Abstract: "The Causaloid framework [1] is useful to study Theories with Indefinite Causality; since Quantum Gravity is expected to marry the radical aspects of General Relativity (dynamic causality) and Quantum Theory (probabilistic-ness). To operationally study physical theories one finds the minimum set of quantities required to perform any calculation through physical compression. In this framework, there are three levels of compression: 1) Tomographic Compression, 2) Compositional Compression and 3) Meta Compression.

We present a diagrammatic representation of the Causaloid framework to facilitate exposition and study Meta compression. We show that there is a hierarchy of theories with respect to Meta compression and characterise its general form. Next, we populate the hierarchy. The theory of circuits forms the simplest case, which we express diagrammatically through Duotensors, following which we construct Triotensors using hyper3wires (hyperedges connecting three operations) for the next rung in the hierarchy. Finally, we discuss the implications for the field of Indefinite Causality.

Hierarchy of Theories with Indefinite Causal Structures:
A Second Look at the Causaloid Framework

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(Ongoing work with Lucien Hardy)

Quantizing Time (2021)
Operational Methodology

Data:
$\mathbf{x}$: location
$\mathbf{a}$: actions
$\mathbf{b}$: observations

Sets:
$\mathbf{R}_i$: Elementary Region
$\mathbf{F}_i$: Procedure
$\mathbf{Y}_i$: Outcome

$\mathbf{R}_i \supseteq \mathbf{F}_i \supseteq \mathbf{Y}_i$

“Thinking Inside the box”
Pre-Compression \( (0^{th} \text{ level}) \)

We want: \( P( Y_{R_i}^{d_{a_i}} U Y_{R-R_i}^{R_{a_i}} \mid F_{R_i}^{a_{R_i}} U F_{R-R_i}) = P_{d_{R_i}} = R_{a_{R_i}} P \)

\( (Y_{R-R_i}, F_{R-R_i}) \) generalised prep for \( R_i \)

\( (Y_{R_i}, F_{R_i}) \) measurement for \( R_i \)

\( R = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \), \( P = \begin{pmatrix} P_{d_{R_i}} \end{pmatrix} \)

\( d_{R_i} \in T_{R_i} \)
Pre-Compression ($0^{th}$ level)

We want: $P(Y_{R}^{a_{i}} U Y_{R-R} \mid F_{R}^{a_{i}} U F_{R-R}) = P_{\alpha_{R_{i}}} = R_{\alpha_{R_{i}}} P$

$(Y_{R-R_{i}}, F_{R-R_{i}})$
- generalised prep for $R_{i}$

$(Y_{R_{i}}, F_{R_{i}})$
- measurement for $R_{i}$

$R = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, P = \begin{pmatrix} P_{R_{i}} \\ \vdots \\ P_{R_{i}} \end{pmatrix}$

$\alpha_{R_{i}} \in T_{R_{i}}$
Consider three regions → $R = R_1 \cup R_2 \cup R_3$

we will focus on this diagram to:

1) Explain the Causaloid framework
2) Introduce a Hierarchy for physical theories
Tomographic Compression

We want: \[ P(Y_{R_i}^{d_{\alpha_i}} U Y_{R-R_i}^{d_{\alpha_i}} \mid F_{R_i}^{d_{\alpha_i}} U F_{R-R_i}) = P_{d_{R_i}} = \langle \alpha_{R_i} \cdot P \rangle \]

\[ r_{\alpha_{R_1}} \rightarrow \alpha_{R_1} \in \Gamma_{R_1} \rightarrow \Lambda l_{R_1} \rightarrow l_{R_1} \in \Omega_{R_1} \rightarrow \text{fiducial set} \]

\[ \equiv \sqrt{d_{\alpha_{R_i}}} = \sum_{l_{R_i} \in \Omega_{R_1}} \lambda_{l_{R_i}}^{d_{\alpha_{R_i}}} \chi_{l_{R_i}} \]
Consider three regions \( R = R_1 \cup R_2 \cup R_3 \)

We will focus on this diagram to:

1) Explain the Causaloid framework
2) Introduce a Hierarchy for physical theories
Compositional Compression (2nd level)

