Title: Kappa-Minkowski: physics with noncommutative time

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Abstract: The kappa-Minkowski noncommutative spacetime has been studied for a long time as an example of quantum spacetime with nontrivial commutation relations between spatial and temporal coordinates which, at first sight, seem to break PoincarÃ© invariance. However kappa-Minkowski is invariant under a Hopf-algebra deformation of the PoincarÃ© group, which involves some noncommutative structures that prevent the sharp localization of reference frames. I will describe recent progress towards the consistent construction of quantum field theories on this spacetime, and the identification of physical predictions that genuinely distinguish kappa-Minkowski from ordinary, commutative Minkowski spacetime.

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κ-Minkowski: Physics with Noncommutative Time

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"Quantizing Time" Workshop

Perimeter Institute for Theoretical Physics

PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

Noncommutative spaces

Gel'Fand–Naimark 1940's:

Commutative Algebras ⟷ Topological Spaces

Connes 1970's:

Commutative Algebras ⟷ Topological Spaces

↓   ↓

Noncommutative Algebras ⟷ ???

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Topology, algebrized

Commutative algebra of continuous functions on a manifold $C(\mathcal{M})$:

$$(f \cdot g)(x) = f(x) \cdot g(x) \,,$$

this algebra defines a topology on the manifold.

E.g., a cylinder: $f(x, y + 2\pi) = f(x, y)$,

or a 2-torus: $f(x + 2\pi\, n, y + 2\pi\, m) = f(x, y)$.

Coordinate functions x^i are elements of $C(\mathcal{M})$ that distinguish points, *i.e.:*

$$x^i(p) \neq x^i(p') \;\Leftrightarrow\; p \neq p' \,.$$

Probability measures belong to the set $S[C(\mathcal{M})]$ of *states* on $C(\mathcal{M})$ (positive linear functionals blah blah...).

Pure states are *evaluation functionals* $E_x(f) = f(x)$, *i.e.* delta-function probability distributions $\rightarrow$ *points.*

Generalization to noncommutative algebras

$C(\mathbb{R}^2)$: commutative algebra of functions on phase space, generated by the coordinate functions (p, q).

$C_\hbar(\mathbb{R}^2)$: non Abelian deformation of $C(\mathbb{R}^2)$ into the *Heisenberg algebra:*

$$[\hat{\boldsymbol{p}}, \hat{\boldsymbol{q}}] = i\,\hbar\,, \qquad \hat{\boldsymbol{p}}^\dagger = \hat{\boldsymbol{p}}\,, \qquad \hat{\boldsymbol{q}}^\dagger = \hat{\boldsymbol{q}}\,.$$

$C_\hbar(\mathbb{R}^2)$ can be thought of as the closure of the algebra of ordered polynomials in $\hat{\boldsymbol{p}}$, $\hat{\boldsymbol{q}}$. Popular quantum operators are (limits of) polynomials:

$$\hat{\boldsymbol{H}} = \frac{1}{2}(\hat{\boldsymbol{p}}^2 + \hat{\boldsymbol{q}}^2)\,, \qquad \hat{\vec{\boldsymbol{L}}} = \frac{1}{2}(\vec{\hat{\boldsymbol{p}}} \times \vec{\hat{\boldsymbol{q}}} - \vec{\hat{\boldsymbol{q}}} \times \vec{\hat{\boldsymbol{p}}})\,, \qquad V(\hat{\boldsymbol{q}}) = -\frac{1}{\hat{\boldsymbol{q}}}\,,$$

Now states are mappings from observables (self-adjoint elements of $C_\hbar(\mathbb{R}^2)$,
e.g. $\hat{\boldsymbol{H}}$, $\hat{\vec{\boldsymbol{L}}}$ *etc.*) to their expectation values.

The set of states $S[C_\hbar(\mathbb{R}^2)]$ does not contain
any state that is perfectly localized in $\hat{\boldsymbol{p}}$ and $\hat{\boldsymbol{q}}$.

Now we would like to interpret noncommutative algebras as algebras of "functions" on "quantum manifolds".

κ-Minkowski

Motivation:

3D Quantum Gravity (a topological QFT, better understood than full 4D QG).

