Title: Relational observables and quantum diffeomorphisms on the worldline

Speakers: Leonardo Chataignier

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Abstract: "Candidate theories of quantum gravity must answer the questions: how can the dynamics of quantum states of matter and geometry be defined in a diffeomorphism-invariant way? How are the quantum states related to probabilities in the absence of a preferred time?Â To address these issues, we discuss the construction and interpretation of relational observables in quantum theories with worldline diffeomorphism invariance, which serve as toy models of quantum gravity. In this context, we present a method of construction of quantum relational observables which is analogous to the construction of gauge-invariant extensions of noninvariant quantities in usual gauge (Yang-Mills) theories. Furthermore, we discuss how the notion of a physical propagator may be used to define a unitary evolution in the quantum theory, which is to be understood in terms of a generalized clock, as is the classical theory.Â We also discuss under which circumstances this formalism can be related to the use of conditional probabilities in a generalization of the Page-Wootters approach. Finally, we also examine how our formalism can be adapted to calculations of quantum-gravitational effects in the early Universe.
Refs.: L. Chataignier, Phys. Rev. DÂ 101,Â 086001 (2020); 103,Â 026013 (2021); 103,Â 066005 (2021)"

Relational Observables and Quantum Diffeomorphisms on the Worldline

Leonardo Chataignier
Institut für Theoretische Physik, Universität zu Köln

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Outline

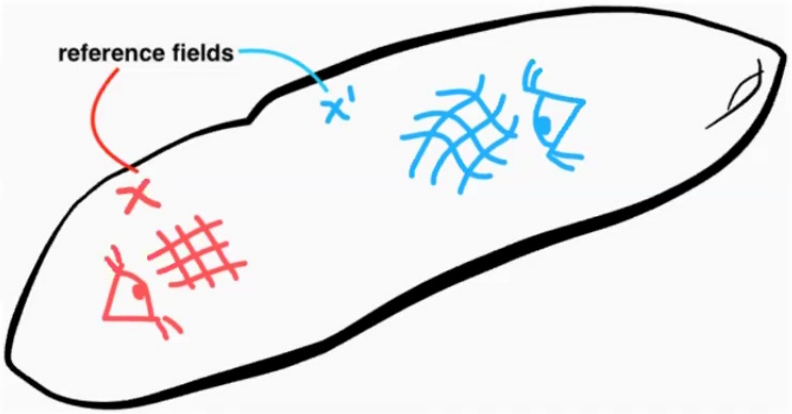
- Classical diffeomorphism invariance on the worldline
  - How to understand the dynamics in a diffeomorphism-invariant way?
- Quantum diffeomorphism invariance on the worldline
  - How to CONSTRUCT quantum relational observables and a notion of relational quantum dynamics?
  - Possible postulates
  - Connection with the Page-Wootters approach
- Effects in the early Universe
- Conclusions

Problems that approaches to quantum gravity face

- Diffeomorphisms are a symmetry of the field equations in GR.
- Approaches to QG must face (or have provided tentative answers to) the questions:[1]
  - What is the relevant space of physical states and which operators act on it?
  - How are the quantum states related to probabilities in the absence of a preferred time? What is their dynamics? ("Problem of Time")
  - What is the origin/nature of probabilities and the Born rule? ("Measurement Problem")
- Root of the problems: how to deal with diffeomorphism invariance at the quantum level?
  - Goal: address this issue in mechanical toy models ('worldline') by direct analogy to techniques and concepts used in classical canonical gauge systems

[1] See, e.g., Claus Kiefer, *Quantum Gravity*, 3rd ed. (Oxford 2012).

Relational observables and generalized reference frames



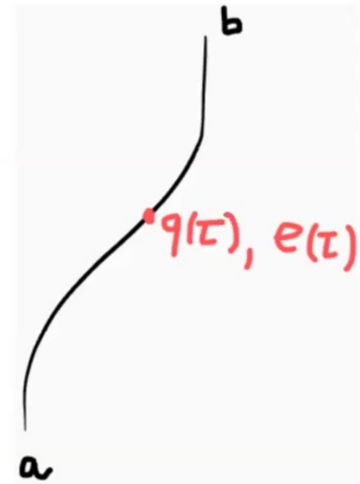
- Observers record the dynamics of fields in local regions using reference fields χ ('generalized clocks and rods') that define 'generalized reference frames'

- The outcomes of their experiments are the values of Φ relative to χ $\Rightarrow$ 'relational observables'

- The relational observables (denoted by $\mathcal{O}[\Phi|\chi]$)
  - completely encode the dynamics in the region of the experiment
  - are **conditional** quantities and yield predictions (the value of Φ) based on a certain condition (the observed value of χ).
  - make no reference to the abstract point $p \in \mathcal{M}$ and can thus be seen as constant spacetime scalars for each fixed value of $\chi \Rightarrow$ diffeomorphism invariants! *(invariant extensions of tensor fields)* *(e.g., Henneaux & Teitelboim 1992)*

Classical diffeomorphism invariance on the worldline

- Consider a $(0+1)$-dimensional spacetime: worldline.



