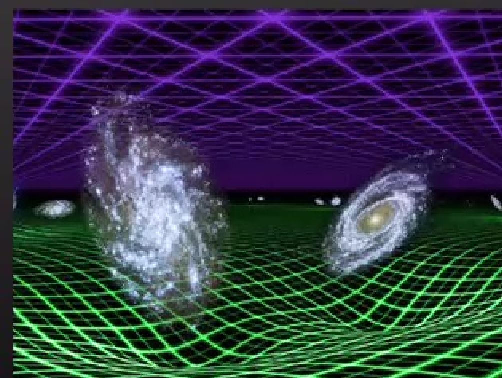
2:40 PM Thu Sep 2

94%



Plan of the talk

- Intro: 3D gravity with $\Lambda \longrightarrow$ deformed phase space
- Kinematics - Classical deformed holonomy-flux phase space
  - The deformed spinors
- Quantum holonomy and quantum flux
  - Quantum deformed spinors
- Dynamics - Hamiltonian constraint in spinor representation



© NASA/JPL-Caltech

9 / 72

2:43 PM Thu Sep 2

3d gravity with a cosmological constant

i, j, k: for internal indices

$\wedge$: for forms on spacetime

$8\pi G = 1$

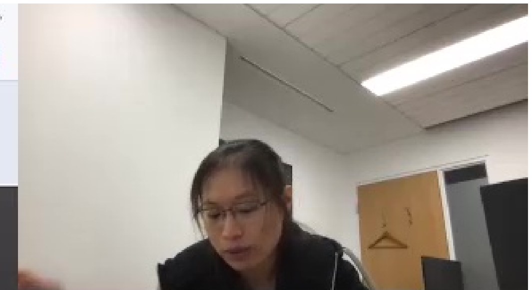
- Space-time $\mathscr{M} = \Sigma \times \mathbb{R}$ Euclidean signature $\eta_{\mu\nu} = \mathrm{diag}(+,+,+)$

- Action $S[A, e] = \frac{1}{2}\int_{\mathscr{M}} \left(e^i \wedge F_i + \frac{\Lambda}{6}\epsilon_{ijk}\, e^i \wedge e^j \wedge e^k \right)$

- EoM
$$\mathrm{d}A^i + \frac{1}{2}\epsilon^i{}_{jk} A^j \wedge A^k + \frac{1}{2}\Lambda \epsilon^i{}_{jk} e^j \wedge e^k = 0 \quad \xrightarrow{\text{curvature}} \quad \text{translation dep. of } \Lambda$$
$$\mathrm{d}e^i + \epsilon^i{}_{jk} A^j \wedge e^k = 0 \quad \xrightarrow{\text{torsion}} \quad \text{gauge transformation indep. of } \Lambda$$

↓ discretization + quantization ?

quantum group structure



3d gravity with a cosmological constant

- Space-time $\mathscr{M} = \Sigma \times \mathbb{R}$ Euclidean signature $\eta_{\mu\nu} = \text{diag}(+,+,+)$

i, j, k: for internal indices

$\wedge$: for forms on spacetime

$8\pi G = 1$

- Action $S[A,e] = \frac{1}{2}\int_{\mathscr{M}} \left(e^i \wedge F_i + \frac{\Lambda}{6}\epsilon_{ijk}\, e^i \wedge e^j \wedge e^k \right) + \frac{1}{2}\int_{\partial\mathscr{M}} \epsilon_{ijk}\,(e^i \wedge e^j) n^k$

$n^2 = -\Lambda > 0$

canonical transformation   new connection: $\omega^i = A^i - \epsilon^i{}_{jk}\, e^j n^k$ parametrized by a constant vector n^i

c.f. K. Noui, A. Perez, D. Pranzetti

- EoM $F^i[\omega] + \epsilon^i{}_{jk}\, e^j \wedge \mathrm{d}_\omega n^k = 0$

$\mathrm{d}_\omega e^i - \left(e^i \wedge e^j\right) n_j = 0$

new curvature → translation dep. of Λ

new torsion → gauge transformation dep. of Λ

↓ discretization + quantization

quantum group structure

[M. Dupuis, L. Freidel, F. Girelli, M. Osumanu, J. Rennert, 2020]

3d gravity with a cosmological constant

- Space-time $\mathscr{M} = \Sigma \times \mathbb{R}$ Euclidean signature $\eta_{\mu\nu} = \mathrm{diag}(+,+,+)$

i, j, k: for internal indices

$\wedge$: for forms on spacetime

$8\pi G = 1$

- Action $S[A,e] = \frac{1}{2}\int_{\mathscr{M}} \left(e^i \wedge F_i + \frac{\Lambda}{6}\epsilon_{ijk} e^i \wedge e^j \wedge e^k \right) + \frac{1}{2}\int_{\partial\mathscr{M}} \epsilon_{ijk}(e^i \wedge e^j) n^k$

$n^2 = -\Lambda > 0$

canonical transformation   new connection: $\omega^i = A^i - \epsilon^i{}_{jk} e^j n^k$ parametrized by a constant vector n^i

- EoM $F^i[\omega] + \epsilon^i{}_{jk} e^j \wedge \mathrm{d}_\omega n^k = 0$ new curvature → translation dep. of Λ

$\mathrm{d}_\omega e^i - (e^i \wedge e^j)\, n_j = 0$ new torsion → gauge transformation dep. of Λ

take $n^i = (0, 0, \kappa)$ $\kappa = \sqrt{-\Lambda}$

Algebra $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2) \bowtie \mathfrak{an}(2)$ (Iwasawa decomposition)

$\mathfrak{su}(2)$ generators σ_i : $[\sigma_i, \sigma_j] = 2i\epsilon_{ij}{}^k \sigma_k$

$\mathfrak{an}(2)$ generators ρ^i : $[\rho^i, \rho^j] = 2i\kappa(\delta^i_k\delta^j_3 - \delta^i_3\delta^j_k)\rho^k$

$[\sigma_i, \rho^j] = 2i\kappa(\delta^j_i\delta^k_3 - \delta^j_3\delta^k_i)\sigma_k + 2i\epsilon^j_{i\,k}\rho^k$

New symplectic form $\Omega = \int_\Sigma \langle \delta e \wedge \delta\omega \rangle$

[M. Dupuis, L. Freidel, F. Girelli, M. Osumanu, J. Rennert, 2020]

3d gravity with a cosmological constant

i, j, k: for internal indices

$\wedge$: for forms on spacetime

- Space-time $\mathcal{M} = \Sigma \times \mathbb{R}$ Euclidean signature $\eta_{\mu\nu} = \mathrm{diag}(+,+,+)$

- Action $S[A,e] = \frac{1}{2}\int_{\mathcal{M}} \left(e^i \wedge F_i + \frac{\Lambda}{6}\epsilon_{ijk} e^i \wedge e^j \wedge e^k \right) + \frac{1}{2}\int_{\partial\mathcal{M}} \epsilon_{ijk}(e^i \wedge e^j) n^k$

$8\pi G = 1$

$n^2 = -\Lambda > 0$

canonical transformation new connection: $\omega^i = A^i - \epsilon^i{}_{jk} e^j n^k$ parametrized by a constant vector n^i

- EoM $F^i[\omega] + \epsilon^i{}_{jk} e^j \wedge \mathrm{d}_\omega n^k = 0$ —new curvature→ translation dep. of Λ

$\mathrm{d}_\omega e^i - \left(e^i \wedge e^j\right) n_j = 0$ —new torsion→ gauge transformation dep. of Λ

take $n^i = (0,0,\kappa)$ $\kappa = \sqrt{-\Lambda}$

Algebra $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2) \bowtie \mathfrak{an}(2)$ (Iwasawa decomposition)

$\mathfrak{su}(2)$ generators σ_i : $[\sigma_i, \sigma_j] = 2i\epsilon_{ij}{}^k \sigma_k$

$\mathfrak{an}(2)$ generators ρ^i : $[\rho^i, \rho^j] = 2i\kappa(\delta^i_k \delta^j_3 - \delta^i_3 \delta^j_k)\rho^k$

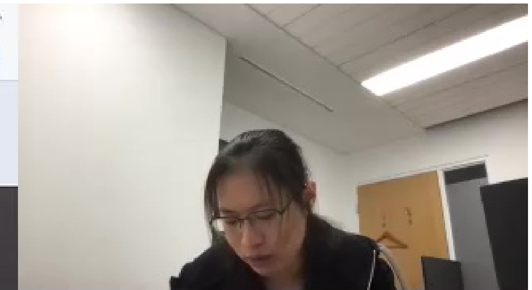
$[\sigma_i, \rho^j] = 2i\kappa(\delta^j_i \delta^k_3 - \delta^j_3 \delta^k_i)\sigma_k + 2i\epsilon_i{}^j{}_k \rho^k$

$\langle \rho^i, \sigma_j \rangle = \delta^i_j, \langle \rho^i, \rho^j \rangle = \langle \sigma_i, \sigma_j \rangle = 0$

$\mathfrak{su}(2)^* = \mathfrak{an}(2)$

Lie bialgebra structure

[M. Dupuis, L. Freidel, F. Girelli, M. Osumanu, J. Rennert, 2020]



3d gravity with a cosmological constant

- Space-time $\mathscr{M} = \Sigma \times \mathbb{R}$ Euclidean signature $\eta_{\mu\nu} = \mathrm{diag}(+,+,+)$

i, j, k: for internal indices

$\wedge$: for forms on spacetime

$8\pi G = 1$

- Action $S[A, e] = \frac{1}{2}\int_{\mathscr{M}} \left(e^i \wedge F_i + \frac{\Lambda}{6}\epsilon_{ijk}\, e^i \wedge e^j \wedge e^k\right) + \frac{1}{2}\int_{\partial\mathscr{M}} \epsilon_{ijk}\,(e^i \wedge e^j) n^k$

$n^2 = -\Lambda > 0$

canonical transformation new connection: $\omega^i = A^i - \epsilon^i{}_{jk}\, e^j n^k$ parametrized by a constant vector n^i

- EoM $F^i[\omega] + \epsilon^i{}_{jk}\, e^j \wedge \mathrm{d}_\omega n^k = 0$ —new curvature→ translation dep. of Λ

$\mathrm{d}_\omega e^i - (e^i \wedge e^j)\, n_j = 0$ —new torsion→ gauge transformation dep. of Λ

take $n^i = (0, 0, \kappa)$ $\kappa = \sqrt{-\Lambda}$

Algebra $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2) \bowtie \mathfrak{an}(2)$ (Iwasawa decomposition)

$\mathfrak{su}(2)$ generators σ_i : $[\sigma_i, \sigma_j] = 2i\epsilon_{ij}{}^k \sigma_k$

$\mathfrak{an}(2)$ generators ρ^i : $[\rho^i, \rho^j] = 2i\kappa\,(\delta^i_k \delta^j_3 - \delta^i_3 \delta^j_k)\,\rho^k$

$[\sigma_i, \rho^j] = 2i\kappa\,(\delta^j_i \delta^k_3 - \delta^j_3 \delta^k_i)\,\sigma_k + 2i\epsilon^j_{i\,k}\,\rho^k$

$\langle \rho^i, \sigma_j \rangle = \delta^i_j,\ \langle \rho^i, \rho^j \rangle = \langle \sigma_i, \sigma_j \rangle = 0$

$\mathfrak{su}(2)^* = \mathfrak{an}(2)$

Lie bialgebra structure

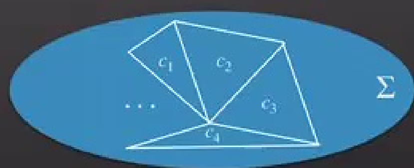
New symplectic form $\Omega = \int_\Sigma \langle \delta e \wedge \delta\omega \rangle$

[M. Dupuis, L. Freidel, F. Girelli, M. Osumanu, J. Rennert, 2020]

3d gravity with a cosmological constant

Discretization — cell decomposition of Σ + truncation of DoF

[L. Freidel, F. Girelli, B. Shonshany, 2019]



Truncation: solution to the constraints

$$\Omega = \int_\Sigma \langle \delta e \wedge \delta\omega \rangle \xrightarrow{\text{subdivision}} \Omega = \sum_\alpha \int_{c_\alpha} \langle \delta e \wedge \delta\omega \rangle = -\sum_\alpha \int_e \delta \langle \ell_{c_\alpha}^{-1} \mathrm{d}\ell_{c_\alpha}, \delta u_{c_\alpha} u_{c_\alpha}^{-1} \rangle$$

$\ell_{c_\alpha} \in \mathrm{AN}(2)$

$u_{c_\alpha} \in \mathrm{SU}(2)$

discrete phase space

$(\ell, u) \in (\mathrm{AN}(2), \mathrm{SU}(2))$ $\xrightarrow{\Lambda = 0}$ $(\ell, u) \in (\mathbb{R}^3, \mathrm{SU}(2))$

$d = \ell u \in \mathrm{SL}(2,\mathbb{C}) = \mathrm{SU}(2) \bowtie \mathrm{AN}(2)$ $\quad T^*\mathrm{SU}(2) = \mathrm{SU}(2) \ltimes \mathbb{R}^3$

[M. Dupuis, L. Freidel, F. Girelli, M. Osumanu, J. Rennert, 2020]

3d deformed loop gravity phase space

- Lie group $\mathscr{D}(\mathrm{SU}(2)) = \mathrm{SU}(2) \bowtie \mathrm{AN}(2) \cong \mathrm{SL}(2,\mathbb{C})$
- The phase space is a Heisenberg double $(\mathscr{D}(\mathrm{SU}(2)), \pi_H)$

Poisson bracket: $\{d_1, d_2\} = -r_{21}d_1d_2 + d_1d_2r, \quad \forall d \in \mathrm{SL}(2,\mathbb{C})$

Classical r-matrix

$\mathrm{SU}(2)^* = \mathrm{AN}(2)$

$d_1 = d \otimes \mathbb{I}, \quad d_2 = \mathbb{I} \otimes d$

$r = \sum_i e_i \otimes f^i, \; r_{21} = \sum_i f^i \otimes e_i$

[V. Bonzom, M. Dupuis, F. Girelli, E. Livine, 2014]