Coupling 3D QG to point "particles" (conical defects) and integrating away the gravitational degrees of freedom, one is left with an effective theory with κ-Minkowski commutation relations. [Matschull, Welling, CQG 15 1998]

A similar argument, but with scalar fields, gives rise to a noncommutative QFT with κ-Minkowski commutation relations [Freidel, Livine, PRL 96 (2006)]

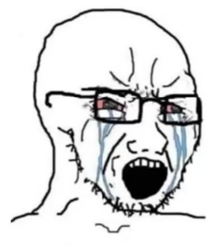
The most popular version of the κ-Minkowski algebra is:

$$[\hat{x}^0, \hat{x}^i] = \frac{i}{\kappa} \hat{x}^i \,, \qquad [\hat{x}^i, \hat{x}^j] = 0 \,, \qquad \kappa \sim L_p^{-1} ?$$

(in 2D it's also the only nonabelian real lie algebra $[\hat{t}, \hat{x}] \propto \hat{x}$).

κ-Minkowski

Remember undergrad QM courses, where they taught us that:



Nooooooooo you can't just make time into a quantum operator!!!!

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So, why can we ignore the wisdom of our teachers and just go $[\hat{x}^0, \hat{x}^i] = \frac{i}{\kappa}\hat{x}^i$?

The problems with defining a quantum $\hat{t}$ operator:

- Pauli: if $\hat{t}^\dagger = \hat{t}$ then $\hat{H}$ cannot be bounded from below
- Time-energy uncertainty relations not universal as position-momentum ones
- Difficulties with defining $\hat{t}$'s domain and the Schroedinger equation

stem from the request that $\hat{t}$ be treated, in nonrelativistic QM, on the same footing as $\hat{x}$ = the "position" observable of a point particle.

But, in QFT, neither time nor position coordinates can be observables:

Malament's theorem (1966): (translational covariance + energy condition + locality) are incompatible with *localizability, i.e.* the existence of projection operators representing the operation of localizing a particle in a certain spatial region, which are orthogonal for disjoint regions.

Conclusion: both t and x^i are not observables, but rather c-number parameters of a QFT, that are not directly observable.

In what sense, in κ-Minkowski, we promote x^μ to noncommuting operators?

We do not intend to interpret $\hat{x}^\mu$ as *observables.* They remain as unobservable as they were in ordinary QFT. Instead, we want to replace the commutative algebra of functions on Minkowski space with a noncommutative algebra, and perform a path integral on the space of noncommutative functions.

I will now show that one can define a noncommutative QFT in terms of N-point functions, admitting the same interpretational framework as commutative QFT.

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Generalized κ-Minkowski algebra

[Majid, Ruegg, PLB 334 (1994)] [Zakrzewski, JPA 27 (1993)]
[Lukierski, Nowicki, Ruegg, PLB 264 (1991), 293 (1992)]

$$[\hat{x}^\mu, \hat{x}^\nu] = \frac{i}{\kappa}(v^\mu \hat{x}^\nu - v^\nu \hat{x}^\mu)\,, \qquad \mu = 0, \ldots, d\,, \qquad (\hat{x}^\mu)^\dagger = \hat{x}^\mu\,,$$

$$v^\mu \in \mathbb{R}^{d+1}\,, \qquad [\kappa] = \text{length}^{-1} \text{ scale.}$$

Most-studied (“timelike”) case: $v^\mu = \delta^\mu{}_0$.

$\hat{x}^\mu \in \mathcal{A}$ = coordinate algebra = ‘noncommutative functions’ = scalar fields.

$[\hat{x}^\mu, \hat{x}^\nu] = \frac{i}{\kappa}(v^\mu \hat{x}^\nu - v^\nu \hat{x}^\mu)$ **break Lorentz (Poincaré) invariance, right?**

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Poincaré invariance: "Doubly Special Relativity"