- If the functional form of the action remains the same under diffeos,

$$S = \int_a^b \mathrm{d}\tau \; \mathcal{L}(q, \dot{q}; e) = \int_a^b \mathrm{d}\tau \; \mathcal{L}(\phi^* q, \phi^* \dot{q}; \phi^* e) \; ,$$

then

$$\mathcal{L}(q, \dot{q}; e) = \frac{\partial \mathcal{L}}{\partial \dot{q}^i}\dot{q}^i + \frac{\partial \mathcal{L}}{\partial e} e = p_i \dot{q}^i - eC(q,p) \; ,$$

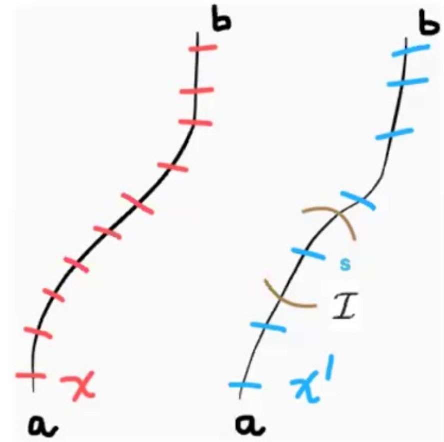
where $C(q,p)$ is the (first-class) Hamiltonian constraint.

- Proper time: $\eta = \int_{\mathcal{I}} \mathrm{d}\tau e(\tau) \Rightarrow \dot{\eta} = e$; canonical proper-time: $\{\eta(q,p), C\} = 1$.

Worldline relational observables

- Choosing a reference field = finding the diffeomorphism ϕ for which $\phi^*\chi = \tau$ ('χ is the clock').
- Relational observables can be seen as constant worldline scalars

$$\mathcal{O}[f|\chi = s] : \mathcal{M} \to \mathbb{R}$$
$$p \mapsto \mathcal{O}[f|\chi = s](p) = \phi^* f \ (\forall p) .$$



- It is convenient to write

$$\mathcal{O} \equiv \mathcal{O}[f|\chi = s] = \phi^* f = \Delta_\chi \int_{\mathcal{I}} \mathrm{d}\tau \ \delta(\chi(\tau) - s) f(\tau) ,$$

DeWitt 1962
Marolf 1995
Hartle & Marolf 1997
Giddings, Marolf & Hartle 2006

where $\Delta_\chi^{-1} = \int_{\mathcal{I}} \mathrm{d}\tau \ \delta(\chi(\tau) - s)$ is the inverse 'Faddeev-Popov (FP) determinant'. $\Delta_\chi \sim \phi^*\{\chi, C\}$

- One can show that $\delta_{\epsilon(\tau)}\mathcal{O}|_{C=0} = \epsilon(\tau) e\{\mathcal{O}, C\}|_{C=0} = 0 \ \Rightarrow$ diffeomorphism invariant!

The physical Hilbert space

- Adopt **Dirac quantization**: auxiliary Hilbert space $\mathcal{H}$ equipped with inner product (IP) $\langle\cdot|\cdot\rangle$ w.r.t. which the canonical quantization of C is self-adjoint:

$$\hat{C}\,|E,\mathrm{k}\rangle = E\,|E,\mathrm{k}\rangle \ , \ \langle E',\mathrm{k}'|E,\mathrm{k}\rangle = \delta(E',E)\delta(\mathrm{k}',\mathrm{k}) \ .$$

- Physical states $|\Psi\rangle$ are superpositions of $|E=0,\mathrm{k}\rangle$ so as to obey the 'quantum constraint' $\hat{C}\,|\Psi\rangle = 0$ ['Wheeler-DeWitt (WDW) equation'].

- If $E=0$ is in the continuous part of the spectrum of $\hat{C}$, then physical states are not normalizable w.r.t $\langle\cdot|\cdot\rangle$ [$\delta(0,0)=\infty$]. Define the physical IP as

$$(E=0,\mathrm{k}'|E=0,\mathrm{k}) := \delta(\mathrm{k}',\mathrm{k}) \ \rightarrow \ (\text{'Rieffel induced IP'}\bullet)$$

Nambu 1950
Feynman 1950
Rieffel 1974
Henneaux & Teitelboim 1982
Marolf 2000

- The space of solutions of $\hat{C}\,|\Psi\rangle = 0$ that are normalizable w.r.t $(\cdot|\cdot)$ is the physical Hilbert space $\mathcal{H}_{\mathrm{phys}}$.

