Title: Complete set of quasi-conserved quantities for spinning particles around Kerr

Speakers: Adrien Druart

Collection: The 24th Capra meeting on Radiation Reaction in General Relativity

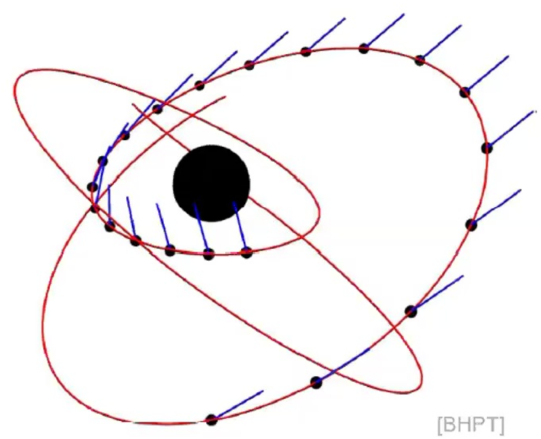
Date: June 10, 2021 - 11:00 AM

URL: http://pirsa.org/21060061

Abstract: I will revisit the conserved quantities of the Mathisson-Papapetrou-Tulczyjew equations describing the motion of spinning particles around a fixed background. Assuming Ricci-flatness and the existence of a Killing-Yano tensor, I obtain three non-trivial quasi-conserved quantities, i.e. conserved at linear order in the spin, thereby completing the two quasi-constants of motion found by RÃ¼diger with one new independent quasi-constant of motion. Finally, I will discuss the implications for the motion of spinning particles in the Kerr geometry.

Complete set of conserved quantities for spinning particles around Kerr

[arXiv:2105.12454, with Geoffrey Compère]



[BHPT]

Adrien Druart

{Physique théorique et mathématique,

Université Libre de Bruxelles}

24th CAPRA Meeting on Radiation Reaction in General Relativity

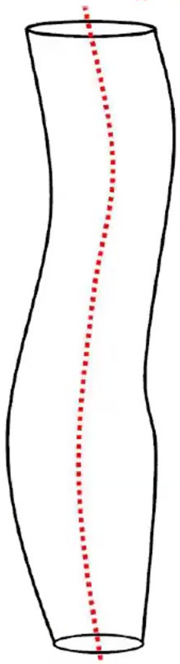
*** June 10, 2021 ***

fnrs LA LIBERTÉ DE CHERCHER UNIVERSITÉ LIBRE DE BRUXELLES ULB

Spinning test-particle in curved spacetime

- **Skeletonization of compact object** + **stress-energy conservation** lead to **Mathisson-Papapetrou (MP) equations**:

$$\left.\begin{aligned} p^\mu &\triangleq \int_{x^0=\text{constant}} \mathrm{d}^3x \sqrt{-g}\, T^{\mu 0} \\ S^{\mu\nu} &\triangleq \int_{x^0=\text{constant}} \mathrm{d}^3x \sqrt{-g}\left(\delta x^\mu T^{\nu 0} - \delta x^\nu T^{\mu 0}\right)\end{aligned}\right\} \overset{\nabla_\mu T^{\mu\nu}=0}{\Longrightarrow} \boxed{\begin{aligned} \frac{\mathrm{D}p^\mu}{\mathrm{d}\lambda} &= -\frac{1}{2} R^\mu{}_{\nu\alpha\beta} v^\nu S^{\alpha\beta} \\ \frac{\mathrm{D}S^{\mu\nu}}{\mathrm{d}\lambda} &= 2p^{[\mu} v^{\nu]}\end{aligned}}$$



- **Tulczyjew SSC**: specify the worldline ($\equiv$ COM seen by an observer of 4-velocity $\propto p^\mu$):

$$\boxed{S^{\mu\nu} p_\mu = 0}$$

It yields $S^{\mu\nu} = S^{\mu\nu}(p^\alpha, S^\alpha) = -\epsilon^{\mu\nu\alpha\beta} \underbrace{\hat{p}_\alpha}_{\triangleq p_\alpha/\mu} S_\beta, \quad v^\mu = v^\mu(p^\alpha, S^\alpha),$

with $\mu \triangleq \sqrt{-p_\alpha p^\alpha}$ + orthogonality condition $p_\mu S^\mu = 0$.