3d deformed loop gravity phase space

- Lie group $\mathscr{D}(\mathrm{SU}(2)) = \mathrm{SU}(2) \bowtie \mathrm{AN}(2) \cong \mathrm{SL}(2,\mathbb{C}) \;\; = \mathrm{AN}(2) \bowtie \mathrm{SU}(2) = \mathscr{D}(\mathrm{AN}(2))$
- The phase space is a Heisenberg double $(\mathscr{D}(\mathrm{SU}(2)), \pi_H)$

SU(2)* = AN(2) | AN(2)* = SU(2)

Poisson bracket: $\{d_1, d_2\} = -\, r_{21} d_1 d_2 + d_1 d_2 r\,, \quad \forall d \in \mathrm{SL}(2,\mathbb{C})$

$$d_1 = d \otimes \mathbb{I}, \quad d_2 = \mathbb{I} \otimes d$$
$$r = \sum_i e_i \otimes f^i, \; r_{21} = \sum_i f^i \otimes e_i$$

- Iwasawa decomposition of $\mathrm{SL}(2,\mathbb{C})$ elements: $d = \ell u = \tilde{u} \tilde{\ell}$ (left, right)

$\ell, \tilde{\ell} \in \mathrm{AN}(2)$, $u, \tilde{u} \in \mathrm{SU}(2)$ $\quad \ell = \begin{pmatrix} \lambda & 0 \\ z & \lambda^{-1} \end{pmatrix}, \quad \tilde{\ell} = \begin{pmatrix} \tilde{\lambda} & 0 \\ \tilde{z} & \tilde{\lambda}^{-1} \end{pmatrix}, \quad \lambda, \tilde{\lambda} \in \mathbb{R}^+;\; z, \tilde{z} \in \mathbb{C}$

Poisson bracket: $\{\ell_1, \ell_2\} = -[r_{21}, \ell_1 \ell_2]\,, \quad \{u_1, u_2\} = -[r, u_1 u_2]\,, \quad \{\tilde{\ell}_1, \tilde{\ell}_2\} = [r_{21}, \tilde{\ell}_1 \tilde{\ell}_2]\,, \quad \{\tilde{u}_1, \tilde{u}_2\} = [r, \tilde{u}_1 \tilde{u}_2]$ & cross brackets

[V. Bonzom, M. Dupuis, F. Girelli, E. Livine, 2014]

3d deformed loop gravity phase space

- Lie group $\mathscr{D}(\mathrm{SU}(2)) = \mathrm{SU}(2) \bowtie \mathrm{AN}(2) \cong \mathrm{SL}(2,\mathbb{C}) = \mathrm{AN}(2) \bowtie \mathrm{SU}(2) = \mathscr{D}(\mathrm{AN}(2))$
- The phase space is a Heisenberg double $(\mathscr{D}(\mathrm{SU}(2)), \pi_H)$

SU(2)* = AN(2)　AN(2)* = SU(2)

Poisson bracket: $\{d_1, d_2\} = -r_{21} d_1 d_2 + d_1 d_2 r, \quad \forall d \in \mathrm{SL}(2,\mathbb{C})$

$d_1 = d \otimes \mathbb{I}, \quad d_2 = \mathbb{I} \otimes d$

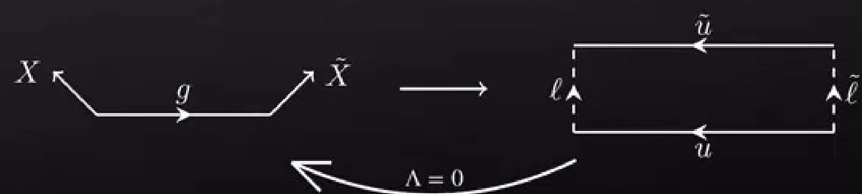
$r = \sum_i e_i \otimes f^i, \; r_{21} = \sum_i f^i \otimes e_i$

- Iwasawa decomposition of $\mathrm{SL}(2,\mathbb{C})$ elements: $d = \ell u = \tilde{u}\tilde{\ell}$

$\ell, \tilde{\ell} \in \mathrm{AN}(2)$, $u, \tilde{u} \in \mathrm{SU}(2)$　$\ell = \begin{pmatrix} \lambda & 0 \\ z & \lambda^{-1} \end{pmatrix}, \quad \tilde{\ell} = \begin{pmatrix} \tilde{\lambda} & 0 \\ \tilde{z} & \tilde{\lambda}^{-1} \end{pmatrix}, \quad \lambda, \tilde{\lambda} \in \mathbb{R}^+; \, z, \tilde{z} \in \mathbb{C}$

Poisson bracket: $\{\ell_1, \ell_2\} = -[r_{21}, \ell_1 \ell_2], \quad \{u_1, u_2\} = -[r, u_1 u_2], \quad \{\tilde{\ell}_1, \tilde{\ell}_2\} = [r_{21}, \tilde{\ell}_1 \tilde{\ell}_2], \quad \{\tilde{u}_1, \tilde{u}_2\} = [r, \tilde{u}_1 \tilde{u}_2]$ & cross brackets

- A link becomes a ribbon



$\ell = (X, 1), \tilde{\ell} = (\tilde{X}, 1) \in \mathbb{R}^3$

$u = \tilde{u} = g^{-1} \in \mathrm{SU}(2)$

[V. Bonzom, M. Dupuis, F. Girelli, E. Livine, 2014]

2:54 PM Thu Sep 2

3d deformed loop gravity phase space

- Lie group $\mathscr{D}(\mathrm{SU}(2)) = \mathrm{SU}(2) \bowtie \mathrm{AN}(2) \cong \mathrm{SL}(2,\mathbb{C})$ $= \mathrm{AN}(2) \bowtie \mathrm{SU}(2) = \mathscr{D}(\mathrm{AN}(2))$
- The phase space is a Heisenberg double $(\mathscr{D}(\mathrm{SU}(2)), \pi_H)$

SU(2)* = AN(2) | AN(2)* = SU(2)

Poisson bracket: $\{d_1, d_2\} = -r_{21} d_1 d_2 + d_1 d_2 r, \quad \forall d \in \mathrm{SL}(2,\mathbb{C})$

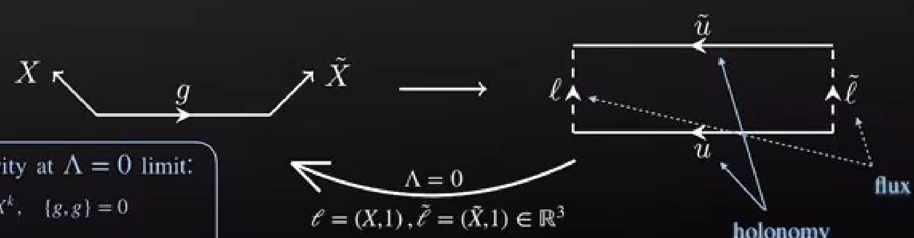
$$d_1 = d \otimes \mathbb{I}, \quad d_2 = \mathbb{I} \otimes d$$
$$r = \sum_i e_i \otimes f^i, \; r_{21} = \sum_i f^i \otimes e_i$$

- Iwasawa decomposition of SL(2,ℂ) elements: $d = \ell u = \tilde{u}\tilde{\ell}$

$$\begin{cases} \ell, \tilde{\ell} \in \mathrm{AN}(2) \\ u, \tilde{u} \in \mathrm{SU}(2) \end{cases} \quad \ell = \begin{pmatrix} \lambda & 0 \\ z & \lambda^{-1} \end{pmatrix}, \quad \tilde{\ell} = \begin{pmatrix} \tilde{\lambda} & 0 \\ \tilde{z} & \tilde{\lambda}^{-1} \end{pmatrix}, \quad \lambda, \tilde{\lambda} \in \mathbb{R}^+; \; z, \tilde{z} \in \mathbb{C}$$

Poisson bracket: $\{\ell_1, \ell_2\} = -[r_{21}, \ell_1 \ell_2]$, $\{u_1, u_2\} = -[r, u_1 u_2]$, $\{\tilde{\ell}_1, \tilde{\ell}_2\} = [r_{21}, \tilde{\ell}_1 \tilde{\ell}_2]$, $\{\tilde{u}_1, \tilde{u}_2\} = [r, \tilde{u}_1 \tilde{u}_2]$ & cross brackets

- A link becomes a ribbon



$\ell = (X, 1), \tilde{\ell} = (\tilde{X}, 1) \in \mathbb{R}^3$
$u = \tilde{u} = g^{-1} \in \mathrm{SU}(2)$

Recovers the Poisson structure of loop gravity at $\Lambda = 0$ limit:

$$\{X^i, g\} = \frac{i}{2}\sigma^i g, \quad \{X^i, X^j\} = \epsilon^{ij}{}_k X^k, \quad \{g, g\} = 0$$
$$\{\tilde{X}^i, g\} = \frac{i}{2} g \sigma^i, \quad \{\tilde{X}^i, \tilde{X}^j\} = \epsilon^{ij}{}_k \tilde{X}^k$$

[V. Bonzom, M. Dupuis, F. Girelli, E. Livine, 2014]

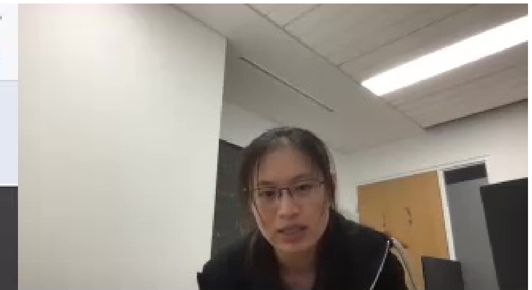
27 / 72

3d deformed loop gravity phase space

$T^*\mathsf{SU}(2) = \mathsf{SU}(2) \ltimes \mathbb{R}^3$ $\qquad$ $\mathsf{SL}(2,\mathbb{C}) = \mathsf{SU}(2) \bowtie \mathsf{AN}(2)$

new feature: back reaction of the flux on holonomy!

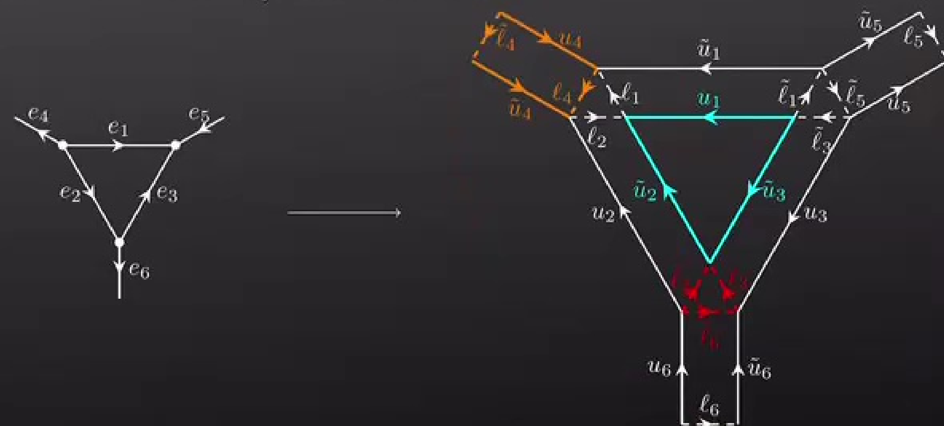
$$(u, \tilde{u}) \triangleright (\ell, \tilde{\ell}) \qquad (\ell, \tilde{\ell}) \triangleright (u, \tilde{u})$$



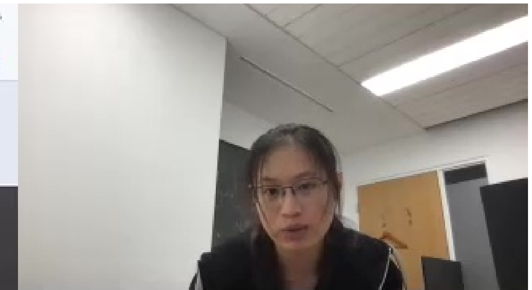
2:55 PM Thu Sep 2

3d deformed loop gravity phase space

- A graph becomes a ribbon graph, which carries all the holonomy and flux variables.



[V. Bonzom, M. Dupuis, F. Girelli, E. Livine, 2014]



2:56 PM Thu Sep 2

3d deformed loop gravity phase space

- A graph becomes a ribbon graph, which carries all the holonomy and flux variables.