Quantum relational observables

- $|t,\mathrm{k}\rangle := \sum_E \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}Et}\,|E,\mathrm{k}\rangle/\sqrt{2\pi\hbar}$ are "conjugate" to $|E,\mathrm{k}\rangle$, but not necessarily a complete orthonormal system w.r.t. $\langle\cdot|\cdot\rangle$. Busch, Grabowski & Lahti 1995; Busch, Lahti, Pellonpää & Ylinen 2016; Höhn, Smith& Lock 2019 (See also Philipp's talk on Wednesday at PIRSA)
- Define the invariant operator

$$\hat{\mathcal{O}}_{\mathrm{inv}}[f|\chi=s] := \pi\hbar\sum_E \hat{P}_E[\hat{f},\hat{P}_{t=s}]_+\hat{P}_E \overset{\mathrm{spec}(\hat{C})=\mathbb{R}}{\to} \frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}\tau\;\hat{f}(\tau)\hat{P}_{t=s-\tau} + \mathrm{h.c.}\,,$$

where $\hat{f}(\tau)$ is a Heisenberg-picture operator and $\hat{P}_{t=s} = \sum_{\mathrm{k}}|t,\mathrm{k}\rangle\langle t,\mathrm{k}|$ is analogous to $\delta(\chi(\tau)-s)$. By construction, $\hat{P}_{E=0}\hat{\mathcal{O}}_{\mathrm{inv}}[1|\chi=s] = \hat{P}_{E=0}$ (on-shell FP resolution of the identity).

- Observables have the correct (*quantum*) dynamics:

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}s}\hat{\mathcal{O}}[f|\chi=s] = \hat{\mathcal{O}}\left[\mathrm{i}\hbar\frac{\partial f}{\partial s}+[f,C]\middle|\,\chi=s\right]\,.$$

- Other gauges: $\hat{\mathcal{O}}[f|\chi=s] := \pi\hbar\sum_\sigma\hat{\Omega}_t^\sigma[\hat{f},\hat{P}_{t=s}]_+\hat{\Omega}_t^\sigma$, where $\hat{\Omega}_t^\sigma$ is a FP operator, $\left(\hat{\Omega}_t^\sigma\right)^{-2} := 2\pi\hbar\hat{P}_{E=0}^\sigma\hat{P}_{t=s}\hat{P}_{E=0}^\sigma \sim \Delta_\chi^{-1} = \int\mathrm{d}\tau\delta(\chi(\tau)-s)$, and σ is a discrete degeneracy (e.g., positive/negative frequencies).

The relativistic particle as an archetypical example

- Mass-shell constraint $C = \frac{1}{2}(-p_0^2 + \vec{p}^{\,2} + \frac{m^2}{2})$.

- Minkowski time: $\mathcal{O}[\vec{q}\,|q^0 = s] = \vec{q} + \vec{p}\,\frac{q^0 - s}{p_0}$ leads to

$$\left(\Psi_{(1)}\,\middle|\hat{\mathcal{O}}[\vec{q}\,|q^0 = s]\middle|\,\Psi_{(2)}\right) = \int \mathrm{d}p_0 \mathrm{d}\vec{p}\;\Psi^*_{(1)}(p_0,\vec{p})\delta\left(-\frac{p_0^2}{2} + \frac{\vec{p}^{\,2}}{2} + \frac{m^2}{2}\right)$$
$$\times\left[\mathrm{i}\hbar\frac{\partial}{\partial\vec{p}} + \mathrm{i}\hbar\frac{\vec{p}}{p_0}\frac{\partial}{\partial p_0} - \frac{\vec{p}}{p_0}s - \mathrm{i}\hbar\frac{\vec{p}}{2p_0^2}\right]\Psi_{(2)}(p_0,\vec{p})$$

for two test states $\left(\psi(p_0 = 0, \vec{p}) = 0\right)$.

