Title: Three ways to classicalize (nearly) any probabilistic theory

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Abstract: It is commonplace that quantum theory can be viewed as a ``non-classical" probability calculus. This observation has inspired the study of more general non-classical probabilistic theories modeled on QM, the so-called generalized probabilistic theories or GPTs. However, the boundary between these putatively non-classical probabilistic theories and classical probability theory is somewhat blurry, and perhaps even conventional. This is because, as is well known, any probabilistic model can be understood in classical terms if we are willing to embrace some form of contextuality. In this talk, I want to stress that this can often be done functorially: given a category $\Cat$ of probabilistic models, there are functors $F : \Cat \rightarrow \Set_{\Delta}$ where $\Set_{\Delta}$ is the category of sets and stochastic maps. In addition to the familiar Beltrametti-Bugajski representation, I'll exhibit two others that are less well known, one involving the ``semi-classical cover" and another, slightly more special, that allows one to represent a probabilistic model with sufficiently strong symmetry properties by a model having a completely classical probabilistic structure, in which any ``non-classicality" is moved into the dynamics, in roughly the spirit of Bohmian mechanics.

(Based on http://philsci-archive.pitt.edu/16721/)

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Three ways to Classicalize (nearly) Any Probabilistic Theory

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June 2021

A commonplace:

Quantum mechanics is a non-classical probability theory

This suggests the study of more general non-classical probabilistic theories (GPTs) modeled on QM, which has been very fruitful.

However, there two caveats:

(a) A probabilistic theory isn't *a probability theory*. (The latter can be viewed as the study *of* probabilistic theories.)

(b) The boundary between "classical" and "non-classical" probabilistic theories is a bit blurry, perhaps even conventional.

In this talk, I'll focus on (b), by considering three ways of interpreting a GPT in terms of classical PT. In descending order of familiarity:

(1) A general Beltrametti-Bugajski style representation;

(2) A representation involving the "semi-classical cover";

(3) A representation of probabilistic models with enough symmetry in terms of a classical probabilistic structure, with the non-classicality is moved into the dynamics (somewhat in the spirit of Bohmian mechanics).

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(http://philsci-archive.pitt.edu/16721/ =: AW2020)

Outline:

I. Background

- Probabilistic models
- Probabilistic theories (= categories of probabilistic models)
- Classical probability theory and classical embeddings

II. Classical Extensions

III. Semiclassical Extensions

IV. Dynamical classical representations

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V. Composite systems (if time)

I. Background

In discrete classical probability theory, a *probabilistic model* is a pair (E, μ):

- E an *outcome-set*,
- μ a probability weight on E.

Obvious generalization: Allow both E and μ to vary. Start with E:

Definition: A *test space* is a collection $\mathcal{M} = \{E, F, ...\}$ of (outcome-sets of) possible experiments, *tests*, etc.

- Mathematically, $\mathcal{M}$ is just a hypergraph.
- Idea due to **C. H. Randall** (1928-1987) and **D. J. Foulis** (1930-2018). Also called *contextuality scenarios* in some more recent literature.

Let $X := \bigcup \mathcal{M}$, i.e., space of *all* outcomes.

Definition: A *probability weight* on $\mathcal{M}$:

$$\boldsymbol{\alpha} : X \to [0,1] \quad \text{with} \quad \sum_{x \in E} \boldsymbol{\alpha}(x) = 1 \ \forall E \in \mathcal{M}.$$

Write $\Pr(\mathcal{M})$ for the set of all these.

- Tests can overlap, but probability weights are *non-contextual*.
- $\Pr(\mathcal{M}) \subseteq [0,1]^X$ is convex, and closed (so, compact) if all tests are finite.
- More structure can be added as needed.