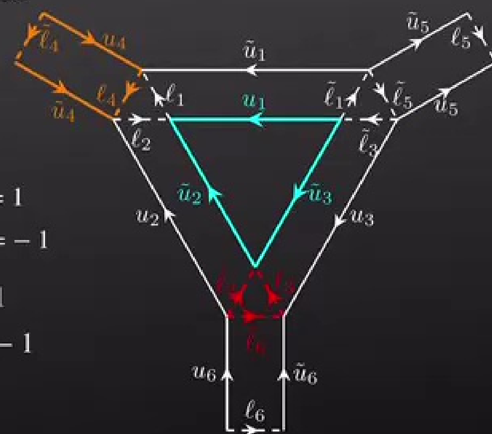
ribbon constraint: $\mathscr{C} = \ell_4 u_4 \tilde{\ell}_4^{-1} \tilde{u}_4^{-1}$ (second class)

Gauss constraint: $\mathscr{G} = \tilde{\ell}_2^{-1} \tilde{\ell}_6^{-1} \ell_3$ (first class) → in general

flatness constraint: $\mathscr{F} = \tilde{u}_2^{-1} \tilde{u}_3^{-1} u_1$ (first class)

$$\mathscr{G}_v = \overrightarrow{\prod}_e \mathscr{L}_e^{o_e}, \quad \mathscr{L}_e^{o_e} = \begin{cases} \ell_e & \text{if } o_e = 1 \\ \tilde{\ell}_e^{-1} & \text{if } o_e = -1 \end{cases}$$

$$\mathscr{F}_f = \overrightarrow{\prod}_e \mathscr{U}_e^{o_e}, \quad \mathscr{U}_e^{o_e} = \begin{cases} u_e & \text{if } o_e = 1 \\ \tilde{u}_e^{-1} & \text{if } o_e = -1 \end{cases}$$



[V. Bonzom, M. Dupuis, F. Girelli, E. Livine, 2014]

30 / 72

2:56 PM Thu Sep 2

3d deformed loop gravity phase space

- A graph becomes a ribbon graph, which carries all the holonomy and flux variables.

ribbon constraint: $\mathscr{C} = \ell_4 u_4 \tilde{\ell}_4^{-1} \tilde{u}_4^{-1}$ (second class)

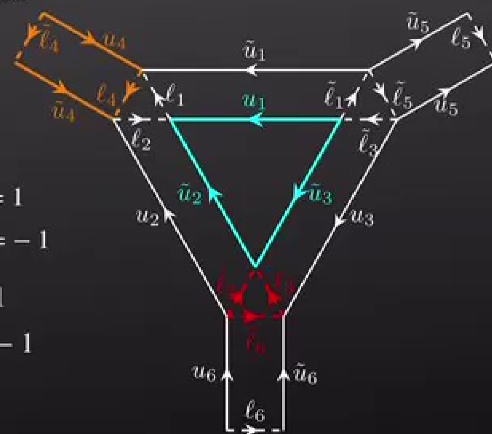
Gauss constraint: $\mathscr{G} = \tilde{\ell}_2^{-1} \tilde{\ell}_6^{-1} \ell_3$ (first class)

flatness constraint: $\mathscr{F} = \tilde{u}_2^{-1} \tilde{u}_3^{-1} u_1$ (first class)

in general

$$\mathscr{G}_v = \overrightarrow{\prod}_e \mathscr{L}_e^{o_e}, \quad \mathscr{L}_e^{o_e} = \begin{cases} \ell_e & \text{if } o_e = 1 \\ \tilde{\ell}_e^{-1} & \text{if } o_e = -1 \end{cases}$$

$$\mathscr{F}_f = \overrightarrow{\prod}_e \mathscr{U}_e^{o_e}, \quad \mathscr{U}_e^{o_e} = \begin{cases} u_e & \text{if } o_e = 1 \\ \tilde{u}_e^{-1} & \text{if } o_e = -1 \end{cases}$$



[V. Bonzom, M. Dupuis, F. Girelli, E. Livine, 2014]

30 / 72

2:57 PM Thu Sep 2

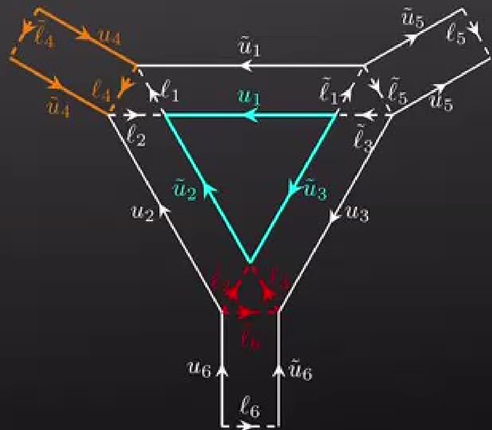
3d deformed loop gravity phase space

- A graph becomes a ribbon graph, which carries all the holonomy and flux variables.

ribbon constraint: $\mathscr{C} = \ell_4 u_4 \tilde{\ell}_4^{-1} \tilde{u}_4^{-1}$ (second class)

Gauss constraint: $\mathscr{G} = \tilde{\ell}_2^{-1} \tilde{\ell}_6^{-1} \ell_3$ (first class)

flatness constraint: $\mathscr{F} = \tilde{u}_2^{-1} \tilde{u}_3^{-1} u_1$ (first class)



- Geometrical interpretation of the Gauss constraint

$\longrightarrow$ discrete hyperbolic geometry

$$\cos\theta_{26} = \frac{\cosh x_2 \cosh x_6 - \cosh x_3}{\sinh x_2 \sinh x_6}$$

Curved(!) building blocks

[V. Bonzom, M. Dupuis, F. Girelli, E. Livine, 2014]

31 / 72

2:58 PM Thu Sep 2

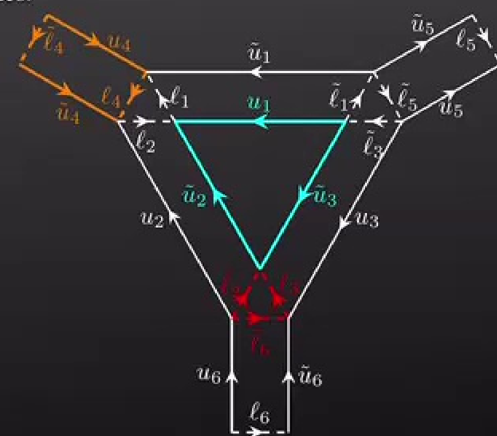
3d deformed loop gravity phase space

- A graph becomes a ribbon graph, which carries all the holonomy and flux variables.

ribbon constraint: $\mathscr{C} = \ell_4 u_4 \tilde{\ell}_4^{-1} \tilde{u}_4^{-1}$ (second class)

Gauss constraint: $\mathscr{G} = \tilde{\ell}_2^{-1} \tilde{\ell}_6^{-1} \ell_3$ (first class)

flatness constraint: $\mathscr{F} = \tilde{u}_2^{-1} \tilde{u}_3^{-1} u_1$ (first class)



- The constraints all have the holonomy nature in different groups:

$$\mathrm{SL}(2,\mathbb{C}),\ \mathrm{AN}(2),\ \mathrm{SU}(2)$$

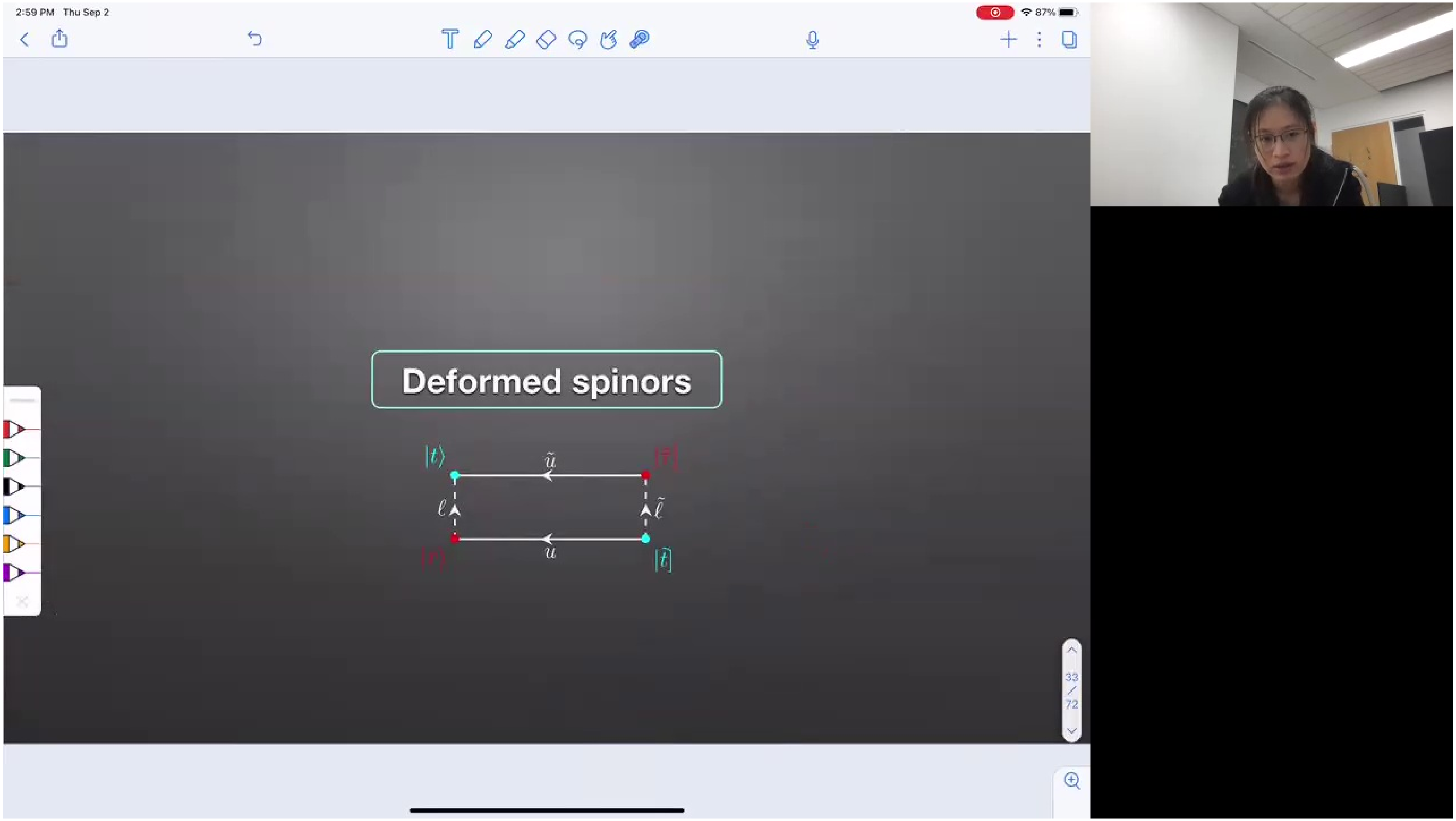
- The Gauss constraint generates the SU(2) gauge transformation (rotation) and the flatness constraint generates the deformed translation.

$$\delta_R h = \prod_e \lambda_e^{-2} \{ \mathrm{Tr} W \mathscr{G} \mathscr{G}^\dagger , h \} , \qquad \delta_T h = \mathrm{Tr} M \mathscr{F}^{-1} \{ \mathscr{F} , h \}$$

$$\mathscr{L}_e = \begin{pmatrix} \lambda_e & 0 \\ z_e & \lambda_e^{-1} \end{pmatrix} \quad W = \begin{pmatrix} 2\epsilon_z & \epsilon_- \\ \epsilon_+ & 0 \end{pmatrix} \quad M = \begin{pmatrix} 2\epsilon_z & \epsilon_- \\ \epsilon_+ & -2\epsilon_z \end{pmatrix}$$

[V. Bonzom, M. Dupuis, F. Girelli, E. Livine, 2014]

32 / 72


2:59 PM Thu Sep 2
87%
Deformed spinors
$|t\rangle$
$\tilde{u}$
ℓ
$\tilde{\ell}$
u
$|\tilde{t}]$
33
72

Deformed spinorial phase space

WHY spinors?

- Canonical variables with a simple Poisson structure $\{\zeta_A, \bar{\zeta}_B\} = i\delta_{AB}$
- Its quantization gives the harmonic oscillators
  - $\longrightarrow$ A good tool to construct the coherent spin network states and the coherent intertwiners
    - —— well-developed in the LQG and spinfoam model
  - $\longrightarrow$ local $U(N)$ symmetry, the algebra of the observables acting on the intertwiner space
- A different aspect of the discrete/quantum geometry

[Many works by V. Bonzom, M. Dupuis, L. Freidel, F. Girelli, E. Livine, S. Speziale, J. Tambornino, W. Wieland, etc. 2005-]

Deformed spinorial phase space

WHY spinors?

- Canonical variables with a simple Poisson structure $\{\zeta_A, \bar{\zeta}_B\} = i\delta_{AB}$
- Its quantization gives the harmonic oscillators
  - $\longrightarrow$ A good tool to construct the coherent spin network states and the coherent intertwiners
    - —— well-developed in the LQG and spinfoam model
  - $\longrightarrow$ local $U(N)$ symmetry, the algebra of the observables acting on the intertwiner space
- A different aspect of the discrete/quantum geometry

WHY deformed spinors?

- Curved discrete geometry!

[Many works by V. Bonzom, M. Dupuis, L. Freidel, F. Girelli, E. Livine, S. Speziale, J. Tambornino, W. Wieland, etc. 2005-]

Spinorial phase space

- Let us recall the spinors in loop gravity

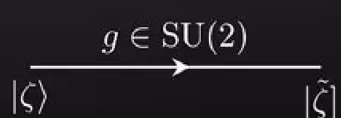
$$|\zeta\rangle = \begin{pmatrix} \zeta_0 \\ \zeta_1 \end{pmatrix}, \quad |\zeta] = \begin{pmatrix} -\bar{\zeta}_1 \\ \bar{\zeta}_0 \end{pmatrix}, \quad \{\zeta_A, \bar{\zeta}_B\} = i\delta_{AB}, \quad N_A = \zeta_A \bar{\zeta}_A .$$

$$(N = N_0 + N_1)$$

- Assign a spinor to each half-link. They reconstruct the holonomy-flux algebra:

$$X^i = \langle \zeta | \sigma^i | \zeta \rangle, \quad \tilde{X}^i = [\tilde{\zeta} | \sigma^i | \tilde{\zeta}], \quad g = \frac{|\tilde{\zeta}]\langle \zeta| - |\tilde{\zeta}\rangle [\zeta|}{\sqrt{\langle \zeta | \zeta \rangle \langle \tilde{\zeta} | \tilde{\zeta} \rangle}} \quad \Longrightarrow \quad \begin{vmatrix} \{X^i, g\} = \frac{i}{2}\sigma^i g, \quad \{X^i, X^j\} = \epsilon^{ij}{}_k X^k, \quad \{g, g\} = 0 \\ \{\tilde{X}^i, g\} = \frac{i}{2} g \sigma^i, \quad \{\tilde{X}^i, \tilde{X}^j\} = \epsilon^{ij}{}_k \tilde{X}^k \end{vmatrix}$$

Norm matching constraint: $\mathscr{M} = \langle \zeta | \zeta \rangle - [\tilde{\zeta} | \tilde{\zeta}]$



Deformed spinorial phase space

- Introduce a κ-deformation of the spinor variables $(\kappa = \sqrt{-\Lambda})$

$$\zeta_A^\kappa = \zeta_A \sqrt{\frac{\sinh(\frac{\kappa}{2}N_A)}{\frac{\kappa}{2}N_A}}, \quad |\zeta^\kappa\rangle = \begin{pmatrix} \zeta_1^\kappa \\ \zeta_1^\kappa \end{pmatrix}, \quad |\zeta^\kappa] = \begin{pmatrix} -\bar{\zeta}_1^\kappa \\ \bar{\zeta}_0^\kappa \end{pmatrix}, \quad \{\zeta_A^\kappa, \bar{\zeta}_B^\kappa\} = -i\delta_{AB}\cosh(\frac{\kappa N_A}{2})$$