We want:

\[ P(Y_{R_1}^{d_{R_1}} U Y_{R_2}^{d_{R_2}} U Y_{R_1-R_2}^{d_{R_1-R_2}} \mid \mathcal{F}_{R_1}^{d_{R_1}} U \mathcal{F}_{R_2}^{d_{R_2}} U \mathcal{F}_{R_1-R_2}) = P_{d_{R_1}} \cdot d_{R_2} = \mathcal{V}_{d_{R_1}} \cdot d_{R_2} \cdot \mathcal{P} \]

\[
\Lambda_{l_{R_1}}^{k_{R_1},l_{R_2}} \quad l_{R_1} \in \Omega_{R_1} \\
\Lambda_{l_{R_2}}^{k_{R_2},l_{R_2}} \quad l_{R_2} \in \Omega_{R_2} \\
\Omega_{R_1,R_2} \subseteq \Omega_{R_1} \times \Omega_{R_2}
\]

\[
\mathcal{C}^{l_{R_1},l_{R_2}} = \sum_{k_{R_1},k_{R_2}} \Lambda_{l_{R_1},l_{R_2}}^{k_{R_1},k_{R_2}} \mathcal{V}_{k_{R_1}} \cdot k_{R_2}
\]
Meta Compression (3rd level)...

We want:
\[
\{ \Lambda_{\alpha_{R_1}^2}, \Lambda_{\alpha_{R_2}^2}, \Lambda_{\alpha_{R_3}^2} \}, \Lambda_{\alpha_{R_1}^2, \alpha_{R_2}^2, \alpha_{R_3}^2}, ... \}
\rightarrow \Lambda \rightarrow \bigotimes_{\alpha_{R_1}, \alpha_{R_2}, \alpha_{R_3}} \rho_{\alpha_{R_1}} \otimes \rho_{\alpha_{R_2}} \otimes \rho_{\alpha_{R_3}}...
\]
Meta Compression (3rd level)...

We want:
\{ \{ \Lambda_{e_1} \}, \{ \Lambda_{e_2} \}, \{ \Lambda_{e_3} \}, ... \} \rightarrow \Lambda \rightarrow \tau_1 \otimes \tau_2 \otimes \ldots \rightarrow \{ \{ \Lambda_{e_1} \}, \{ \Lambda_{e_2} \}, \{ \Lambda_{e_3} \} \} \rightarrow \Lambda

\[ P = \sum \sum \sum \sum \sum \sum \sum \Lambda_{k_{i_1} k_{i_2}} \Lambda_{k_{i_3} k_{i_4}} \Lambda_{k_{i_5} k_{i_6}} \Lambda_{k_{i_7} k_{i_8}} = \sum \sum \sum \sum \Lambda_{k_{i_1} k_{i_2}} \Lambda_{k_{i_3} k_{i_4}} \Lambda_{k_{i_5} k_{i_6}} \Lambda_{k_{i_7} k_{i_8}} \Lambda_{k_{i_9} k_{i_{10}}}

\text{when} \quad R_1 \rightarrow R_2 \rightarrow R_3 \quad \sum \Lambda_{k_{i_1} k_{i_2}} = \sum \sum \Lambda_{k_{i_1} k_{i_2}} \Lambda_{k_{i_3} k_{i_4}}

\text{such that} \quad \Omega_{R_1, R_2} \equiv (\Omega_{R_1 \times R_2} \times \Omega_{R_3}) \Lambda (\Omega_{R_1 \times R_2} \times \Omega_{R_3})
\[ \sum_{l_2, l_3} \lambda_{j_{a_2}, j_{a_3}} = \sum_{k_{a_2}, k_{a_3}} \lambda_{k_{a_2}, k_{a_3}} \sum_{l_2, l_3} \lambda_{l_{a_2}, l_{a_3}} \]

such that

\[ \Omega_{R_1, R_2, R_3} = (\Omega_{R_1, R_2} \times \Omega_{R_3}) \wedge (\Omega_{R_1} \times \Omega_{R_2, R_3}) \]
Hierarchies in $d<n$

\[ \Omega_{R_1, R_2, \ldots, R_n} = \bigcap_{i \neq j \neq \ldots \neq d^{\text{index}}} \mathcal{O}(F_n(\Omega_{R_i, R_j, \ldots, R_d^{\text{index}}})) \]

- $d=1$: No causal structure
- $d=2$: Duotensors (Including Q Theory, C Prob Theory)
- $d=3$: Triotensors (No Quantum analogue)
- $d=4$: Quadrotensors (Quantum generalization linked to Quaternions)
Hierarchies in $d<n$

\[
\Omega_{R_1, R_2, \ldots, R_n} = \bigcap_{i \neq j \neq \ldots \neq d^{th} \text{ index}} O(F_n(\Omega_{R_i, R_j, \ldots, R_{d^{th} \text{ index}}}))
\]

- **$d=1$**  
  No causal structure

- **$d=2$**  
  Duotensors (Including Q Theory, C Prob Theory)

- **$d=3$**  
  Triotensors (no Quantum analogue)

- **$d=4$**  
  Quadrotensors (Quantum generalisation linked to Quaternions)
Thank you!

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