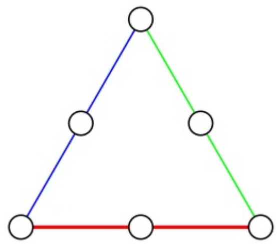
So we have a natural generalization of the received idea of a probabilistic model:

Definition: A *probabilistic model* A is a pair $(\mathcal{M}(A), \Omega(A))$ where $\mathcal{M}(A)$ is a test space and $\emptyset \neq \Omega(A) \subseteq \Pr(\mathcal{M}(A))$.

$\Omega(A)$ is the state space, and elements of $\Omega(A)$ are the *states*, of A.

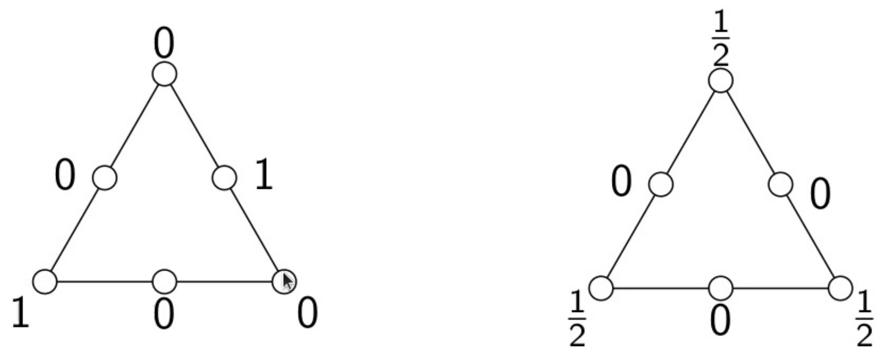
- Idea goes back at least to Mackey (1957!), with $\mathcal{M}(A)$ consisting of two-outcome tests ("questions").
- Also considered by Foulis-Randall under the much cooler name "stochastic entity"

Toy Example



X = nodes; $\mathcal{M}$ = sides

Sample probability weights:



Note: Both of these are pure!

Real Examples

Classical models:

(i) *Simple:* $A = (\{E\}, \Delta(E))$ — one test, all probability weights.

(ii) *Kolmogrovian:* if S is a measurable space, let $\mathcal{D}(S) =$ set of all finite measurable partitions of S Every probability weight $\mu \in \Delta(S)$ defines a probability weight on $\mathcal{D}(S)$, so $(\mathcal{D}(S), \Delta(S))$ is a probabilistic model.

To simplify notation, let's write $\mathcal{D}(S)$ for this entire model.

Real Examples

A simple **quantum model**: For a (f.d.) Hilbert space $\mathcal{H}$, let

- $\mathcal{M}(\mathcal{H})$ = set of ONBs for $\mathcal{H}$; $X(\mathcal{H})$ = unit sphere.
- $\Omega(\mathcal{H})$ = all probability weights states of the form

$$\alpha(x) = \langle Wx, x \rangle,$$

W a density operator on $\mathcal{H}$

Models from convex sets

Let K be a convex set (an abstract state-space). An *effect* on K is an affine functional $a : K \to [0,1]$. The *unit effect* is $u_K : \alpha \mapsto 1$ $\forall \alpha \in K$. Write $[0, u_K]$ for the set of all effects.

Definition: A (discrete) *observable* over K is a finitely-indexed collection $\{a_i\}_{i \in I}$ of nonzero effects with $\sum_{i \in I} a_i = u_K$.

We can organize observables into a test space. Let

(a) $X^o(K) := \mathbb{N} \times (0, u_K]$,

(b) $\mathcal{M}^o(K) =$ graphs of observables indexed by finite $I \subseteq \mathbb{N}$.

For $\alpha \in K$, define $\alpha^o(i, a) = a(\alpha)$: we then have a model $(\mathcal{M}^o(K), K^o)$ with $K^o \simeq K$.

I'll use $\mathcal{M}^o(K)$ to stand for this entire model.