- They can be used to reconstruct the fluxes as functions of the κ-deformed spinors

$$\ell(\zeta_0^\kappa, \zeta_1^\kappa, \bar{\zeta}_0^\kappa, \bar{\zeta}_1^\kappa) = \begin{pmatrix} \lambda & 0 \\ z & \lambda^{-1} \end{pmatrix}, \quad \tilde{\ell}(\tilde{\zeta}_0^\kappa, \tilde{\zeta}_1^\kappa, \bar{\tilde{\zeta}}_0^\kappa, \bar{\tilde{\zeta}}_1^\kappa) = \begin{pmatrix} \tilde{\lambda} & 0 \\ \tilde{z} & \tilde{\lambda}^{-1} \end{pmatrix} \quad \left| \begin{array}{l} \lambda = \exp(\frac{\kappa}{4}(N_1 - N_0)), \quad z = -\kappa\bar{\zeta}_0^\kappa\zeta_1^\kappa \\ \tilde{\lambda} = \exp(\frac{\kappa}{4}(\tilde{N}_1 - \tilde{N}_0)), \quad z = -\kappa\bar{\tilde{\zeta}}_0^\kappa\tilde{\zeta}_1^\kappa \end{array} \right.$$

[M. Dupuis, F. Girelli, E. Livine, 2014; V. Bonzom, M. Dupuis, F. Girelli, QP, in prep]

3:04 PM Thu Sep 2

Deformed spinorial phase space

- Introduce a κ-deformation of the spinor variables $(\kappa = \sqrt{-\Lambda})$

$$\zeta_A^\kappa = \zeta_A \sqrt{\frac{\sinh(\frac{\kappa}{2}N_A)}{\frac{\kappa}{2}N_A}}, \quad |\zeta^\kappa\rangle = \begin{pmatrix} \zeta_1^\kappa \\ \zeta_1^\kappa \end{pmatrix}, \quad |\zeta^\kappa] = \begin{pmatrix} -\bar{\zeta}_1^\kappa \\ \bar{\zeta}_0^\kappa \end{pmatrix}, \quad \{\zeta_A^\kappa, \bar{\zeta}_B^\kappa\} = -i\delta_{AB}\cosh(\frac{\kappa N_A}{2})$$

- They can be used to reconstruct the fluxes as functions of the κ-deformed spinors

$$\ell(\zeta_0^\kappa, \zeta_z^\kappa, \bar{\zeta}_0^\kappa, \bar{\zeta}_1^\kappa) = \begin{pmatrix} \lambda & 0 \\ z & \lambda^{-1} \end{pmatrix}, \quad \tilde{\ell}(\tilde{\zeta}_0^\kappa, \tilde{\zeta}_z^\kappa, \bar{\tilde{\zeta}}_0^\kappa, \bar{\tilde{\zeta}}_1^\kappa) = \begin{pmatrix} \tilde{\lambda} & 0 \\ \tilde{z} & \tilde{\lambda}^{-1} \end{pmatrix} \quad \Bigg| \begin{array}{l} \lambda = \exp(\frac{\kappa}{4}(N_1 - N_0)), \quad z = -\kappa\bar{\zeta}_0^\kappa\zeta_1^\kappa \\ \tilde{\lambda} = \exp(\frac{\kappa}{4}(\tilde{N}_1 - \tilde{N}_0)), \quad z = -\kappa\bar{\tilde{\zeta}}_0^\kappa\tilde{\zeta}_1^\kappa \end{array}$$

- BUT,

  ⟶ These are not SU(2)-covariant spinors

$$w = \mathbb{I} + i\,\vec{\epsilon}\cdot\vec{\sigma},$$
$$\delta_\epsilon|t\rangle := -\lambda^{-2}\kappa^{-1}\{\mathrm{Tr}W\ell\ell^\dagger, |t\rangle\} = (w - \mathbb{I})|t\rangle$$
$$\delta_\epsilon|\tilde{t}\rangle := -\tilde{\lambda}^2\kappa^{-1}\{\mathrm{Tr}W(\tilde{\ell}^\dagger\tilde{\ell})^{-1}, |\tilde{t}\rangle\} = (w - \mathbb{I})|\tilde{t}\rangle$$

$$W = \begin{pmatrix} 2\epsilon_z & \epsilon_- \\ \epsilon_+ & 0 \end{pmatrix}$$

  ⟶ Ambiguity in placing them in a ribbon graph

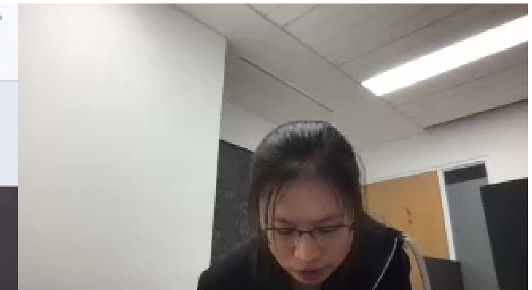
- Need a second deformation

$$|t\rangle = \begin{pmatrix} t_+ \\ t_- \end{pmatrix} := \begin{pmatrix} e^{\frac{\kappa N_1}{4}}\zeta_0^\kappa \\ e^{-\frac{\kappa N_0}{4}}\zeta_1^\kappa \end{pmatrix}, \quad |\tilde{t}\rangle = \begin{pmatrix} \tilde{t}_+ \\ \tilde{t}_- \end{pmatrix} := \begin{pmatrix} e^{\frac{\kappa \tilde{N}_1}{4}}\tilde{\zeta}_0^\kappa \\ e^{-\frac{\kappa \tilde{N}_0}{4}}\tilde{\zeta}_1^\kappa \end{pmatrix}$$

solution

[M. Dupuis, F. Girelli, E. Livine, 2014; V. Bonzom, M. Dupuis, F. Girelli, QP, in prep]

39 / 72



Deformed spinorial phase space

- Spinors on the opposite corners of the ribbon

$$|t\rangle = \begin{pmatrix} t_+ \\ t_- \end{pmatrix} := \begin{pmatrix} e^{\frac{\kappa N_1}{4}} \zeta_0^\kappa \\ e^{-\frac{\kappa N_0}{4}} \zeta_1^\kappa \end{pmatrix}, \qquad |\tilde{t}\rangle = \begin{pmatrix} \tilde{t}_+ \\ \tilde{t}_- \end{pmatrix} := \begin{pmatrix} e^{\frac{\kappa \tilde{N}_1}{4}} \tilde{\zeta}_0^\kappa \\ e^{-\frac{\kappa \tilde{N}_0}{4}} \tilde{\zeta}_1^\kappa \end{pmatrix}$$

$$w = \mathbb{I} + i\,\vec{\epsilon} \cdot \vec{\sigma},$$

$$\delta_\epsilon |t\rangle := -\lambda^{-2}\kappa^{-1}\{\mathrm{Tr} W \ell \ell^\dagger, |t\rangle\} = (w - \mathbb{I})|t\rangle$$

$$\delta_\epsilon |\tilde{t}\rangle := -\tilde{\lambda}^2\kappa^{-1}\{\mathrm{Tr} W (\tilde{\ell}^\dagger \tilde{\ell})^{-1}, |\tilde{t}\rangle\} = (w - \mathbb{I})|\tilde{t}\rangle$$

- Braided spinors on the other two corners

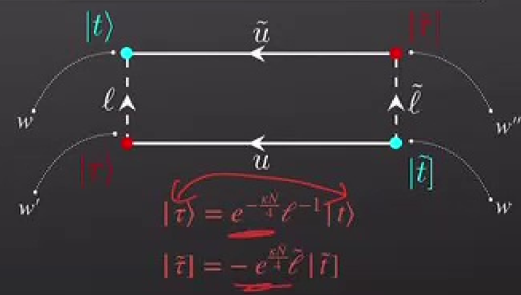
$$|\tau\rangle = \begin{pmatrix} \tau_+ \\ \tau_- \end{pmatrix} := \begin{pmatrix} e^{-\frac{\kappa N_1}{4}} \zeta_0^\kappa \\ e^{\frac{\kappa N_0}{4}} \zeta_1^\kappa \end{pmatrix}, \qquad |\tilde{\tau}\rangle = \begin{pmatrix} \tilde{\tau}_+ \\ \tilde{\tau}_- \end{pmatrix} := \begin{pmatrix} e^{-\frac{\kappa \tilde{N}_1}{4}} \tilde{\zeta}_0^\kappa \\ e^{\frac{\kappa \tilde{N}_0}{4}} \tilde{\zeta}_1^\kappa \end{pmatrix}$$

$${}^{(w)}\ell = w \ell w'^{-1} \in \mathrm{AN}(2) \qquad w' = \mathbb{I} + i\,\vec{\epsilon}' \cdot \vec{\sigma}$$

$$\tilde{\ell}^{(w)} = w''^{-1} \tilde{\ell} w \in \mathrm{AN}(2) \qquad w'' = \mathbb{I} + i\,\vec{\epsilon}'' \cdot \vec{\sigma}$$

$$\delta_\epsilon |\tau\rangle := -\lambda^{-2}\kappa^{-1}\{\mathrm{Tr} W \ell \ell^\dagger, |\tau\rangle\} = (w' - \mathbb{I})|\tau\rangle$$

$$\delta_\epsilon |\tilde{\tau}\rangle := -\tilde{\lambda}^2\kappa^{-1}\{\mathrm{Tr} W (\tilde{\ell}^\dagger \tilde{\ell})^{-1}, |\tilde{\tau}\rangle\} = (w'' - \mathbb{I})|\tilde{\tau}\rangle$$



[M. Dupuis, F. Girelli, E. Livine, 2014; V. Bonzom, M. Dupuis, F. Girelli, QP, in prep]

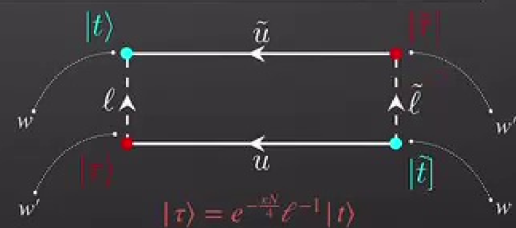
3:07 PM Thu Sep 2

Deformed spinorial phase space

- Spinors on the opposite corners of the ribbon

$$|t\rangle = \begin{pmatrix} t_+ \\ t_- \end{pmatrix} := \begin{pmatrix} e^{\frac{\kappa N_1}{4}} \zeta_0^\kappa \\ e^{-\frac{\kappa N_0}{4}} \zeta_1^\kappa \end{pmatrix}, \qquad |\tilde{t}\rangle = \begin{pmatrix} \tilde{t}_+ \\ \tilde{t}_- \end{pmatrix} := \begin{pmatrix} e^{\frac{\kappa \tilde{N}_1}{4}} \tilde{\zeta}_0^\kappa \\ e^{-\frac{\kappa \tilde{N}_0}{4}} \tilde{\zeta}_1^\kappa \end{pmatrix}$$

$$w = \mathbb{I} + i\,\vec{\epsilon} \cdot \vec{\sigma},$$
$$\delta_\epsilon |t\rangle := -\lambda^{-2}\kappa^{-1}\{\mathrm{Tr}W\ell\ell^\dagger, |t\rangle\} = (w - \mathbb{I})|t\rangle$$
$$\delta_\epsilon |\tilde{t}\rangle := -\tilde{\lambda}^{2}\kappa^{-1}\{\mathrm{Tr}W(\tilde{\ell}^\dagger\tilde{\ell})^{-1}, |\tilde{t}\rangle\} = (w - \mathbb{I})|\tilde{t}\rangle$$



$$|\tau\rangle = e^{-\frac{\kappa N}{4}}\ell^{-1}|t\rangle$$
$$|\tilde{\tau}] = -e^{\frac{\kappa \tilde{N}}{4}}\tilde{\ell}\,|\tilde{t}]$$

- Braided spinors on the other two corners

$$|\tau\rangle = \begin{pmatrix} \tau_+ \\ \tau_- \end{pmatrix} := \begin{pmatrix} e^{-\frac{\kappa N_1}{4}} \zeta_0^\kappa \\ e^{\frac{\kappa N_0}{4}} \zeta_1^\kappa \end{pmatrix}, \qquad |\tilde{\tau}\rangle = \begin{pmatrix} \tilde{\tau}_+ \\ \tilde{\tau}_- \end{pmatrix} := \begin{pmatrix} e^{-\frac{\kappa \tilde{N}_1}{4}} \tilde{\zeta}_0^\kappa \\ e^{\frac{\kappa \tilde{N}_0}{4}} \tilde{\zeta}_1^\kappa \end{pmatrix}$$

$$^{(w)}\ell = w\ell w'^{-1} \in \mathrm{AN}(2) \qquad w' = \mathbb{I} + i\,\vec{\epsilon}' \cdot \vec{\sigma}$$
$$\tilde{\ell}^{(w)} = w''^{-1}\tilde{\ell}w \in \mathrm{AN}(2) \qquad w'' = \mathbb{I} + i\,\vec{\epsilon}'' \cdot \vec{\sigma}$$
$$\delta_\epsilon |\tau\rangle := -\lambda^{-2}\kappa^{-1}\{\mathrm{Tr}W\ell\ell^\dagger, |\tau\rangle\} = (w' - \mathbb{I})|\tau\rangle$$
$$\delta_\epsilon |\tilde{\tau}\rangle := -\tilde{\lambda}^{2}\kappa^{-1}\{\mathrm{Tr}W(\tilde{\ell}^\dagger\tilde{\ell})^{-1}, |\tilde{\tau}\rangle\} = (w'' - \mathbb{I})|\tilde{\tau}\rangle$$

The holonomies can be reconstructed with these four deformed spinors

$$u = \frac{|\tau\rangle[\tilde{t}| - |\tau]\langle\tilde{t}|}{\sqrt{\langle\tau|\tau\rangle\langle\tilde{t}|\tilde{t}\rangle}}, \qquad \tilde{u} = \frac{|t]\langle\tilde{\tau}| - |t\rangle[\tilde{\tau}|}{\sqrt{\langle t|t\rangle\langle\tilde{\tau}|\tilde{\tau}\rangle}} \in \mathrm{SU}(2)$$

(with the norm matching constraint $N = \tilde{N}$)

Flux-holonomy Poisson structure is recovered

Fluxes can also parallel transport spinors

[M. Dupuis, F. Girelli, E. Livine, 2014; V. Bonzom, M. Dupuis, F. Girelli, QP, in prep]