Wrong. The standard Poincaré transformation law:

$$\hat{\boldsymbol{x}}'^{\mu} = \hat{\boldsymbol{\Lambda}}^{\mu}{}_{\nu}\hat{\boldsymbol{x}}^{\nu} + \hat{\boldsymbol{a}}^{\mu}\,,$$

where $\hat{\boldsymbol{\Lambda}}^{\mu}{}_{\nu}$ and $\hat{\boldsymbol{a}}^{\nu}$ are assumed to be noncommutative operators which commute with $\hat{\boldsymbol{x}}^{\mu}$:

$$[\hat{\boldsymbol{\Lambda}}^{\mu}{}_{\nu}, \hat{\boldsymbol{x}}^{\rho}] = 0 = [\hat{\boldsymbol{a}}^{\mu}, \hat{\boldsymbol{x}}^{\nu}]\,,$$

will leave the κ-Minkowski commutators invariant:

$$[\hat{\boldsymbol{x}}'^{\mu}, \hat{\boldsymbol{x}}'^{\nu}] = \frac{i}{\kappa}(v^{\mu}\hat{\boldsymbol{x}}'^{\nu} - v^{\nu}\hat{\boldsymbol{x}}'^{\mu})\,,$$

if the Lorentz matrices and translation vector satisfy:

$$[\hat{\boldsymbol{\Lambda}}^{\mu}{}_{\nu}\hat{\boldsymbol{\Lambda}}^{\alpha}{}_{\beta}] = 0, \qquad [\hat{\boldsymbol{a}}^{\mu}, \hat{\boldsymbol{a}}^{\nu}] = \frac{i}{\kappa}(v^{\mu}\,\hat{\boldsymbol{a}}^{\nu} - v^{\nu}\,\hat{\boldsymbol{a}}^{\mu})\,,$$

$$[\hat{\boldsymbol{\Lambda}}^{\mu}{}_{\nu}, \hat{\boldsymbol{a}}^{\gamma}] = \frac{i}{\kappa}\left[\left(\hat{\boldsymbol{\Lambda}}^{\mu}{}_{\alpha}\,v^{\alpha} - v^{\mu}\right)\hat{\boldsymbol{\Lambda}}^{\gamma}{}_{\nu} + \left(\hat{\boldsymbol{\Lambda}}^{\alpha}{}_{\nu}g_{\alpha\beta} - g_{\nu\beta}\right)v^{\beta}g^{\mu\gamma}\right],$$

$$\hat{\boldsymbol{\Lambda}}^{\mu}{}_{\alpha}\hat{\boldsymbol{\Lambda}}^{\nu}{}_{\beta}g^{\alpha\beta} = g^{\mu\nu},\;\; \hat{\boldsymbol{\Lambda}}^{\rho}{}_{\mu}\hat{\boldsymbol{\Lambda}}^{\sigma}{}_{\nu}g_{\rho\sigma} = g_{\mu\nu}\,,$$

for an arbitrary real invertible c-number metric $g_{\mu\nu}$.

If $g_{\mu\nu}$ is Lorentzian, $\hat{\Lambda}^{\mu}{}_{\nu}$ and $\hat{a}^{\mu}$ close an algebra which is a noncommutative deformation of the *algebra of functions on the Poincaré group.*

Doubly Special Relativity principle: just like Special Relativity deforms Galilean Relativity by introducing an observer-independent velocity scale, there could be a regime of Quantum Gravity in which inertial observers are related by transformation laws that deform those of SR, by introducing a further invariant scale (a length scale).

[Amelino-Camelia, gr-qc/0012051; Nature 418 (2002)]

κ^{-1} here plays the role of observer-independent length scale:

Alice	**Bob**
$[\hat{x}^{\mu}, \hat{x}^{\nu}] = \frac{i}{\kappa}(v^{\mu}\hat{x}^{\nu} - v^{\nu}\hat{x}^{\mu})\,,$	$[\hat{x}'^{\mu}, \hat{x}'^{\nu}] = \frac{i}{\kappa}(v^{\mu}\hat{x}'^{\nu} - v^{\nu}\hat{x}'^{\mu})$

[Lizzi, Manfredonia, FM, PRD 99 (2019)]

Field theory on κ-Minkowski?

$\mathcal{A}$ = noncommutative deformation of the algebra of functions on spacetime

$\Downarrow$

algebra of scalar fields $\phi(x)\psi(x) \neq \psi(x)\phi(x)$.