Probabilistic Theories

An *event* of A is a subset of a test $E \in \mathcal{M}(A)$. Write $\mathcal{E}(A)$ for the set of events. If $\boldsymbol{\alpha} \in \Omega(A)$ and $a \in \mathcal{E}(A)$, let $\boldsymbol{\alpha}(a) = \sum_{x \in A} \boldsymbol{\alpha}(x)$.

Definition (cf FR 1978): An *interpretation* from A to B is a mapping $\phi : X(A) \longrightarrow \mathcal{E}(B)$ such that

(a) for every $E \in \mathcal{M}(A)$, $\{\phi(x) | x \in E\}$ is pairwise disjoint and $\bigcup_{x \in E} \phi(x) \in \mathcal{M}(B)$;

(b) for every $\boldsymbol{\beta} \in \Omega(B)$, $\phi^*(\boldsymbol{\beta})(x) = \boldsymbol{\beta}(\phi(x))$ defines a state in $\Omega(A)$.

In other words: ϕ maps tests of A to possibly *coarse-grained* versions of tests in B, in such a way that states on B pull back to states on A.

Interpretations $A \xrightarrow{\phi} B \xrightarrow{\psi} C$ compose according to

$$(\psi \circ \phi)(x) \;=\; \bigcup_{y \in \phi(x)} \psi(y).$$

This gives us a category, **Int**. Any subcategory of **Int** is a probabilistic theory.

Examples:

(a) Let S and T be measuable spaces and $f : S \to T$, measurable. Let $\phi : \mathcal{D}(T) \to \mathcal{D}(S)$ by

$$\phi(b) = \begin{cases} \{f^{-1}(b)\} & f^{-1}(b) \neq \emptyset \\ \emptyset & \phi(b) = \emptyset \end{cases}$$

(b) If K_1, K_2 are compact convex sets and $f : K_1 \to K_2$ an affine mapping. Define $\phi : \mathcal{M}^o(K_2) \to \mathcal{M}^o(K_1)$ by

$$\phi(i, b) = \begin{cases} \{(i, b \circ f)\} & b \circ f \neq 0 \\ \emptyset & b \circ f = 0 \end{cases}$$

Classical Embeddings

If $|\phi(x)| = 1$ for every $x \in X(A)$, then we can regard an interpretation $\phi : A \to B$ as a mapping $\phi : X(A) \to X(B)$. Call such an interpretation an *embedding* iff this is injective and the mapping $\phi^* : \Omega(B) \to \Omega(A)$ is surjective.

When does an abstract model A admit a *classical embedding* $\phi : A \to (\mathcal{D}(S), \Delta(S))$?

Definition

- $\boldsymbol{\alpha} \in \Pr(\mathcal{M}(A))$ is *dispersion-free* (DF) iff $\boldsymbol{\alpha}(x) \in \{0, 1\}$ for every $x \in X(A)$.
- Let $S(A)$ stand for the set of all DF probability weights on $\mathcal{M}(A)$. A is *unitally dispersion-free* (UDF) iff
  (i) $\forall x \in X(A)$, $\exists \delta \in S(A)$ with $\delta(x) = 1$,
  (ii) $\Omega(A)$ is contained in the closed convex hull of $S(A)$.

Versions of the following can be found in many places in the quantum-logical literature.

Theorem: *A model admits a classical embedding iff it's UDF. In that case, one can take $S = S(A)$.*

Not enough for $\mathcal{M}(A)$ to have a lot of dispersion-free probability weights: all states $\alpha \in \Omega(A)$ must be averages of these.