42 / 72

3:08 PM Thu Sep 2

Scalar product of deformed spinors

- Scalar product of spinors from different links

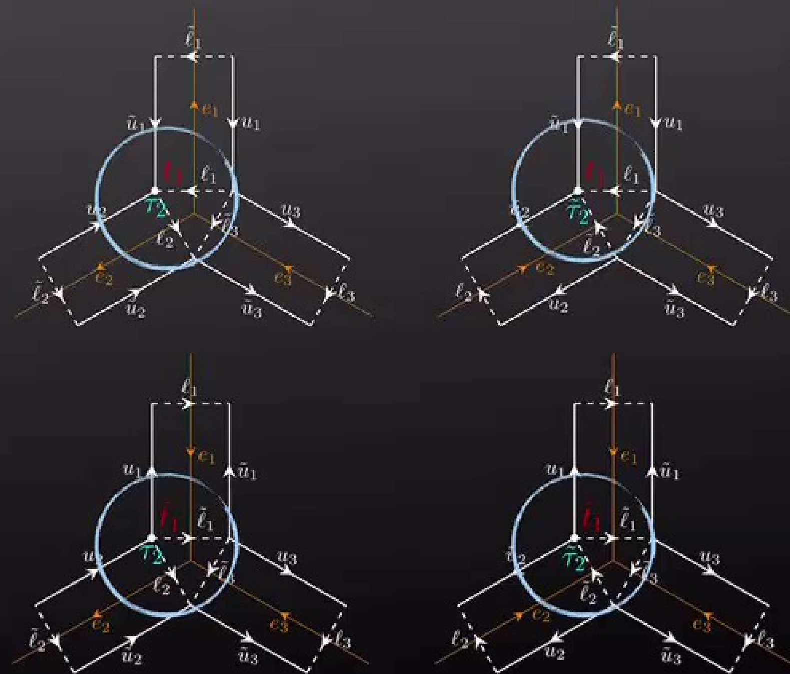
↪ Four pairs of spinors depending on the orientation

↪ Each pair has four possibilities

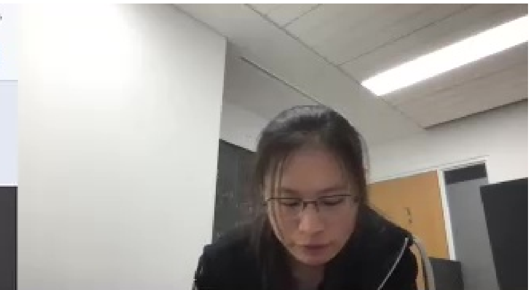
$$E_{12}^{\epsilon_1,\epsilon_2} = \begin{cases} [t_1 | \tau_2\rangle & \text{for} \quad \epsilon_1 = \epsilon_2 = - \\ [t_1 | \tau_2] & \text{for} \quad \epsilon_1 = -\,, \epsilon_2 = + \\ \langle t_1 | \tau_2\rangle & \text{for} \quad \epsilon_1 = +\,, \epsilon_2 = - \\ \langle t_1 | \tau_2] & \text{for} \quad \epsilon_1 = \epsilon_2 = + \end{cases}$$

↪ They are SU(2)-invariant observables

$$\{\mathrm{Tr} W \mathcal{G}\mathcal{G}^\dagger, E_{12}^{\epsilon_1,\epsilon_2}\} = 0$$



[V. Bonzom, M. Dupuis, F. Girelli, QP, in prep]



Quantization

$$A \longrightarrow \hat{A}$$
$$\{A, B\} \longrightarrow i\hbar[\hat{A}, \hat{B}]$$

3:10 PM Thu Sep 2

Quantum fluxes and holonomies

- Quantize the fluxes and the holonomies into operators

$$\ell \longrightarrow L, \quad u \longrightarrow U, \quad \tilde{\ell} \longrightarrow \tilde{L}, \quad \tilde{u} \longrightarrow \tilde{U}$$

- Quantize the classical r-matrix into the quantum $\mathscr{R}$-matrix

$$\mathscr{R} = \mathbb{I} \otimes \mathbb{I} + i\hbar r + O(\hbar^2)$$

In the fundamental representation, introduce $q = e^{\kappa\hbar} = 1 + \kappa\hbar + O(\hbar^2)$

$$r = \frac{i\kappa}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 4 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow R = \begin{pmatrix} q^{\frac{1}{4}} & 0 & 0 & 0 \\ 1 & q^{-\frac{1}{4}} & q^{-\frac{1}{4}}(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) & 0 \\ 0 & 0 & q^{-\frac{1}{4}} & 0 \\ 0 & 0 & 0 & q^{\frac{1}{4}} \end{pmatrix}$$

- Quantize the Poisson brackets into commutation relations

$$e.g. \begin{array}{l} \{\ell_1, \ell_2\} = [r, \ell_1\ell_2] \\ \{\tilde{\ell}_1, \tilde{\ell}_2\} = -[r, \tilde{\ell}_1\tilde{\ell}_2] \end{array} \longrightarrow \begin{array}{l} L_1L_2R^{-1} = R^{-1}L_2L_1 \\ \tilde{L}_1\tilde{L}_2R = R\tilde{L}_2\tilde{L}_1 \end{array}$$

$$\begin{array}{l} \{u_1, u_2\} = -[r, u_1u_2] \\ \{\tilde{u}_1, \tilde{u}_2\} = [r, \tilde{u}_1\tilde{u}_2] \end{array} \longrightarrow \begin{array}{l} R^{-1}U_1U_2 = U_2U_1R^{-1} \\ R\tilde{U}_1\tilde{U}_2 = \tilde{U}_2\tilde{U}_1R \end{array}$$

[V. Bonzom, M. Dupuis, F. Girelli, 2014; V. Bonzom, M. Dupuis, F. Girelli, QP, in prep]

46 / 72

3:10 PM Thu Sep 2

Quantum fluxes and holonomies

- Quantize the fluxes and the holonomies into operators

$$\ell \longrightarrow L, \quad u \longrightarrow U, \quad \tilde{\ell} \longrightarrow \tilde{L}, \quad \tilde{u} \longrightarrow \tilde{U}$$

- Quantize the classical r-matrix into the quantum $\mathscr{R}$-matrix

$$\mathscr{R} = \mathbb{I} \otimes \mathbb{I} + i\hbar r + O(\hbar^2)$$

the $\mathscr{R}$-matrix of $\mathscr{U}_q(\mathfrak{su}(2))$

In the fundamental representation, introduce $q = e^{\kappa\hbar} = 1 + \kappa\hbar + O(\hbar^2)$

$$r = \frac{i\kappa}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 4 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow R = \begin{pmatrix} q^{\frac{1}{4}} & 0 & 0 & 0 \\ 1 & q^{-\frac{1}{4}} & q^{-\frac{1}{4}}(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) & 0 \\ 0 & 0 & q^{-\frac{1}{4}} & 0 \\ 0 & 0 & 0 & q^{\frac{1}{4}} \end{pmatrix}$$

- Quantize the Poisson brackets into commutation relations

$$e.g. \begin{matrix} \{\ell_1, \ell_2\} = [r, \ell_1\ell_2] \\ \{\tilde{\ell}_1, \tilde{\ell}_2\} = -[r, \tilde{\ell}_1\tilde{\ell}_2] \end{matrix} \longrightarrow \begin{matrix} L_1L_2R^{-1} = R^{-1}L_2L_1 \\ \tilde{L}_1\tilde{L}_2R = R\tilde{L}_2\tilde{L}_1 \end{matrix}$$

$$\begin{matrix} \{u_1, u_2\} = -[r, u_1u_2] \\ \{\tilde{u}_1, \tilde{u}_2\} = [r, \tilde{u}_1\tilde{u}_2] \end{matrix} \longrightarrow \begin{matrix} R^{-1}U_1U_2 = U_2U_1R^{-1} \\ R\tilde{U}_1\tilde{U}_2 = \tilde{U}_2\tilde{U}_1R \end{matrix}$$

What quantum group are they?

[V. Bonzom, M. Dupuis, F. Girelli, 2014; V. Bonzom, M. Dupuis, F. Girelli, QP, in prep]

47 / 72

Quantum fluxes and holonomies

- Quantize the fluxes and the holonomies into operators

$$\ell \longrightarrow L, \quad u \longrightarrow U, \quad \tilde{\ell} \longrightarrow \tilde{L}, \quad \tilde{u} \longrightarrow \tilde{U}$$

- Quantize the classical r-matrix into the quantum $\mathscr{R}$-matrix

$$\mathscr{R} = \mathbb{I} \otimes \mathbb{I} + i\hbar r + O(\hbar^2)$$

the $\mathscr{R}$-matrix of $\mathscr{U}_q(\mathfrak{su}(2))$

In the fundamental representation, introduce $q = e^{\kappa\hbar} = 1 + \kappa\hbar + O(\hbar^2)$

$$r = \frac{i\kappa}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 4 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow R = \begin{pmatrix} q^{\frac{1}{4}} & 0 & 0 & 0 \\ 1 & q^{-\frac{1}{4}} & q^{-\frac{1}{4}}(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) & 0 \\ 0 & 0 & q^{-\frac{1}{4}} & 0 \\ 0 & 0 & 0 & q^{\frac{1}{4}} \end{pmatrix}$$

- Quantize the Poisson brackets into commutation relations

$$e.g. \begin{array}{l} \{\ell_1, \ell_2\} = [r, \ell_1\ell_2] \\ \{\tilde{\ell}_1, \tilde{\ell}_2\} = -[r, \tilde{\ell}_1\tilde{\ell}_2] \end{array} \longrightarrow \begin{array}{l} L_1L_2R^{-1} = R^{-1}L_2L_1 \\ \tilde{L}_1\tilde{L}_2R = R\tilde{L}_2\tilde{L}_1 \end{array}$$

$$\begin{array}{l} \{u_1, u_2\} = -[r, u_1u_2] \\ \{\tilde{u}_1, \tilde{u}_2\} = [r, \tilde{u}_1\tilde{u}_2] \end{array} \longrightarrow \begin{array}{ll} R^{-1}U_1U_2 = U_2U_1R^{-1} & \longrightarrow \mathrm{SU}_{q^{-1}}(2) \\ R\tilde{U}_1\tilde{U}_2 = \tilde{U}_2\tilde{U}_1R & \longrightarrow \mathrm{SU}_q(2) \end{array}$$

[V. Bonzom, M. Dupuis, F. Girelli, 2014; V. Bonzom, M. Dupuis, F. Girelli, QP, in prep]

3:11 PM Thu Sep 2 82%

Quantum fluxes and holonomies

- Quantize the fluxes and the holonomies into operators

$$\ell \longrightarrow L\,,\quad u \longrightarrow U\,,\quad \tilde{\ell} \longrightarrow \tilde{L}\,,\quad \tilde{u} \longrightarrow \tilde{U}$$

- Quantize the classical r-matrix into the quantum $\mathscr{R}$-matrix

$$\mathscr{R} = \mathbb{I} \otimes \mathbb{I} + i\hbar r + O(\hbar^2)$$

the $\mathscr{R}$-matrix of $\mathscr{U}_q(\mathfrak{su}(2))$

In the fundamental representation, introduce $q = e^{\kappa\hbar} = 1 + \kappa\hbar + O(\hbar^2)$

$$r = \frac{i\kappa}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 4 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow R = \begin{pmatrix} q^{\frac{1}{4}} & 0 & 0 & 0 \\ 1 & q^{-\frac{1}{4}} & q^{-\frac{1}{4}}(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) & 0 \\ 0 & 0 & q^{-\frac{1}{4}} & 0 \\ 0 & 0 & 0 & q^{\frac{1}{4}} \end{pmatrix}$$

$$L = \begin{pmatrix} K^{-1} & 0 \\ -q^{\frac{1}{4}}(q^{\frac{1}{2}} - q^{-\frac{1}{2}})J_+ & K \end{pmatrix},\quad \tilde{L} = \begin{pmatrix} K & 0 \\ q^{-\frac{1}{4}}(q^{\frac{1}{2}} - q^{-\frac{1}{2}})J_+ & K^{-1} \end{pmatrix}$$

$$KJ_\pm K^{-1} = q^{\pm\frac{1}{2}}J_\pm\,,\quad [J_+, J_-] = [2J_z] \quad [n] := \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$$

- Quantize the Poisson brackets into commutation relations

$$e.g.\ \begin{matrix} \{\ell_1, \ell_2\} = [r, \ell_1\ell_2] \\ \{\tilde{\ell}_1, \tilde{\ell}_2\} = -[r, \tilde{\ell}_1\tilde{\ell}_2] \end{matrix} \longrightarrow \begin{matrix} L_1L_2R^{-1} = R^{-1}L_2L_1 \longrightarrow \mathscr{U}_{q^{-1}}(\mathfrak{su}(2)) \\ \tilde{L}_1\tilde{L}_2R = R\tilde{L}_2\tilde{L}_1 \longrightarrow \mathscr{U}_q(\mathfrak{su}(2)) \end{matrix}$$

$$\begin{matrix} \{u_1, u_2\} = -[r, u_1u_2] \\ \{\tilde{u}_1, \tilde{u}_2\} = [r, \tilde{u}_1\tilde{u}_2] \end{matrix} \longrightarrow \begin{matrix} R^{-1}U_1U_2 = U_2U_1R^{-1} \longrightarrow \mathrm{SU}_{q^{-1}}(2) \\ R\tilde{U}_1\tilde{U}_2 = \tilde{U}_2\tilde{U}_1R \longrightarrow \mathrm{SU}_q(2) \end{matrix}$$

[V. Bonzom, M. Dupuis, F. Girelli, 2014; V. Bonzom, M. Dupuis, F. Girelli, QP, in prep]

49 / 72

3:12 PM Thu Sep 2

Quantum fluxes and holonomies

- Quantize the fluxes and the holonomies into operators

$$\ell \longrightarrow L, \quad u \longrightarrow U, \quad \tilde{\ell} \longrightarrow \tilde{L}, \quad \tilde{u} \longrightarrow \tilde{U}$$