Tentative postulates. Quantum reference frames

1. The quantum state of a diffeomorphism-invariant quantum system corresponds to a ray in the physical Hilbert space $\mathcal{H}_{\text{phys}}$. Observables are self-adjoint on-shell operators.
2. Observers who employ a certain generalized clock record the dynamics of worldline tensor fields according to the relational Heisenberg-picture operators. This defines the quantum generalized reference frame associated to the observer's choice of clock.
3. If the system is in the state $|\Psi\rangle$, a measurement of $\hat{f}$ relative to the generalized clock results in an eigenvalue $f(s, \mathrm{n})$ of $\hat{\mathcal{O}}[f|\chi = s]$ with probability

$$p_\Psi = \sum_\sigma \frac{|(\sigma, \mathrm{n}; s|\Psi)|^2}{(\Psi|\Psi)} ,$$

   where $|\sigma, \mathrm{n}; s\rangle$ are the eigenstates of $\hat{\mathcal{O}}[f|\chi = s]$.
4. After the measurement, the state of the system is updated to $|\sigma, \mathrm{n}; s\rangle$ in the generalized reference frame of the observer.
5. A change of *quantum* reference frame is a change of basis in $\mathcal{H}_{\text{phys}}$: $(\sigma_1, \mathrm{n}_1; \chi_1|\Psi) = \sum_{\sigma_2, \mathrm{n}_2} (\sigma_1, \mathrm{n}_1; \chi_1|\sigma_2, \mathrm{n}_2; \chi_2)(\sigma_2, \mathrm{n}_2; \chi_2|\Psi)$.

Page-Wootters case

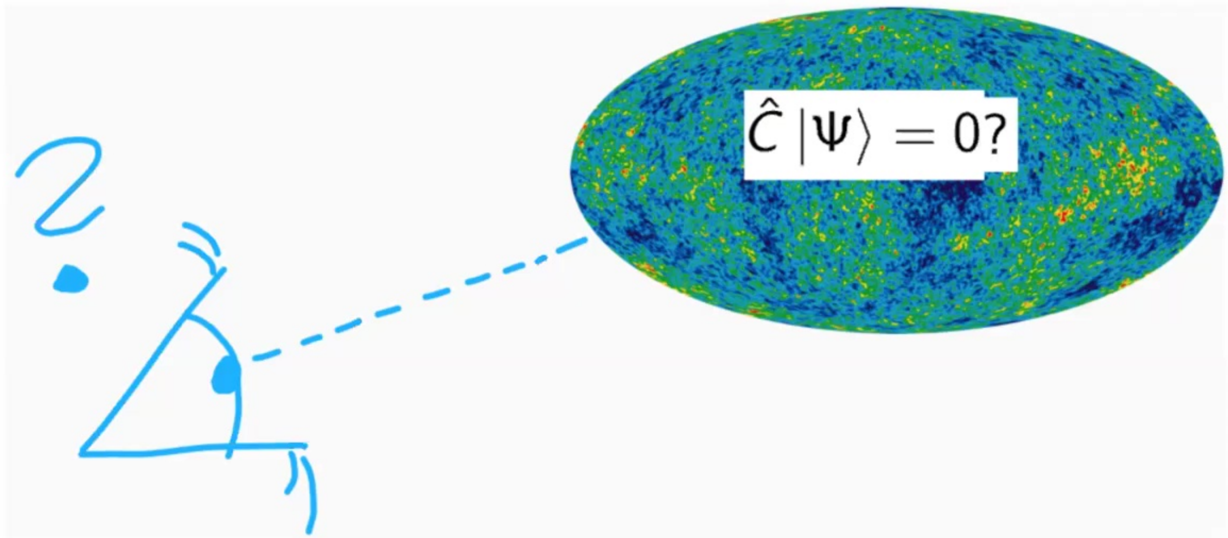
- The Page-Wootters (PW) formalism[2] uses conditional probabilities. How does our formalism relate to it?
- Define conditional expectation values in the usual way: $\mathrm{E}_\Psi[f|\chi=s] = \frac{\langle\Psi|\hat{f}\hat{P}_{t=s}|\Psi\rangle}{\langle\Psi|\hat{P}_{t=s}|\Psi\rangle}$. Then it is possible to show:

$$\langle\hat{\mathcal{O}}[f|\chi=s]\rangle_\Psi = \frac{\left(\Psi\left|\hat{\mathcal{O}}[f|\chi=s]\right|\Psi\right)}{(\Psi|\Psi)} = \sum_\sigma p_\Psi(\sigma)\mathrm{E}_{\Psi_\sigma}[f|\chi=s]\ ,$$