Long and interesting history of studies, *e.g.* (nonexhaustive list!):

[Kosinski, Lukierski, Maslankla PRD 62 (2000); NP-PS 102 (2001)]
[Amelino-Camelia, Arzano PRD, 65 (2002)]
[Dimitrijevic,Jonke,Moeller,Tsouchnika,Wess,Wohlgenannt, EPJC 31 (2003)]
[Agostini, Amelino-Camelia, Arzano, CQG 21 (2004)]
[Dimitrijevic, Jonke, Moeller, JHEP 0509 (2005)]
[Arzano, Marcianò, PRD 76 (2007)]
[Daszkiewicz, Lukierski, Woronowicz PRD 77 (2008)]
[Freidel,Kowalski-Glikman, Nowak, IJMPA 23 (2008)]
[Daszkiewicz, Lukierski, Woronowicz, JPA 42,(2009)]
[Amelino-Camelia, Gubitosi, Marcianò, Martinetti, FM, PLB 67 (2009)]
[Borowiec, Pachol, SIGMA 6 (2010)]
[Arzano, Kowalski-Glikman, Walkus, CQG 27 (2010)]
[Dimitrijevic, Jonke, Pachol, SIGMA 10 (2014)]
[Juric, Meljanac, Samsarov, JP-CS 634 (2015)]
[Arzano, Kowalski-Glikman, PLB 771 (2017)]
[Arzano, Consoli, PRD 98, (2018)]
[FM, Sergola, PRD 98 (2018); PLB 787 (2018)]
[Poulain, Wallet, Phys. Rev. D 98, 025002 (2018); JHEP 01 (2019)]
[Mathieu, Wallet, JHEP 05 (2020)]

Many proposals for formulating a QFT on κ-Minkowski.
None 100 % satisfactory so far.

A popular way to get physical predictions without having to discuss the basic ontology and the interpretational framework is to use a nonlocal *star product* between commutative fields:

$$\phi(x) * \psi(x) = \sum_{n,m} c^{\mu_1 \dots \mu_n, \nu_1 \dots \nu_m} \frac{\partial^n \phi(x)}{\partial x^{\mu_1} \dots \partial x^{\mu_n}} \frac{\partial^m \psi(x)}{\partial x^{\nu_1} \dots \partial x^{\nu_m}} ,$$

which realizes a representation of the commutation relations:

$$x^\mu * x^\nu - x^\nu * x^\mu = \frac{i}{\kappa}(v^\mu x^\nu - v^\nu x^\mu) .$$

Then a κ-Poincaré covariant action is written with the help of the star product as a commutative, but nonlocal field action, *e.g.*

$$\int d^4x \left[\frac{1}{2}(P_\mu \phi(x)) * (P_\mu \phi(x)) - \frac{1}{2} m^2 \phi(x) * \phi(x) + \lambda \phi(x)^{*4}\right] ,$$

then N-point functions can be calculated in the standard way, as variational derivatives of the commutative generating functional for this nonlocal QFT.

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I have several issues with this:

- The star product is not unique.

- What happens to noncommutative, κ-Poincaré covariance?

- Fields are expanded in a basis of *ordered* plane waves, the ordering choice should be arbitrary and the physical predictions independent of the ordering choice. Are they?

- Related to the above: is there invariance under change of coordinate system in *momentum space* (see below)?

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"Multilocal functions" and braiding

To properly define N-point functions, we need to algebrize the concept of a function of multiple variables.

Commutative case:

$C(\mathcal{M})$ = commutative algebra of functions on $\mathcal{M}$.

$C(\mathcal{M}) \otimes C(\mathcal{M})$ = algebra of functions of two points.

$C(\mathcal{M}) \otimes C(\mathcal{M})$ is generated by the coordinate functions $x_1^\mu = x^\mu \otimes 1$, $x_2^\mu = 1 \otimes x^\mu$ and the identity $1 \otimes 1$.