- Quantize the classical r-matrix into the quantum $\mathscr{R}$-matrix

$$\mathscr{R} = \mathbb{I} \otimes \mathbb{I} + i\hbar r + O(\hbar^2)$$

the $\mathscr{R}$-matrix of $\mathscr{U}_q(\mathfrak{su}(2))$

In the fundamental representation, introduce $q = e^{\kappa\hbar} = 1 + \kappa\hbar + O(\hbar^2)$

$$r = \frac{i\kappa}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 4 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow R = \begin{pmatrix} q^{\frac{1}{4}} & 0 & 0 & 0 \\ 1 & q^{-\frac{1}{4}} & q^{-\frac{1}{4}}(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) & 0 \\ 0 & 0 & q^{-\frac{1}{4}} & 0 \\ 0 & 0 & 0 & q^{\frac{1}{4}} \end{pmatrix}$$

$$L = \begin{pmatrix} K^{-1} & 0 \\ -q^{\frac{1}{4}}(q^{\frac{1}{2}} - q^{-\frac{1}{2}})J_+ & K \end{pmatrix}, \quad \tilde{L} = \begin{pmatrix} K & 0 \\ q^{-\frac{1}{4}}(q^{\frac{1}{2}} - q^{-\frac{1}{2}})J_+ & K^{-1} \end{pmatrix}$$

$$KJ_\pm K^{-1} = q^{\pm\frac{1}{2}}J_\pm, \quad [J_+, J_-] = [2J_z] \quad [n] := \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$$

$K = q^{\frac{J_z}{2}}$

- Quantize the Poisson brackets into commutation relations

e.g. $\{\ell_1, \ell_2\} = [r, \ell_1\ell_2] \longrightarrow L_1L_2R^{-1} = R^{-1}L_2L_1 \longrightarrow \mathscr{U}_{q^{-1}}(\mathfrak{su}(2))$

$\{\tilde{\ell}_1, \tilde{\ell}_2\} = -[r, \tilde{\ell}_1\tilde{\ell}_2] \longrightarrow \tilde{L}_1\tilde{L}_2R = R\tilde{L}_2\tilde{L}_1 \longrightarrow \mathscr{U}_q(\mathfrak{su}(2))$

$\{u_1, u_2\} = -[r, u_1u_2] \longrightarrow R^{-1}U_1U_2 = U_2U_1R^{-1} \longrightarrow \mathrm{SU}_{q^{-1}}(2)$

$\{\tilde{u}_1, \tilde{u}_2\} = [r, \tilde{u}_1\tilde{u}_2] \longrightarrow R\tilde{U}_1\tilde{U}_2 = \tilde{U}_2\tilde{U}_1R \longrightarrow \mathrm{SU}_q(2)$

[V. Bonzom, M. Dupuis, F. Girelli, 2014; V. Bonzom, M. Dupuis, F. Girelli, QP, in prep]

49 / 72

Quantum fluxes and holonomies

- The commutation relations of L's, U's, $\tilde{L}$'s and $\tilde{U}$'s imply:

$$\begin{array}{lcl} \ell \in \mathrm{AN}(2) & \longrightarrow & L \in \mathrm{Fun}_{q^{-1}}(\mathrm{AN}(2)) \cong \mathscr{U}_{q^{-1}}(\mathfrak{su}(2)) \\ u \in \mathrm{SU}(2) & \longrightarrow & U \in \mathrm{SU}_{q^{-1}}(2) \\ \tilde{\ell} \in \mathrm{AN}(2) & \longrightarrow & \tilde{L} \in \mathrm{Fun}_{q}(\mathrm{AN}(2)) \cong \mathscr{U}_{q}(\mathfrak{su}(2)) \\ \tilde{u} \in \mathrm{SU}(2) & \longrightarrow & \tilde{U} \in \mathrm{SU}_{q}(2) \end{array}$$

Duality between $\mathscr{U}_q(\mathfrak{su}(2))$ and $\mathrm{SU}_q(2)$:
$\langle U_1, L_2 \rangle = R^{-1}, \quad \langle \tilde{U}_1, \tilde{L}_2 \rangle = R$

Both q and q^{-1}-deformation in the ribbon picture

Duality between holonomy and flux extends from Lie bialgebra to quantum group

[V. Bonzom, M. Dupuis, F. Girelli, QP, in prep]

Quantum fluxes and holonomies

- The geometrical interpretation of the $\mathscr{R}$-matrix - "quantum holonomy"

$$\tilde{L}_1\tilde{L}_2R = R\tilde{L}_2\tilde{L}_1$$
$$R\tilde{U}_1\tilde{U}_2 = \tilde{U}_2\tilde{U}_1R$$

They are nothing but the quantum Yang-Baxter equations in particular representations

$$(\tilde{L}_{kl})_{ij} = \mathscr{R}_{ij,kl} = (\tilde{U}_{ij})_{kl}$$

$k, l = \pm\frac{1}{2}$ $\qquad$ $i, j = \pm\frac{1}{2}$

$$\mathscr{R}_{12}\mathscr{R}_{13}\mathscr{R}_{23} = \mathscr{R}_{23}\mathscr{R}_{13}\mathscr{R}_{12}$$

$$\mathscr{R} = \sum \mathscr{R}_{(1)} \otimes \mathscr{R}_{(2)}$$

quantum holonomy $\qquad$ quantum flux

The $\mathscr{R}$-matrix contains flux and holonomy information

[V. Bonzom, M. Dupuis, F. Girelli, QP, in prep]

Quantum fluxes and holonomies

- The geometrical interpretation of the $\mathscr{R}$-matrix - "quantum holonomy"

$$\tilde{L}_1\tilde{L}_2R = R\tilde{L}_2\tilde{L}_1$$
$$R\tilde{U}_1\tilde{U}_2 = \tilde{U}_2\tilde{U}_1R$$

They are nothing but the quantum Yang-Baxter equations in particular representations

$$(\tilde{L}_{kl})_{ij} = \mathscr{R}_{ij,kl} = (\tilde{U}_{ij})_{kl}$$

$k,l = \pm\frac{1}{2}$ $\quad$ $i,j = \pm\frac{1}{2}$

$$\mathscr{R}_{12}\mathscr{R}_{13}\mathscr{R}_{23} = \mathscr{R}_{23}\mathscr{R}_{13}\mathscr{R}_{12}$$

$$\mathscr{R} = \sum \mathscr{R}_{(1)} \otimes \mathscr{R}_{(2)}$$

quantum holonomy $\quad$ quantum flux

- The quantized Gauss constraint is from the coproduct of quantum matrices L or $\tilde{L}$

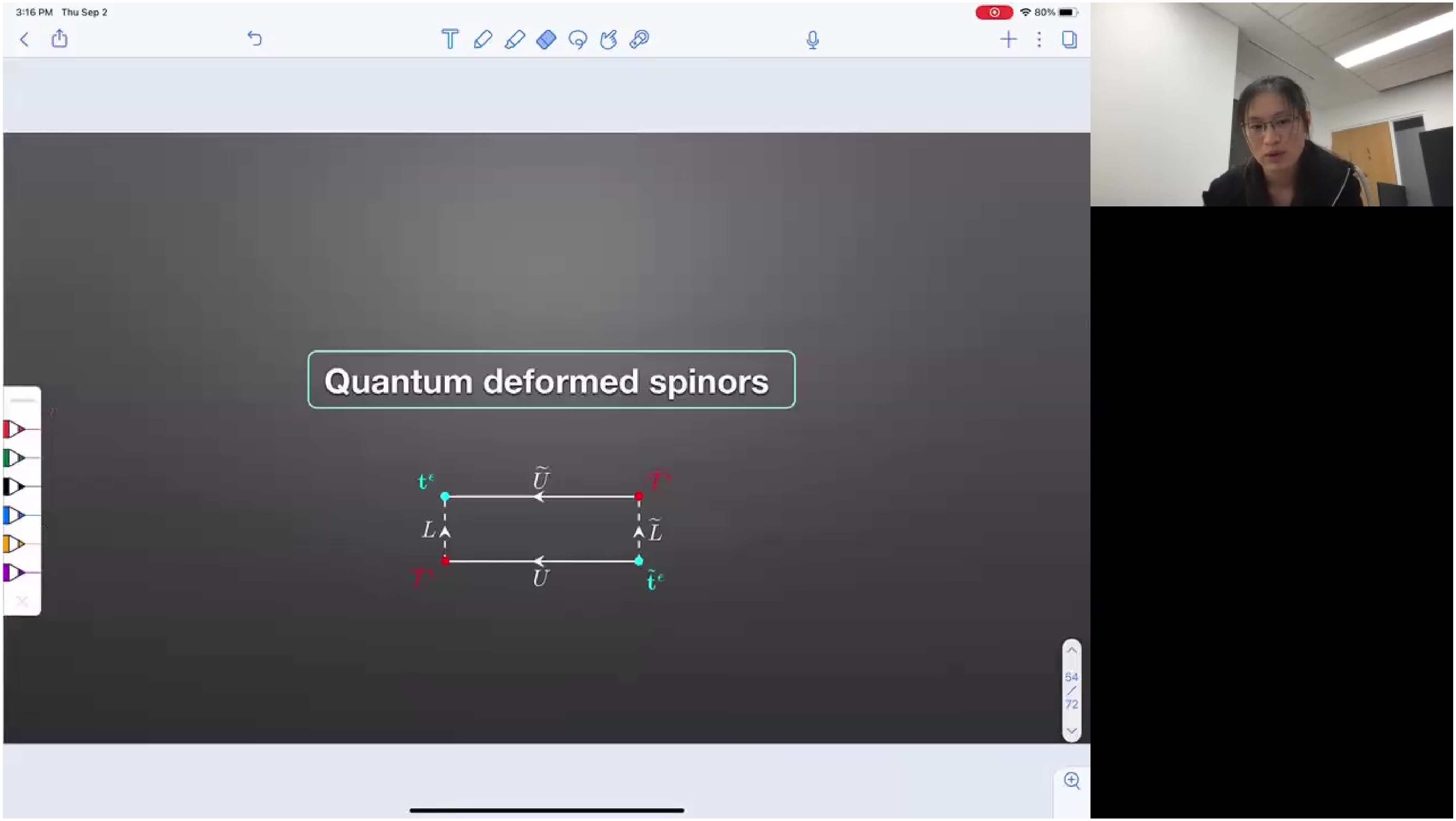
$$\mathscr{G} = \mathscr{L}_1\mathscr{L}_2\mathscr{L}_3 \longrightarrow \hat{\mathscr{G}}_{ij} = \sum_{kl} \hat{L}_{ik} \otimes \hat{L}_{kl} \otimes \hat{L}_{lj}$$

- The solutions are the $\mathscr{U}_q(\mathfrak{su}(2))$-intertwiners

$$\hat{\mathscr{G}}\, i_{j_1j_2j_3} = i_{j_1j_2j_3}$$

The $\mathscr{R}$-matrix contains flux and holonomy information

[V. Bonzom, M. Dupuis, F. Girelli, QP, in prep]


3:16 PM Thu Sep 2
80%
Quantum deformed spinors
t^ϵ
$\widetilde{U}$
T^ϵ
L
$\widetilde{L}$
T^ϵ
U
$\tilde{t}^\epsilon$
54 / 72

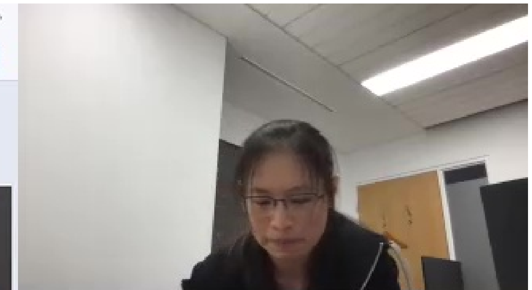
3:18 PM Thu Sep 2 79%

Quantum spinors

- Quantize the κ-deformed spinor variables to the q-harmonic oscillators
  $(\zeta_0^\kappa, \zeta_1^\kappa) \to (a, b)\,, \quad (\bar\zeta_0^\kappa, \bar\zeta_1^\kappa) \to (a^\dagger, b^\dagger)\,, \quad (N_0, N_1) \to (N_a, N_b)$
- They obey the commutation rules
  $$aa^\dagger - q^{\mp\frac{1}{2}}a^\dagger a = q^{\pm\frac{N_a}{2}}\,, \quad [N_a, a^\dagger] = a^\dagger\,, \quad [N_a, a] = -a$$
- The Jordan map is recovered
  $$J_+ = a^\dagger b\,, \quad J_- = ab^\dagger\,, \quad J_z = \frac{N_a - N_b}{2}$$
- Quantize the deformed spinors to quantum spinors
  $$|t\rangle \to \mathbf{t}^-\,, \quad |t] \to \mathbf{t}^+\,, \quad |\tau\rangle \to \mathscr{T}^-\,, \quad |\tau] \to \mathscr{T}^+$$
  $$|\tilde t\rangle \to \tilde{\mathbf{t}}^-\,, \quad |\tilde t] \to \tilde{\mathbf{t}}^+\,, \quad |\tilde\tau\rangle \to \tilde{\mathscr{T}}^-\,, \quad |\tilde\tau] \to \tilde{\mathscr{T}}^+$$

In what sense can we call them quantum spinors?
$J_\pm \cdot \mathbf{T}_\mp = 0\,, \quad J_\pm \cdot \mathbf{T}_\pm = \mathbf{T}_\mp\,, \quad K \cdot \mathbf{T}_\pm = q^{\pm\frac{1}{4}}\mathbf{T}_\pm$

56 / 72



3:18 PM Thu Sep 2

Quantum spinors

- Quantize the κ-deformed spinor variables to the q-harmonic oscillators
  $$(\zeta_0^\kappa, \zeta_1^\kappa) \to (a, b), \quad (\bar{\zeta}_0^\kappa, \bar{\zeta}_1^\kappa) \to (a^\dagger, b^\dagger), \quad (N_0, N_1) \to (N_a, N_b)$$
- They obey the commutation rules
  $$aa^\dagger - q^{\mp\frac{1}{2}} a^\dagger a = q^{\pm\frac{N_a}{2}}, \quad [N_a, a^\dagger] = a^\dagger, \quad [N_a, a] = -a$$
- The Jordan map is recovered
  $$J_+ = a^\dagger b, \quad J_- = ab^\dagger, \quad J_z = \frac{N_a - N_b}{2}$$
- Quantize the deformed spinors to quantum spinors
  $$|t\rangle \to \mathbf{t}^-, \quad |t] \to \mathbf{t}^+, \quad |\tau\rangle \to \mathcal{T}^-, \quad |\tau] \to \mathcal{T}^+$$
  $$|\tilde{t}\rangle \to \tilde{\mathbf{t}}^-, \quad |\tilde{t}] \to \tilde{\mathbf{t}}^+, \quad |\tilde{\tau}\rangle \to \tilde{\mathcal{T}}^-, \quad |\tilde{\tau}] \to \tilde{\mathcal{T}}^+$$