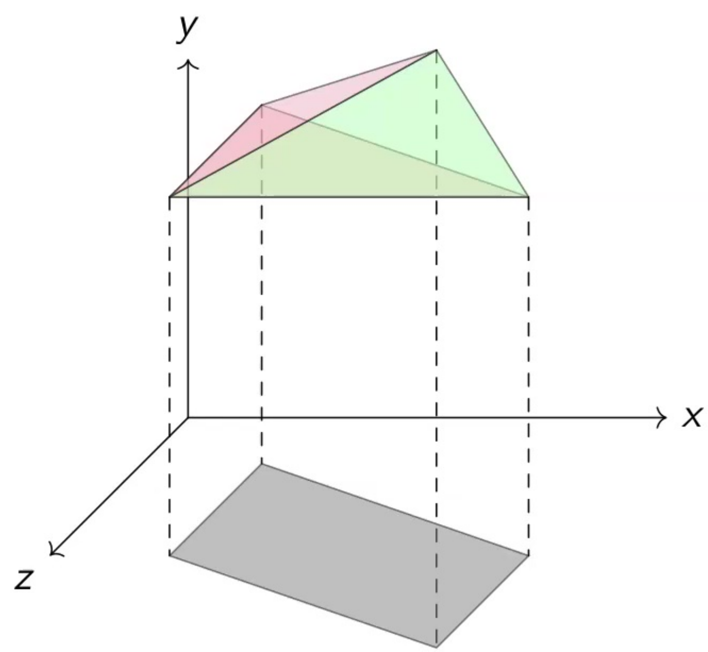
Example: The $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$ state on the Triangle is pure, but not DF. So this state blocks any classical embedding.

If $\mathcal{M}(A)$ has *no* DF probability weights, then a classical embedding is impossible regardless of how we choose $\Omega(A)$. This is the case for quantum models: Gleason's Theorem tells us that $\mathcal{M}(\mathcal{H})$ has no DF states for $\dim(\mathcal{H}) > 2$.

II. Classical Extensions

An embedding isn't the only way of explaining one mathematical object in terms of another. One can also represent one object as a *quotient* of another, more familiar object.

Example: a square can arise as the projection of a tetrahedron (a 4-simplex) on a plane.



Getting this kind of thing to works in infinite dimensional generality is the object of *Choquet Theory*.

If K is a compact convex subset of a locally convex TVS V, let $\Delta_0(K)$ be the set of Baire probability measures on K (an infinite-dimensional simplex).

For each $\mu \in \Delta_o(K)$, the *barycenter* of μ, $\widehat{\mu} \in V^{**}$, is defined by

$$\widehat{\mu}(f) = \int_K f d\mu$$

for all functionals $f \in V^*$.

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Proposition: (see Alfsen, 1971, I.2.1) $\widehat{\mu} \in K$. *Hence,* $\mu \mapsto \widehat{\mu}$ *is an affine surjection* $\Delta_o(K) \to K$.

Definition: A *classical extension* of a probabilistic model A with convex state space $\Omega(A)$ is an affine surjection

$$q : \Delta(S) \to \Omega(A).$$

for some measurable space S.

This makes no reference to $\mathcal{M}(A)$. But the surjection q dualizes to

$$q^* : X(A) \to [0, u_{\Delta(S)}] \qquad q^*(x)(\mu) := q(\mu)(x).$$

This is an interpretation from A into $\mathcal{M}^o(\Delta(S))$, and an embedding as long as $\Omega(A)$ separates outcomes.

Effects on $\Delta(S)$ are "unsharp" (or fuzzy, or noisy) indicator function, so tests in $\mathcal{M}^o(\Delta(S))$ are "unsharp" versions partitions. We can regard q^* as an "unsharp" classical embedding.

As we've just seen, *every* probabilistic model with $\Omega(A)$ compact and convex has a *canonical* classical extension obtained by taking $S = \Omega(A)$ (with its Baire field).

This construction is even functorial:

$$A \mapsto (\mathcal{M}^o(\Delta_0(\Omega(A)))$$

is the object part of a covariant functor from probabilistic models to unsharp Kolmogorovian models. (Ω and $\mathcal{M}^o$ are contravariant, Δ_o is covariant.)