Given a nonabelian algebra like $\mathcal{A}$, there is a canonical algebra structure on the tensor product $\mathcal{A} \otimes \mathcal{A}$:

$$[\hat{x}_1^\mu, \hat{x}_1^\nu] = \frac{i}{\kappa}(v^\mu \hat{x}_1^\nu - v^\nu \hat{x}_1^\mu)\,, \qquad [\hat{x}_2^\mu, \hat{x}_2^\nu] = \frac{i}{\kappa}(v^\mu \hat{x}_2^\nu - v^\nu \hat{x}_2^\mu)\,,$$

$$\boxed{[\hat{x}_1^\mu, \hat{x}_2^\nu] = 0\,,} \qquad [\hat{x}_1^\mu, \hat{1}] = [\hat{x}_2^\nu, \hat{1}] = 0\,.$$

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"Multilocal functions" and braiding

The tensor product algebra construction might be natural, but it's not κ-Poincaré covariant. In fact, κ-Poincaré-transforming both coordinates:

$$\hat{\boldsymbol{x}}'^{\mu}_a = \hat{\boldsymbol{\Lambda}}^{\mu}{}_{\nu}\hat{\boldsymbol{x}}^{\nu}_a + \hat{\boldsymbol{a}}^{\mu} \,, \qquad [\hat{\boldsymbol{\Lambda}}^{\mu}{}_{\nu}, \hat{\boldsymbol{x}}^{\nu}_a] = [\hat{\boldsymbol{a}}^{\mu}, \hat{\boldsymbol{x}}^{\nu}_a] = 0 \,,$$

we get that $\hat{\boldsymbol{x}}^{\mu}_a \to \hat{\boldsymbol{x}}'^{\mu}_a$ doesn't leave the cross-commutators invariant:

$$[\hat{\boldsymbol{x}}'^{\mu}_1, \hat{\boldsymbol{x}}'^{\nu}_2] \neq 0 \,.$$

In jargon: $\hat{\boldsymbol{x}}^{\mu}_a \to \hat{\boldsymbol{x}}'^{\mu}_a$ is not a homomorphism for $\mathcal{A} \otimes \mathcal{A}$,
or $\mathcal{A} \otimes \mathcal{A}$ is not a κ-Poincaré-comodule.

...we need to philosophize differently.

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Braided tensor product algebra

This fact is known by mathematicians: in general, tensor product algebras built from a Hopf-algebra comodule are not comodules themselves [Majid, Foundations of Quantum Group Theory, CUP (1995)].

Inspired by Fiore and Wess' work [PRD 75 (2007)] on a different noncommutative space, in [Lizzi, FM, PRD 103, 126009 (2021)] we relaxed the cross-commutation rules. It turns out that these:

$$\left[\hat{x}_1^\mu, \hat{x}_2^\nu\right] = \frac{i}{\kappa}\left[v^\mu \hat{x}_1^\nu - v^\nu \hat{x}_2^\mu - g^{\mu\nu} g_{\rho\sigma} v^\rho \left(\hat{x}_1^\sigma - \hat{x}_2^\sigma\right)\right] ,$$

are κ-Poincaré covariant:

$$[\hat{x}_1'^\mu, \hat{x}_2'^\nu] = [\hat{x}_1^\mu, \hat{x}_2^\nu]'$$

and can be immediately extended to N points:

$$[\hat{x}_a^\mu, \hat{x}_b^\nu] = \frac{i}{\kappa}\left[v^\mu \hat{x}_a^\nu - v^\nu \hat{x}_b^\mu - g^{\mu\nu} g_{\rho\sigma} v^\rho \left(\hat{x}_a^\sigma - \hat{x}_b^\sigma\right)\right] .$$

We are not prepared yet to consider nonassociative structures,
we therefore impose the Jacobi rule:

$$[\hat{\boldsymbol{x}}_a^\mu, [\hat{\boldsymbol{x}}_b^\nu, \hat{\boldsymbol{x}}_c^\rho]] + [\hat{\boldsymbol{x}}_b^\nu [\hat{\boldsymbol{x}}_c^\rho, \hat{\boldsymbol{x}}_a^\mu]] + [\hat{\boldsymbol{x}}_c^\rho, [\hat{\boldsymbol{x}}_a^\mu, \hat{\boldsymbol{x}}_b^\nu]] =$$

$$-g_{\alpha\beta} v^\alpha v^\beta \left[\eta^{\nu\rho}(\hat{\boldsymbol{x}}_c^\mu - \hat{\boldsymbol{x}}_b^\mu) + \eta^{\rho\mu}(\hat{\boldsymbol{x}}_a^\nu - \hat{\boldsymbol{x}}_c^\nu) + \eta^{\mu\nu}(\hat{\boldsymbol{x}}_b^\rho - \hat{\boldsymbol{x}}_a^\rho) \right] ,$$

and this is satisfied only if $g_{\mu\nu} v^\mu v^\nu = 0$,
this is the so-called *lightlike* κ-Minkowski noncommutativity.