$\mathscr{U}_q(\mathfrak{su}(2))$ quantum spinors | $\mathscr{U}_{q^{-1}}(\mathfrak{su}(2))$ quantum spinors

in the sense of the adjoint right action

$$J_\pm \triangleright \mathcal{O} = -q^{\pm\frac{1}{2}} J_\pm \mathcal{O} K + K \mathcal{O} J_\pm \qquad J_\pm \bar{\triangleright} \mathcal{O} = -q^{\mp\frac{1}{2}} J_\pm \mathcal{O} K^{-1} + K^{-1} \mathcal{O} J_\pm$$
$$K \triangleright \mathcal{O} = K^{-1} \mathcal{O} K \qquad K \bar{\triangleright} \mathcal{O} = K^{-1} \mathcal{O} K$$

57 / 72

Quantum spinors

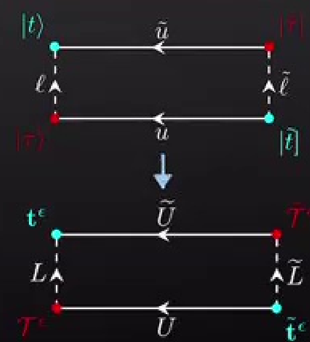
- Quantize the κ-deformed spinor variables to the *q*-harmonic oscillators
  $(\zeta_0^\kappa, \zeta_1^\kappa) \to (a, b), \quad (\bar\zeta_0^\kappa, \bar\zeta_1^\kappa) \to (a^\dagger, b^\dagger), \quad (N_0, N_1) \to (N_a, N_b)$
- They obey the commutation rules
  $$aa^\dagger - q^{\mp\frac{1}{2}} a^\dagger a = q^{\pm\frac{N_a}{2}}, \quad [N_a, a^\dagger] = a^\dagger, \quad [N_a, a] = -a$$
- The Jordan map is recovered
  $$J_+ = a^\dagger b, \quad J_- = ab^\dagger, \quad J_z = \frac{N_a - N_b}{2}$$
- Quantize the deformed spinors to quantum spinors
  $$|t\rangle \to \mathbf{t}^-, \quad |t] \to \mathbf{t}^+, \quad |\tau\rangle \to \mathcal{T}^-, \quad |\tau] \to \mathcal{T}^+$$
  $$|\tilde t\rangle \to \tilde{\mathbf{t}}^-, \quad |\tilde t] \to \tilde{\mathbf{t}}^+, \quad |\tilde\tau\rangle \to \tilde{\mathcal{T}}^-, \quad |\tilde\tau] \to \tilde{\mathcal{T}}^+$$
  $\mathcal{U}_q(\mathfrak{su}(2))$ quantum spinors $\qquad \mathcal{U}_{q^{-1}}(\mathfrak{su}(2))$ quantum spinors

  in the sense of the adjoint right action

  $J_\pm \triangleright \mathcal{O} = -q^{\pm\frac{1}{2}} J_\pm \mathcal{O} K + K \mathcal{O} J_\pm$ $\qquad J_\pm \,\bar\triangleright\, \mathcal{O} = -q^{\mp\frac{1}{2}} J_\pm \mathcal{O} K^{-1} + K^{-1} \mathcal{O} J_\pm$

  $K \triangleright \mathcal{O} = K^{-1} \mathcal{O} K$ $\qquad K \,\bar\triangleright\, \mathcal{O} = K^{-1} \mathcal{O} K$

- These spinors can be used to reconstruct the quantum holonomies
  $$U_{AB} = (-1)^{\frac{1}{2}+B} q^{-\frac{B}{2}} \sum_{\epsilon=\pm} \mathcal{T}_A^\epsilon \tilde{\mathbf{t}}^\epsilon_{-B} \frac{1}{[N+1]}, \quad U \in \mathbf{SU}_{q^{-1}}(2)$$
  $$\tilde U_{AB} = \frac{1}{[\tilde N + 1]} (-1)^{\frac{1}{2}+B} q^{\frac{B}{2}} \sum_{\epsilon=\pm} \mathbf{t}_A^\epsilon \tilde{\mathcal{T}}^\epsilon_{-B}, \quad \tilde U \in \mathbf{SU}_q(2)$$



3:20 PM Thu Sep 2

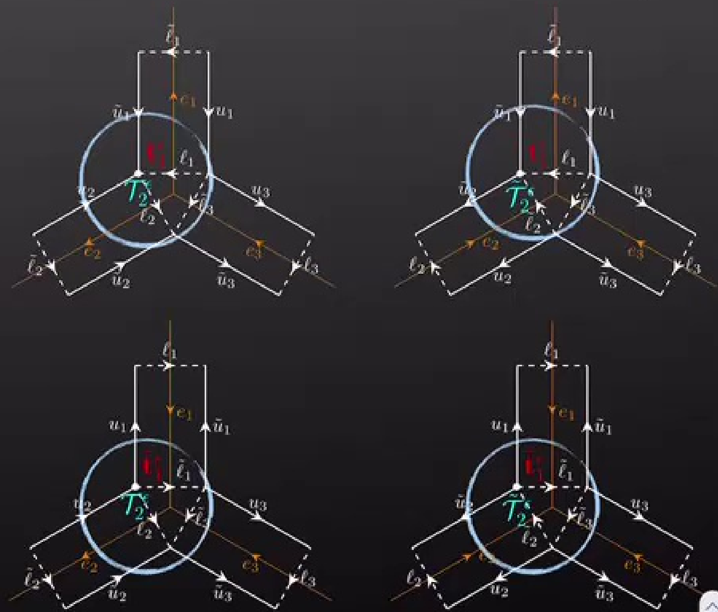
Scalar operator from the quantum spinors

- The scalar product of classical deformed spinors are quantized to the scalar operators

$$E_{12}^{\epsilon_1,\epsilon_2} \longrightarrow \mathbf{E}_{12}^{\epsilon_1,\epsilon_2} = o_1\sqrt{[2]} \sum_{A=\pm\frac{1}{2}} q^{-o_1} C^{\frac{1}{2}\,\frac{1}{2}\,0}_{A\,-A\,0}\, \mathbf{T}^{-o_1\epsilon_1}_{e_1,A} \otimes \mathbf{T}^{-o_2\epsilon_2}_{e_2,-A} \otimes \mathbb{I}$$

↪ They are $\mathcal{U}_q(\mathfrak{su}(2))$-invariant thus live in the intertwiner space

$$\mathbf{E}_{12}^{\epsilon_1,\epsilon_2}\, i_{j_1 j_2 j_3} = \delta_{l_1, j_1+\frac{\epsilon_1}{2}} \delta_{l_2, j_2+\frac{\epsilon_2}{2}} \sqrt{[d_{j_1}][d_{j_2}][d_{l_1}][d_{l_2}]}\,(-1)^{l_1+l_2+j_3} \begin{Bmatrix} l_1 & j_1 & \frac{1}{2} \\ j_2 & l_2 & j_3 \end{Bmatrix}_q i_{l_1 l_2 j_3}$$



60 / 72

3:21 PM Thu Sep 2

Scalar operator from the quantum spinors

- The scalar product of classical deformed spinors are quantized to the scalar operators

$$E_{12}^{\epsilon_1,\epsilon_2} \longrightarrow \mathbf{E}_{12}^{\epsilon_1,\epsilon_2} = o_1\sqrt{[2]} \sum_{A=\pm\frac{1}{2}} q^{-o_1} C^{\frac{1}{2}\,\frac{1}{2}\,0}_{A\,-A\,0}\, \mathbf{T}^{-o_1\epsilon_1}_{e_1,A} \otimes \mathbf{T}^{-o_2\epsilon_2}_{e_2,-A} \otimes \mathbb{I}$$

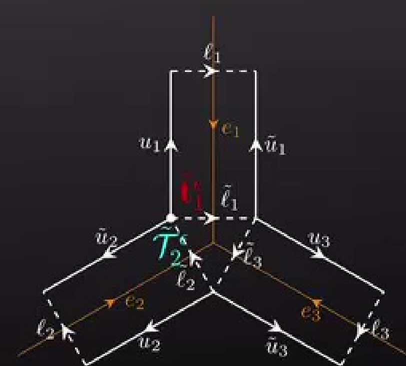
↪ They are $\mathscr{U}_q(\mathfrak{su}(2))$-invariant thus live in the intertwiner space

$$\mathbf{E}_{12}^{\epsilon_1,\epsilon_2}\, i_{j_1 j_2 j_3} = \delta_{l_1, j_1+\frac{\epsilon_1}{2}} \delta_{l_2, j_2+\frac{\epsilon_2}{2}} \sqrt{[d_{j_1}][d_{j_2}][d_{l_1}][d_{l_2}]}\,(-1)^{l_1+l_2+j_3} \begin{Bmatrix} l_1 & j_1 & \frac{1}{2} \\ j_2 & l_2 & j_3 \end{Bmatrix}_q i_{l_1 l_2 j_3}$$

- A standard way to define the scalar operators is in a braided way with the $\mathscr{R}$-matrix, $e.g.$

$$\sum_A {}_qC^{\frac{1}{2}\,\frac{1}{2}\,0}_{A\,-A\,0}\; {}^{(2)}\tilde{\mathcal{T}}^{\epsilon}_{-A}\; {}^{(1)}\tilde{\mathcal{T}}^{\epsilon}_{A} \overset{!}{\propto} \mathbf{E}_{12}^{\epsilon_1,\epsilon_2}$$

$${}^{(2)}\tilde{\mathcal{T}}^{\epsilon}_{A} = \mathscr{R}^{-1}(\mathbb{I} \otimes \tilde{\mathcal{T}}^{\epsilon}_{A})\mathscr{R}, \quad {}^{(1)}\tilde{\mathcal{T}}^{\epsilon}_{A} = \tilde{\mathcal{T}}^{\epsilon}_{A} \otimes \mathbb{I}$$



[V. Rittenberg, M. Scheunert, 1992]

61 / 72

Scalar operator from the quantum spinors

- The scalar product of classical deformed spinors are quantized to the scalar operators

$$E_{12}^{\epsilon_1,\epsilon_2} \longrightarrow \mathbf{E}_{12}^{\epsilon_1,\epsilon_2} = o_1\sqrt{[2]}\sum_{A=\pm\frac{1}{2}} q^{-o_1}C^{\frac{1}{2}\,\frac{1}{2}\,0}_{A\,-A\,0}\,\mathbf{T}^{-o_1\epsilon_1}_{\epsilon_1,A}\otimes\mathbf{T}^{-o_2\epsilon_2}_{\epsilon_2,-A}\otimes\mathbb{I}$$

Why don't we need braided structure here?

Ans: Because the quantum flux plays the role of braiding

$$\ell\,|\tau\rangle \propto |t\rangle \longrightarrow L\mathcal{T}^{\epsilon} \propto \mathbf{t}^{\epsilon}$$

$$\tilde{\ell}\,|\tilde{\tau}] \propto |\tilde{\tau}] \longrightarrow \tilde{L}\tilde{\mathbf{t}}^{\epsilon} \propto \tilde{\mathcal{T}}^{\epsilon}$$

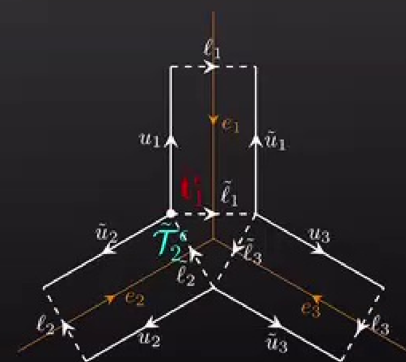
↪ They are $\mathcal{U}_q(\mathfrak{su}(2))$-invariant thus live in the intertwiner space

$$\mathbf{E}_{12}^{\epsilon_1,\epsilon_2}\, i_{j_1j_2j_3} = \delta_{l_1,j_1+\frac{\epsilon_1}{2}}\delta_{l_2,j_2+\frac{\epsilon_2}{2}}\sqrt{[d_{j_1}][d_{j_2}][d_{l_1}][d_{l_2}]}(-1)^{l_1+l_2+j_3}\begin{Bmatrix} l_1 & j_1 & \frac{1}{2} \\ j_2 & l_2 & j_3 \end{Bmatrix}_q i_{l_1l_2j_3}$$

- A standard way to define the scalar operators is in a braided way with the $\mathcal{R}$-matrix, e.g.