  where $p_\Psi(\sigma)$ is the probability of the system to be in the σ sector $\left(= \langle\hat{P}^\sigma_{E=0}\rangle_\Psi\right)$, and $|\Psi_\sigma\rangle := \hat{\Omega}^\sigma_t \bullet |\Psi\rangle$.
- Particular case $\hat{C} = \hat{C}_{\text{clock}} + \hat{C}_{\text{system}} \to$ if solve $E = E_{\text{clock}} + E_{\text{system}}$ for E_{clock} $\to$'clock proper time' gauge.
- Rel. obs.: $\left(\text{spec}(\hat{C}) = \mathbb{R}\right)$ $\hat{\mathcal{O}}[q|\chi=s] = \int_{-\infty}^{\infty}\mathrm{d}\tau\ \hat{q}(\tau)\hat{P}_{t=s-\tau}$ $\to$ usual Heisenberg equations ('Heisenberg-picture PW'); $\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}s}\hat{\mathcal{O}}[q|\chi=s] = [\hat{\mathcal{O}}[q|\chi=s], \hat{C}_{\text{System}}]$.

[2] D. N. Page and W. K. Wootters, Phys. Rev. D **27** 2885 (1983).

What about our Universe?



What if we cannot solve $\hat{C} |\Psi\rangle = 0$ exactly? How to construct $\mathcal{H}_{\text{phys}}$?
What is the (unitary) dynamics?

Weak-coupling expansion

- de Sitter background + perturbations (MS variables). Constraint:
$$\left[\frac{e^{-3\alpha}}{a_0}\left(\frac{\kappa}{2}\frac{\partial^2}{\partial\alpha^2} + a_0^6 e^{6\alpha}\frac{H_0^2}{2\kappa}\right) + \frac{e^{-\alpha}}{a_0}\hat{H}\right]\Psi(\alpha, v) = 0.$$

- Weak-coupling expansion w.r.t κ: $\Psi = e^{\frac{i}{\kappa}\mathcal{W}(\alpha,v)} = e^{\frac{i}{\kappa}\mathcal{W}_0(\alpha)}\psi(\alpha; v)$, where $\mathcal{W} = \sum_{n=0}^{\infty}\mathcal{W}_n\kappa^n =: \mathcal{W}_0 - i\kappa\log\psi$. Solve constraint order by order!

- Up to order κ: $\hat{\mu} = |H_0^2\eta^3|^{-1}\left(1 + \kappa H_0^2\eta^4\hat{H}\right)$, $\tilde{\psi} := \hat{\mu}^{\frac{1}{2}}\psi$,

$$\boxed{i\frac{\partial\tilde{\psi}}{\partial\eta} = \left(\hat{H} - \kappa\frac{H_0^2\eta^4}{2}\hat{H}^2\right)\tilde{\psi} + \mathcal{O}(\kappa^2)}$$

$\Rightarrow$ Unitary w.r.t.
$$(\Psi_{(1)}|\Psi_{(2)}) := \int dv\; \tilde{\psi}^*_{(1)}\tilde{\psi}_{(2)} \overset{\eta(a)\text{ gauge}}{=} \int d\alpha dv \left|\frac{\partial\eta}{\partial\alpha}\right|\left(\hat{\mu}^{\frac{1}{2}}\Psi_{(1)}\right)^*\delta(\eta - s)\hat{\mu}^{\frac{1}{2}}\Psi_{(2)}$$

$\sim \hat{P}_{E=0} = 2\pi\hbar\sum_\sigma \hat{\Omega}_t^\sigma\hat{P}_{t=s}\hat{\Omega}_t^\sigma \Rightarrow \hat{\mu} \equiv |\widehat{\det FP}|$ for single σ (expanding) sector

Towards phenomenology:
$$\mathcal{P}_{S,T}(k) \simeq \mathcal{P}_{S,T;0}(k)\left\{1 + \kappa H_0^2\left(\frac{k_*}{k}\right)^3[2.85 - 2\log(-2k\eta)]\right\}$$

Conclusions

- We have presented a possible formalism for the systematic construction and interpretation of relational observables, both in the classical and quantum theories, in a model-independent way.
- In this formalism:
  - Operator version of the FP resolution of the identity as a defining feature $\to$ choice of generalized clock/ref. frame (regularization of the IP)
  - Quantum rel. obs. have the correct quantum evolution; self-adjointness $\to$ unitarity of the physical propagator
  - Changes of ref. frames = changes of basis in $\mathcal{H}_{\text{phys}}$ "quantum diffeos"
  - Generalization of Page-Wootters: averages of rel. obs. $\leftrightarrow$ conditional expectation values (for more general C)
- It addresses conceptual issues at the interface of quantum theory and gravitation (problems of time and observables) and, furthermore, it provides a useful set of tools for various toy models (quantum cosmologies).
- The future: further explore connections with other approaches, unitary corrections to slow-roll models, generalization to field theory, and more