III. Semiclassical Covers

We can in some respects do better than this, with much less work. If $E \in \mathcal{M}(A)$, let $\widetilde{E} = \{(x, E) | x \in E\} \subseteq X \times \mathcal{M}$, and set

$$\widetilde{\mathcal{M}} = \{ \widetilde{E} \mid E \in \mathcal{M} \}.$$

The outcome-set of $\widetilde{\mathcal{M}}$ is $\widetilde{X} = \{ (x, E) \mid x \in E \in \mathcal{M} \}$. Every $\boldsymbol{\alpha} \in \Omega(A)$ defines a probability weight $\widetilde{\boldsymbol{\alpha}}(x, E) = \boldsymbol{\alpha}(x)$, so we have a model

$$\widetilde{A} = (\widetilde{\mathcal{M}(A)}, \widetilde{\Omega(A)}),$$

the *semi-classical cover* of A. Note there's an outcome-preserving interpretation

$$\pi : \widetilde{A} \longrightarrow A, \quad \pi(x, E) = x.$$

$\widetilde{A}$ is UDF, and so, has a classical embedding. We can therefore represent A as a quotient of a sub-model of a (genuine, not "fuzzy") classical model.

The semiclassical cover is functorial: an interpretation $\phi : A \to B$ yields an interpretation $\widetilde{\phi} : \widetilde{A} \to \widetilde{B}$, accoring to

$$\widetilde{\phi}(x, E) \;=\; \{\, (y, \phi(E)) \mid y \in \phi(x) \,\}.$$

So we have a covariant functor — on objects, $A \mapsto \mathcal{D}(S(\widetilde{A}))$ — from models to (sharp) Kolmogorovian models.

Summarizing:

$$A \longrightarrow \mathcal{D}(S)$$

(Sharp) classical embedding (rarely exists)

$$\begin{array}{c} \mathcal{D}^{o}(\Delta_0(\Omega)) \\ \uparrow \\ A \end{array} \qquad\qquad \begin{array}{ccc} \widetilde{A} & \longrightarrow & \mathcal{D}(S(\widetilde{A})) \\ \downarrow & & \\ A & & \end{array}$$

Unsharp classical embedding

Semiclassical extension

Now consider probabilistic models A equipped with a *dynamical group* $G(A)$ of positive, outcome-preserving, invertible intepretations $A \to A$. Equivalently: of bijection $g : X(A) \to X(A)$ acting on

(a) $G(A)$ is a group of bijections $g : X(A) \to X(A)$ with

(b) $\mathcal{M}(A)$ invariant under $E \mapsto g(E)$,

(c) $\Omega(A)$ invariant under $\boldsymbol{\alpha} \mapsto \boldsymbol{\alpha} \circ g$

Call A, so equipped, a *dynamical* probabilistic model.

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Call A, so equipped, a *dynamical* probabilistic model.

Where $G(A)$ is a Lie group, a reversible *dynamics* for A is a choice of a continuous one-parameter group $g \in \text{Hom}(\mathbb{R}, G(A))$. tracking the system's evolution over time: if $\boldsymbol{\alpha}_o$ is the state at $t = 0$, $g_t(\boldsymbol{\alpha}) = \boldsymbol{\alpha} \circ g_{-t}$ is the state after t units of time.

The assumption that $g_{t+s} = g_t g_s$ encodes a Markovian assumption about the dynamics: a system's later state depends only on its initial state and the amount of time elapsed, rather than on the system's entire history.

Definition: A dynamical model A is *symmetric* iff $G(A)$ acts transitively on the set $\mathcal{M}(A)$ of tests, and the stabilizer, $G(A)_E$, of a (so, of any) test $E \in \mathcal{M}$ acts transitively on E.

The quantum probabilistic model $A(\mathcal{H})$ is symmetric w.r.t. $G(A(\mathcal{H})) = \mathcal{U}(\mathcal{H})$, the unitary group of $\mathcal{H}$.