Unfortunately, the well-studied $v^\mu = \delta^\mu{}_0$ case doesn't work.

However, we have a nontrivial case to study!

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$$-g_{\alpha\beta} v^\alpha v^\beta \left[\eta^{\nu\rho} (\hat{\boldsymbol{x}}^\mu_c - \hat{\boldsymbol{x}}^\mu_b) + \eta^{\rho\mu} (\hat{\boldsymbol{x}}^\nu_a - \hat{\boldsymbol{x}}^\nu_c) + \eta^{\mu\nu} (\hat{\boldsymbol{x}}^\rho_b - \hat{\boldsymbol{x}}^\rho_a) \right] ,$$

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A quasi-commutative algebra

$$[\hat{\boldsymbol{x}}^\mu_a, \hat{\boldsymbol{x}}^\nu_b] = \frac{i}{\kappa}\left[v^\mu \hat{\boldsymbol{x}}^\nu_a - v^\nu \hat{\boldsymbol{x}}^\mu_b - g^{\mu\nu} g_{\rho\sigma} v^\rho \left(\hat{\boldsymbol{x}}^\sigma_a - \hat{\boldsymbol{x}}^\sigma_b\right)\right] ,$$

the coordinate differences close an Abelian subalgebra:

$$\delta\hat{\boldsymbol{x}}^\mu_{ab} = \hat{\boldsymbol{x}}^\mu_a - \hat{\boldsymbol{x}}^\mu_b \,, \qquad [\delta\hat{\boldsymbol{x}}^\mu_{ab}, \delta\hat{\boldsymbol{x}}^\mu_{cd}] = 0 \,.$$

All the noncommutativity is concentrated on four "center-of-mass" degrees of freedom:

$$\hat{\boldsymbol{x}}^\mu_{cm} = \frac{1}{N}\sum_{a=1}^{N} \hat{\boldsymbol{x}}^\mu_a \,, \qquad \hat{\boldsymbol{y}}^\mu_a = \hat{\boldsymbol{x}}^\mu_a - \hat{\boldsymbol{x}}^\mu_{cm} \,,$$

$$[\hat{\boldsymbol{x}}^\mu_{\text{cm}}, \hat{\boldsymbol{y}}^\nu_a] = i\left(g^{\mu\nu} g_{\rho\sigma} v^\rho \hat{\boldsymbol{y}}^\sigma_a - v^\nu \hat{\boldsymbol{y}}^\mu_a\right) , \qquad [\hat{\boldsymbol{x}}^\mu_{\text{cm}}, \hat{\boldsymbol{x}}^\nu_{\text{cm}}] = i\left(v^\mu \hat{\boldsymbol{x}}^\nu_{\text{cm}} - v^\nu \hat{\boldsymbol{x}}^\mu_{\text{cm}}\right) ,$$

$$[\hat{\boldsymbol{y}}^\mu_a, \hat{\boldsymbol{y}}^\nu_b] = 0 \,.$$

One can write an explicit representation of this algebra in terms of Lorentz matrices and dilatation operators.

A case of serendipity

N-point functions are, in the commutative case, Poincaré-invariant distributions which admit a Fourier representation.

A natural noncommutative version of an N-point function is a product of exponentials:

$$f(\hat{\boldsymbol{x}}_a^\mu) = \int d^4k^1 \dots d^4k^N \, \tilde{f}(k_\mu^a) :e^{ik_\mu^1 \hat{\boldsymbol{x}}_1^\mu} : \dots : e^{ik_\mu^N \hat{\boldsymbol{x}}_N^\mu} : ,$$

where : : represents an ordering prescription for the noncommutative coordinates.

A moderately lengthy proof shows that κ-Poincaré invariance implies that $f(\hat{\boldsymbol{x}}_a^\mu)$ depends only on $\hat{\boldsymbol{y}}_a^\mu$, and not on $\hat{\boldsymbol{x}}_{cm}^\mu$.