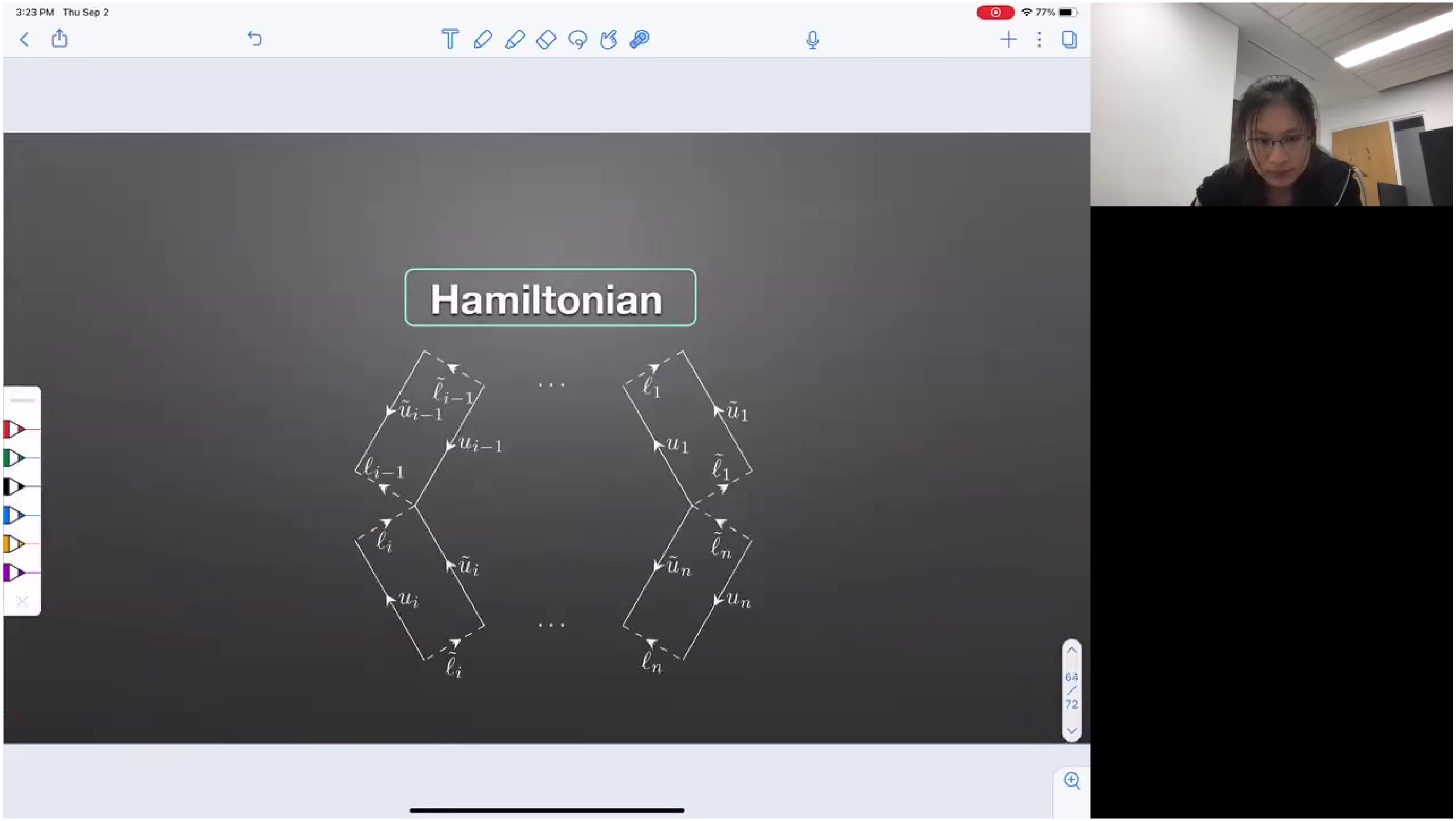
$$\sum_A {}_qC^{\frac{1}{2}\,\frac{1}{2}\,0}_{A\,-A\,0}\;{}^{(2)}\tilde{\mathcal{T}}^{\epsilon}_{-A}\;{}^{(1)}\tilde{\mathcal{T}}^{\epsilon}_{A} \overset{!}{\propto} \mathbf{E}_{12}^{\epsilon_1,\epsilon_2}$$

$${}^{(2)}\tilde{\mathcal{T}}^{\epsilon}_{A} = \mathcal{R}^{-1}(\mathbb{I}\otimes\tilde{\mathcal{T}}^{\epsilon}_{A})\mathcal{R}, \quad {}^{(1)}\tilde{\mathcal{T}}^{\epsilon}_{A} = \tilde{\mathcal{T}}^{\epsilon}_{A}\otimes\mathbb{I}$$



Braiding is parallel transport in the ribbon picture!

[V. Rittenberg, M. Scheunert, 1992]

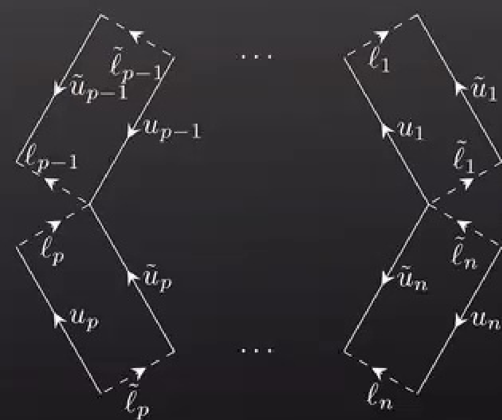

3:23 PM Thu Sep 2
77%
Hamiltonian
$\tilde{\ell}_{i-1}$
$\tilde{u}_{i-1}$
u_{i-1}
ℓ_{i-1}
ℓ_i
$\tilde{u}_i$
u_i
$\tilde{\ell}_i$
ℓ_1
$\tilde{u}_1$
u_1
$\tilde{\ell}_1$
$\tilde{\ell}_n$
$\tilde{u}_n$
u_n
ℓ_n
64 / 72

3:24 PM Thu Sep 2 77%

Spinor representation of the Hamiltonian constraint

- The Hamiltonian constraint

$$\mathscr{F}_f = \overrightarrow{\prod}_e \mathscr{U}_e^{o_e}, \quad \mathscr{U}_e^{o_e} = \begin{cases} u_e & \text{if } o_e = 1 \\ \tilde{u}_e^{-1} & \text{if } o_e = -1 \end{cases}$$



65 / 72

3:24 PM Thu Sep 2

Spinor representation of the Hamiltonian constraint

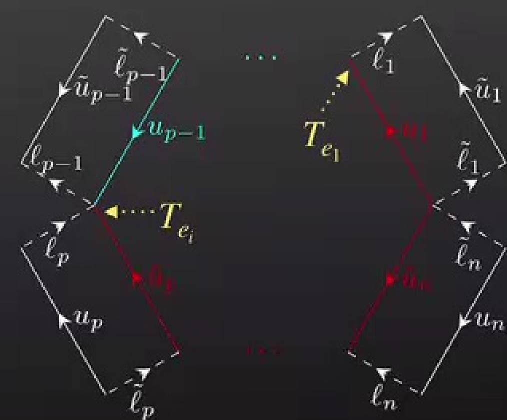
- The Hamiltonian constraint

$$\mathcal{F}_f = \overrightarrow{\prod}_e \mathcal{U}_e^{o_e}, \quad \mathcal{U}_e^{o_e} = \begin{cases} u_e & \text{if } o_e = 1 \\ \tilde{u}_e^{-1} & \text{if } o_e = -1 \end{cases}$$

can be written with purely spinors

$$E_{e_1 \to e_p} = \sum_{A,B} T_{e_p,-A} \left(\mathcal{U}_{e_{p-1}}^{o_{p-1}} \cdots \mathcal{U}_{e_2}^{o_2} \right)_{AB} T_{e_1,B}$$

$$= \sum_{A,B} T_{e_p,-A} \left(\mathcal{U}_{e_1}^{o_1} \mathcal{U}_{e_n}^{o_n} \cdots \mathcal{U}_{e_p}^{o_p} \right)^{-1}_{AB} T_{e_1,B}$$



66 / 72

3:24 PM Thu Sep 2

Spinor representation of the Hamiltonian constraint

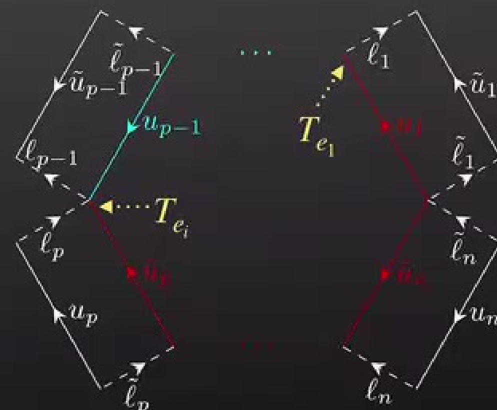
- The Hamiltonian constraint

$$\mathscr{F}_f = \overrightarrow{\prod}_e \mathscr{U}_e^{o_e}, \quad \mathscr{U}_e^{o_e} = \begin{cases} u_e & \text{if } o_e = 1 \\ \tilde{u}_e^{-1} & \text{if } o_e = -1 \end{cases}$$

can be written with purely spinors

$$E_{e_1 \to e_p} = \sum_{A,B} T_{e_p,-A} \left(\mathscr{U}_{e_{p-1}}^{o_{p-1}} \cdots \mathscr{U}_{e_2}^{o_2} \right)_{AB} T_{e_1,B}$$

$$= \sum_{A,B} T_{e_p,-A} \left(\mathscr{U}_{e_1}^{o_1} \mathscr{U}_{e_n}^{o_n} \cdots \mathscr{U}_{e_p}^{o_p} \right)^{-1}_{AB} T_{e_1,B}$$



- The holonomies can be given by the spinors, then we have a flatness constraint written with purely scalar product of spinors

$$h_{e_1,e_p}^{\epsilon_1,\epsilon_p} = \sum_{\epsilon_2,\cdots,\epsilon_{p-1}=\pm} \prod_{i=2}^{p} \frac{o_i \epsilon_i}{N_{e_i}} E_{e_i e_{i-1}}^{\epsilon_i,\epsilon_{i-1}} - (-1)^{d-p} o_1 o_p \epsilon_1 \epsilon_p \frac{N_{e_1}}{N_{e_p}} \sum_{\epsilon_{p+1},\cdots,\epsilon_d=\pm} \prod_{i=p+1}^{d+1} \frac{o_i \epsilon_i}{N_{e_i}} E_{e_i e_{i-1}}^{-\epsilon_i,-\epsilon_{i-1}}$$

67 / 72

3:25 PM Thu Sep 2

Spinor representation of the Hamiltonian constraint

- Quantizing the scalar products to operators, we get a quantum Hamiltonian in terms of quantum spinors

↪ Solutions to the quantum Hamiltonian are topological states!

$$\psi_i(j_1,\cdots,j_6) \propto \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}_q \psi_f(j_3,j_4,j_5)$$

$$\psi_f(j_0,\cdots) \propto \begin{Bmatrix} j_1 & j_2 & j_0 \\ j_3 & j_4 & j_5 \end{Bmatrix}_q \psi_i(j_5,\cdots)$$

68 / 72

3:26 PM Thu Sep 2

Spinor representation of the Hamiltonian constraint

- Quantizing the scalar products to operators, we get a quantum Hamiltonian in terms of quantum spinors

↪ Solutions to the quantum Hamiltonian are topological states!

$$\psi_i(j_1,\cdots,j_6) \propto \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}_q \psi_f(j_3,j_4,j_5)$$

$$\psi_f(j_0,\cdots) \propto \begin{Bmatrix} j_1 & j_2 & j_0 \\ j_3 & j_4 & j_5 \end{Bmatrix}_q \psi_i(j_5,\cdots)$$

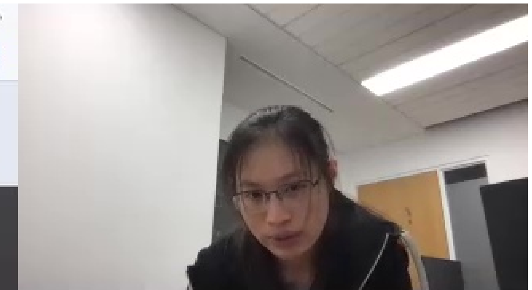
↪ Triangle face $\longrightarrow$ recursion relation of the physical states by spin-$1/2$ $\longrightarrow$ spinfoam dynamics

$$\mathbf{A}_q^-(j_1)\,\psi(j_1-1/2,j_2,\cdots)+\mathbf{A}_q^0(j_1)\,\psi(j_1,j_2,\cdots)+\mathbf{A}_q^+(j_1)\,\psi(j_1+1/2,j_2,\cdots)=0$$

It is a q-deformation of the well-known result in the flat case when $q=1$!

[V. Bonzom, E. Livine, 2011]

69 / 72



Summary

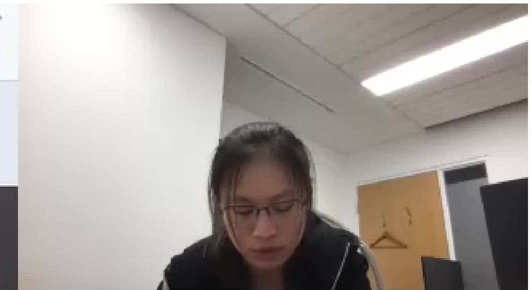
- Quantum group structure emerges from the continuous theory by performing a canonical transformation
- The deformed phase space shows up a ribbon structure $\longrightarrow$ holonomy and flux are of the same footing
  $\longrightarrow$ constraints expressed in a symmetric way
- Hyperbolic geometry at the discrete level
- Quantization inherits the duality between the holonomy and flux and leads to both q and q^{-1}-deformation
- Spinors are deformed to capture the curved geometry, and the quantization also leads to q and q^{-1}-deformation
- The deformed hamiltonian has topological solution and gives the recursion relations of the q-$6j$ symbols
- Such a q-deformed LQG framework connects to the spinfoam and makes it easier to connect to the Chern-Simons quantization

Summary

- Quantum group structure emerges from the continuous theory by performing a canonical transformation
- The deformed phase space shows up a ribbon structure $\longrightarrow$ holonomy and flux are of the same footing
  $\longrightarrow$ constraints expressed in a symmetric way
- Hyperbolic geometry at the discrete level
- Quantization inherits the duality between the holonomy and flux and leads to both q and q^{-1}-deformation
- Spinors are deformed to capture the curved geometry, and the quantization also leads to q and q^{-1}-deformation
- The deformed hamiltonian has topological solution and gives the recursion relations of the q-$6j$ symbols
- Such a q-deformed LQG framework connects to the spinfoam and makes it easier to connect to the Chern-Simons quantization

To explore ...

- Euclidean with $\Lambda > 0$; Lorentzian signature; 4D
- q-deformed coherent states from the deformed spinors
- Spinfoam model from the first principle



Summary

- Quantum group structure emerges from the continuous theory by performing a canonical transformation
- The deformed phase space shows up a ribbon structure $\longrightarrow$ holonomy and flux are of the same footing
  $\longrightarrow$ constraints expressed in a symmetric way
- Hyperbolic geometry at the discrete level
- Quantization inherits the duality between the holonomy and flux and leads to both q and q^{-1}-deformation
- Spinors are deformed to capture the curved geometry, and the quantization also leads to q and q^{-1}-deformation
- The deformed hamiltonian has topological solution and gives the recursion relations of the q-$6j$ symbols
- Such a q-deformed LQG framework connects to the spinfoam and makes it easier to connect to the Chern-Simons quantization

To explore ...

- Euclidean with $\Lambda > 0$; Lorentzian signature; 4D
- q-deformed coherent states from the deformed spinors
- Spinfoam model from the first principle

Thank you for your attention!