Now consider a single classical apparatus/experiment with outcome-set E, coupled probabilistically to a physical system having a state-space Ω, with dynamics given by a Lie group Suppose G (acting on the right). The coupling is given by a function

$$p : \Omega \times E \to [0,1], \quad \sum_{x \in E} p(\boldsymbol{\alpha}, x) = 1 \ \forall \boldsymbol{\alpha} \in \Omega \tag{1}$$

giving the probability $p(\boldsymbol{\alpha}, x)$ to obtain outcome $x \in E$ with the system in state $\boldsymbol{\alpha} \in \Omega$.

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(Note: p is just a discrete Markov kernel. Nothing "nonclassical" here.)

Two further assumptions:

(2) Ω separates outcomes in E:

$$(\ \forall \boldsymbol{\alpha} \in \Omega\ p(\boldsymbol{\alpha}, x) \ = \ p(\boldsymbol{\alpha}, y)\ ) \quad \Rightarrow \quad x = y. \qquad (2)$$

(If not, factor out the obvious equivalence relation on E.)

(3) E and G together separate states:

$$(\forall g \in G, \forall x \in E\ p(\boldsymbol{\alpha} g, x) = p(\boldsymbol{\beta} g, x)) \ \Rightarrow \ \boldsymbol{\alpha} = \boldsymbol{\beta} \ (3)$$

Each state $\boldsymbol{\alpha} \in \Omega$ gives us mappings

$$\widehat{\boldsymbol{\alpha}} : G \to \Delta(E) \text{ and } \widehat{\boldsymbol{\alpha}} : G \times E \to [0,1]$$

defined by

$$\widehat{\boldsymbol{\alpha}}(g)(x) = \widehat{\boldsymbol{\alpha}}(g,x) := p(\boldsymbol{\alpha} g, x).$$

The first is a random probability weight on G, the second, a discrete Markov kernel on $G \times E$. Take your pick: either way, call $\widehat{\boldsymbol{\alpha}}$ the *dynamical state* associated with $\boldsymbol{\alpha} \in \Omega$.

The state-separation assumption (3) tells us that $\boldsymbol{\alpha} \mapsto \widehat{\alpha}$ injective, so we can identify $\boldsymbol{\alpha}$ with $\widehat{\boldsymbol{\alpha}}$.

Say that $g \in G$ *implements* a permutation $\sigma : E \to E$ iff $p(\boldsymbol{\alpha} g, x) = p(\boldsymbol{\alpha}, \sigma x)\ \forall \boldsymbol{\alpha} \in \Omega, x \in E$. By (2), each $g \in G$ implements *at most one* $\sigma = \sigma_g$.

Lemma: *The set H of all $g \in G$ implementing permutations of E is a subgroup of G and $\sigma : H \to Sym(E)$, $g \mapsto \sigma_g$, is a homomorphism.*

So E carries a natural H action. To simplify the notation, write hx for $\sigma_h x$ ($h \in H$ and $x \in E$).

It's natural to consider cases in which E is *transitive* as an H-set: for every $x, y \in E$, $\exists h \in H$ with $hx = y$. This motivates the following

Definition: A *dynamical classical model (DCM)* consists of

(a) Groups $H \leq G$,

(b) A right G-set Ω and a transitive left H-set E,

(c) A Markov kernel $p : \Omega \times E \to [0,1]$ satisfying conditions (2) and (3) above, such that

$$p(\boldsymbol{\alpha}, hx) \;=\; p(\boldsymbol{\alpha} h, x)$$

for all $\boldsymbol{\alpha} \in \Omega$, $h \in H$ and $x \in E$.

So far, nothing we've done takes us outside the range of classical probability theory.

Nevertheless, *any symmetric model* — *including, e.g., any finite-dimensional Quantum model* — gives an example of this scenario: Choose a test $E \in \mathcal{M}(A)$, and let $p(\alpha, x) = \alpha(x)$ and $\widehat{\alpha}(g)(x) = \alpha(gx)$ for all $\alpha \in \Omega(A)$ and any $x \in E$.