$$\boxed{\Rightarrow \text{7 independent DOFs: 4 } p\text{'s} + 3\ S\text{'s}}$$

Invariant mass μ and spin parameter $\mathcal{S}^2 \triangleq S^\alpha S_\alpha = S^{\alpha\beta} S_{\alpha\beta}/2$ constant along the motion.

- **Linearized EOMs**: astrophysically realistic EMRIs: $\frac{\mathcal{S}}{\mu M} \leq \frac{\mu^2}{\mu M} = \frac{\mu}{M} \ll 1$.

$$\boxed{\Rightarrow \text{Neglect all } \mathcal{O}(\mathcal{S}^2) \text{ corrections}}$$

In particular, we recover $p^\mu = \mu v^\mu + \mathcal{O}(\mathcal{S}^2)$.

```
AIM: study the conserved quantities of this system for a Kerr background
     { 6 known for 7 DOFs + chaos/integrability @ linear order ?  }
```

Construction of invariants: general algorithm

''Symmetry implies conservation'' (textbook view)

vs

''Conservation requires symmetry'' [Rüdiger 1981-83]

- Sketch of the generic procedure applied to **geodesic motion**:
  1. Assume the form of the invariant (polynomial):
  $$C_{\rm K}^{(n)} \triangleq K_{\alpha_1\ldots\alpha_n} p^{\alpha_1} \ldots p^{\alpha_n}$$
  2. Plug it into the conservation equation
  $$\dot{C}(p^\alpha) = 0 \quad \Leftrightarrow \quad v^\mu \nabla_\mu C(p^\alpha) = 0$$
  3. Use EOMs to reduce it
  $$p^\mu \nabla_\mu K_{\alpha_1\ldots\alpha_n} p^{\alpha_1} \ldots p^{\alpha_n} = 0$$
  4. Express it in terms of independent variables (trivial here)
  5. Infer the constraint(s):
  $$\nabla_{(\mu} K_{\alpha_1\ldots\alpha_n)} = 0$$
  6. Solve them !

◇ $n=1$: K_μ Killing vector: $\nabla_{(\mu} K_{\nu)} = 0$

◇ $n=2$: $K_{\mu\nu}$ rank-2 Killing tensor ($\nabla_{(\mu} K_{\nu\rho)} = 0$) or $K_{\mu\nu} = g_{\mu\nu}$ ($C_{\rm g}^{(2)} = -\mu^2$)

◇ $n \geq 3$: $K_{\alpha_1\ldots\alpha_n}$ rank-n Killing tensor

- **Spinning case (MPT equations)**: the S's are not independent, because $P_\mu S^\mu = 0$. Trick: introduce a *relaxed spin vector* s^α:
$$S^\mu \equiv \Pi^\mu_\nu s^\nu \quad \text{with} \quad \Pi^\mu_\nu \triangleq \delta^\mu_\nu + \hat{p}^\mu \hat{p}_\nu$$
- **State-of-the art**:
  - ◇ μ and $\mathcal{S}$ are obviously conserved
  - ◇ For any Killing vector field ξ^α,
  $$\mathcal{C}_\xi = \xi_\mu p^\mu + \frac{1}{2}\nabla_\mu \xi_\nu S^{\mu\nu}$$
  is conserved [Semerak 1999]
  - ◇ In presence of a Killing-Yano tensor $\nabla_{(\alpha} Y_{\beta)\gamma} = 0$, one can build quasi-invariants using the general scheme mentionned above: a **'linear'** one [Rüdiger 1981]
  $$\mathcal{Q}_Y = S^*_{\alpha\beta} Y^{\alpha\beta}$$
  and a **'quadratic'** one [Rüdiger 1983]
  $$\mathcal{Q}_R = -L_\alpha L^\alpha - 2\mu S^\alpha \partial_\alpha \mathcal{Z} - 2\mu^{-1} L_\alpha S^\alpha \xi_\beta p^\beta$$
  $[L_\alpha \triangleq Y_{\alpha\lambda} p^\lambda,\ \mathcal{Z} \triangleq \frac{1}{4} Y^*_{\alpha\beta} Y^{\alpha\beta},\ \xi^\alpha \triangleq -\frac{1}{3}\nabla_\lambda Y^{*\lambda\alpha}]$