Therefore *all N-point functions are commutative!*

This hugely simplifies the interpretational framework of the theory: it is just a complicated way to calculate perfectly ordinary N-point functions.

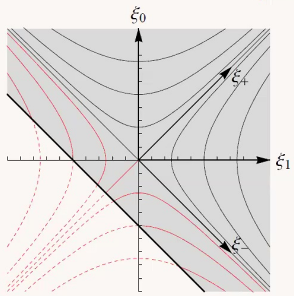
Features of (1+1D) κ-deformed field theory with $v^\mu v^\nu g_{\mu\nu} = 0$

1-point ordered plane waves $\hat{\boldsymbol{e}}_a[k_\mu] = e^{ik_- \hat{\boldsymbol{x}}_a^-} e^{ik_+ \hat{\boldsymbol{x}}_a^+}, \quad k_\pm \in \mathbb{R}$
form a Lie group ($SB(2)$ in 2D, $AN(3)$ in 4D).

Different ordering prescriptions correspond to different coordinate systems on the "momentum space" group manifold (*i.e.* different group factorizations):

$$e^{iq_- \hat{\boldsymbol{x}}_a^- + iq_+ \hat{\boldsymbol{x}}_a^+} = e^{i\left(\frac{1-e^{-2q_+/\kappa}}{2q_+}\right) q_- \hat{\boldsymbol{x}}_a^-} e^{iq_+ \hat{\boldsymbol{x}}^+},$$

The group manifold is a half Minkowski space:



the mass Casimir is deformed in certain coordinates, but it can always be made standard ($p^\mu p_\mu = m^2$) with a general momentum-space coordinate change. Forget about "deformed dispersion relations" here, it's not physical.

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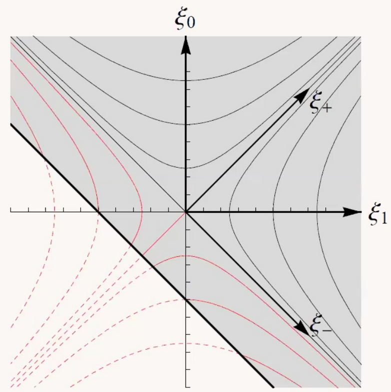
Nonlinear action of the κ-Poincaré group on momentum space:

$$\hat{\boldsymbol{e}}'_a[k] = e^{ik_- \hat{x}'^-_a} e^{ik_+ \hat{x}'^+_a} = \hat{\boldsymbol{e}}_a[\lambda(k, \hat{\Lambda})] e^{ik_- \hat{\boldsymbol{a}}^-} e^{ik_- \hat{\boldsymbol{a}}^+} .$$



Only 2 quadrants are Lorentz-invariant, fortunately including a full mass-shell.

Physics can be defined in a momentum-space-diffeomorphism-invariant way.

Let's look for a Wightman function that is
independent of the ordering choice, and κ-Poincaré-invariant:

$$W_{(+)}(\hat{\boldsymbol{x}}_1 - \hat{\boldsymbol{x}}_2) = \int d^2k \sqrt{-g(k)}\delta(\mathcal{C}(k) - m^2)\Theta(k_+)\, \hat{\boldsymbol{e}}_1[k]\, \hat{\boldsymbol{e}}_2^\dagger[k]\,,$$

$$W_{(-)}(\hat{\boldsymbol{x}}_1 - \hat{\boldsymbol{x}}_2) = \int d^2k \sqrt{-g(k)}\delta(\mathcal{C}(k) - m^2)\Theta(-k_+)\, \hat{\boldsymbol{e}}_1[k]\, \hat{\boldsymbol{e}}_2^\dagger[k]\,,$$

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Triviality?

This result reminds of [Fiore, arXiv:0809.4507] and [Fiore–Wess PRD 75 (2007)] cited above: in the canonical noncommutative spacetime, all N-point functions are undeformed.

Our result, however, holds only for the *non-interacting* theory. One could conjecture that a interacting theory would have κ-dependent corrections to the N-point functions from interaction vertices.

The issue is under investigation.

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