Conversely, given a DCM, one can always construct a symmetric probabilistic model with dynamical group G, containing E as a test. (For the recipe, see the last slide).

So one can convert a DCM into a symmetric probabilistic model (in various ways, parametrized by K).

It would seem that there's little mathematical difference between a "non-classical") symmetric probabilistic model A and an essentially "classical" DCM. Can this be right?

Given a state $\alpha \in \Omega$ plus the system's dynamics, specified by a choice of one-parameter group $g : \mathbb{R} \to G$, we have a path

$$\widehat{\boldsymbol{\alpha}}_t \ := \ \widehat{\alpha}(g_t)$$

in $\Delta(E)$. Dynamical states specify how probabilities change over time, given any possible dynamics.

But these paths are generally *not governed by flows:* there's no one-parameter group of affine mappings $T^s : \Delta(E) \to \Delta(E)$ such that $\widehat{\boldsymbol{\alpha}}_{t+s} = T^s(\boldsymbol{\alpha}_t)$. The observed evolution of probabilities on E is not Markov.

In this respect, it is the *dynamical*, rather than the probabilistic, structure of the DCM that can be regarded as non-classical.

V. Composite Systems

So, in view of all this — is quantum mechanics itself essentially classical?

Standard response: *No* — because nonlocality! entanglement!

So let's briefly consider how the three classical representations we've discussed play with composition.

Without going into details, there's a standard notion of a nonsignaling composite, AB, of probabilistic models A and B. This needn't be locally tomographic.

- In general, $\Omega(AB)$ will contain entangled or (maybe better to say) *non-local* states: states that aren't mixtures of product states.
- Because composites can be non-locally tomographic, this can happen *even if the models A and B are classical*.
- The classical and semiclassical extensions are "non-local" in just this way.

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Moral: Passing from a "non-classical" probabilistic theory to its classical or semiclassical extension shifts the "non-classicality" from the probabilistic structure to the *physics*.

Composites of DCMs are more interesting. One can also write down a plausible definition for a composite AB of DCMs AB (see AW2020). One assumption is that

$$E(AB) = E(A) \times E(B).$$

In general, $\Omega(AB)$ will contain entangled states.

But *how do we detect them?* AB has only the single, classical, measurement $E(A) \times E(B)$.

We can reinterpret A, B and C into symmetric probabilistic models and take (ordinary) composites of these, obtaining enough tests to do the job. But maybe there's a more direct story about the dynamics?

Further Questions and Projects

(a) Classical extensions are preparation contextual. Semiclassical extensions are measurement contextual. *In what way is the DCM representation contextual?*

(b) If $A = (\Omega, G, E)$ is a CDM with G a Lie group, *how does an element of $\mathfrak{g}$, the Lie algebra of G, show up as an "observable"*?

(c) Can we *extend the definition of a DCM to replace the discrete measurement E with $\mathcal{D}(S)$ for a measurable space S*?

(d) *Is the Bohm interpretation of QM just a (suitably extended) DCM representation?*

Appendix: The Recipe

Proposition: Given a DCM as above, let $x_o \in E$ and define

$$\mathcal{K} = \{g \in G | p(\boldsymbol{\alpha} g, x_o) = p(\boldsymbol{\alpha}, x_o)\}.$$

For any subgroup $K \leq \mathcal{K}$ with $K \cap H = H_{x_o}$, there's a well-defined H-equivariant injection $\phi : E \to X := G/K$ such that

(a) $\mathcal{M} := \{ \ g\phi(E) \mid g \in G \ \}$ is a fully G-symmetric test space,

(b) For every $\boldsymbol{\alpha} \in \Omega$,

$$[\boldsymbol{\alpha}](x_g) := \boldsymbol{\alpha}(g, x_o)$$

is a well-defined probability weight on $\mathcal{M}$,

(c) The mapping $\boldsymbol{\alpha} \mapsto [\boldsymbol{\alpha}]$ is a G-equivariant affine injection.