  They satisfy $\dot{\mathcal{Q}}_Y = \mathcal{O}(\mathcal{S}^2)$, $\dot{\mathcal{Q}}_R = \mathcal{O}(\mathcal{S}^2)$.

$\Rightarrow$ **In Kerr**: **6** invariants for linearized MPT, recovered by Hamilton-Jacobi [Witzany 2019] . . . but chaos seems only to appear @ $\mathcal{O}(\mathcal{S}^2)$ [Zelenka et al. 2019]

$7^{\rm th}$ invariant ?? Integrability ??

QUADRATIC QUASI-INVARIANTS FOR SPINNING PARTICLES

GOAL: generalize Rüdiger's quadratic solution: **(1)** **assume** an invariant at most linear in the spin

$$\mathcal{Q} \triangleq K_{\mu\nu} p^\mu p^\nu + L_{\mu\nu\rho} S^{\mu\nu} p^\rho$$

(2-3-4) the **conservation equation** becomes

$$\dot{\mathcal{Q}} \propto \mu^3 U_{\mu\nu\rho} \hat{p}^\mu \hat{p}^\nu \hat{p}^\rho + 2\mu^2 {}^*V^\alpha{}_{\mu\nu\rho} s_\alpha \hat{p}^\mu \hat{p}^\nu \hat{p}^\rho + \mathcal{O}(\mathcal{S}^2)$$

$$\left[U_{\alpha\beta\gamma} \triangleq \nabla_\gamma K_{\alpha\beta} - 2L_{\gamma(\alpha\beta)}, \quad V_{\alpha\beta\gamma\delta} \triangleq \nabla_\delta L_{\alpha\beta\gamma} - K_{\lambda\gamma} R^\lambda{}_{\delta\alpha\beta} + \tfrac{2}{3} K_{\lambda\rho} R^\lambda{}_{\delta[\alpha}{}^\rho g_{\beta]\gamma} \right]$$

(5) **Constraints** read:

$$\mathcal{O}(\mathcal{S}^0) : U_{(\mu\nu\rho)} = 0,$$

$$\mathcal{O}(\mathcal{S}^1) : {}^*V^\alpha{}_{(\mu\nu\rho)} = 0,$$

$$\mathcal{O}(\mathcal{S}^2) : \ \ldots$$

(6) **Solution**: at $\mathcal{O}(\mathcal{S}^0)$, $K_{\mu\nu}$ should be a Killing tensor,

$$\nabla_{(\alpha} K_{\beta\gamma)} = 0$$

At $\mathcal{O}(\mathcal{S}^1)$, the situation is more involved. Assuming Ricci-flatness and the existence of a Killing-Yano tensor $Y_{\mu\nu}$ ($K_{\mu\nu} = Y_{\mu\lambda} Y^\lambda{}_\nu$), there exists a solution

$$\boxed{L_{\alpha\beta\gamma} = D^{\text{tf}}_{\alpha\beta\gamma} + \frac{1}{3}\lambda_{\alpha\beta\gamma} + \epsilon_{\alpha\beta\gamma\delta}\left(Y^\delta + \frac{4}{3}\nabla^\delta Z\right)}, \qquad \text{with } \lambda_{\alpha\beta\gamma} \triangleq 2\nabla_{[\alpha} K_{\beta]\gamma}.$$

Provided that $\boxed{\delta^{[\mu}_{(\beta} W^{\nu\rho]}{}_{\gamma;\delta)} = 0,}$ where $W_{\alpha\beta\gamma} \triangleq D^{\text{tf}}_{\alpha\beta\gamma} - \tilde{Y}_{\alpha\beta\gamma}$. $D^{\text{tf}}_{\alpha\beta\gamma} = D^{\text{tf}}_{[\alpha\beta]\gamma}$ is trace-free and $D^{\text{tf}}_{[\alpha\beta\gamma]}$.

- $D^{\text{tf}}_{\alpha\beta\gamma} = 0$, $Y_\alpha = 0$ is Rüdiger invariant $\mathcal{Q}_R$
- Finding a more general solution appears to be more involved (**special thanks to Justin Vines**)...

Current status

- Use a more general Ansatz for $D^{\rm tf}_{\alpha\beta\gamma}$, e.g.

$$
\begin{aligned}
D^{\rm tf}_{\alpha\beta\gamma} = &\Lambda_1\left(\lambda_{\alpha\beta\gamma} + 2g_{\gamma[\beta}\nabla^{\lambda}K_{\alpha]\lambda}\right) + \Lambda_2\left(\xi_{[\alpha}Y^{*}_{\beta]\gamma} - \frac{1}{3}g_{\gamma[\alpha}\xi^{\delta}Y^{*}_{\beta]\delta}\right) + \Lambda_3\left(\xi^{\lambda}\epsilon_{\lambda\delta\gamma[\alpha}Y^{\delta}{}_{\beta]} - \frac{2}{3}\xi^{\lambda}Y^{*}_{\lambda[\alpha}g_{\beta]\gamma}\right)\\
&+\Lambda_4\left(\xi^{\lambda}\epsilon_{\lambda\alpha\beta\delta}Y^{\delta}{}_{\gamma} + \frac{4}{3}\xi^{\lambda}Y^{*}_{\lambda[\alpha}g_{\beta]\gamma}\right) + \Lambda_5\left(R_{\alpha\beta\mu\nu}\xi^{\mu}Y^{*\nu}{}_{\gamma} + g_{\gamma[\alpha}T^{(5)}_{\beta]}\right)\\
&+\Lambda_6\left(R_{\alpha\beta\mu\nu}\xi^{\lambda}\epsilon_{\lambda\delta\gamma}{}^{\mu}Y^{\delta\nu} + g_{\gamma[\alpha}T^{(6)}_{\beta]}\right) + \Lambda_7\left(R_{\alpha\beta\mu\nu}\xi_{\lambda}\epsilon^{\lambda\mu\nu\delta}Y_{\delta\gamma} + g_{\gamma[\alpha}T^{(7)}_{\beta]}\right)\\
&+\Lambda_8\left(\xi^{\mu}R_{\mu\gamma}{}^{\nu}{}_{[\alpha}Y^{*}_{\beta]\nu} + g_{\gamma[\alpha}T^{(8)}_{\beta]}\right) + \Lambda_9\left(\xi^{\mu}R_{\mu}{}^{\nu}{}_{\gamma[\alpha}Y^{*}_{\beta]\nu} + g_{\gamma[\alpha}T^{(9)}_{\beta]}\right)\\
&+\Lambda_{10}\left({}^{*}R_{\alpha\beta\delta\gamma}\nabla^{\delta}\mathcal{Z} + g_{\gamma[\alpha}T^{(10)}_{\beta]}\right) + \Lambda_{11}\left(\xi_{[\alpha}R_{\beta]\gamma\mu\nu}Y^{*\mu\nu} + g_{\gamma[\alpha}T^{(11)}_{\beta]}\right) + \ldots
\end{aligned}
$$

  ...and try to solve the constraint equation in a non-trivial way !
- At this point, our results are still not conclusive (but we suspect at least one non-trivial invariant to exist)
- Several highly non-trivial identities on KY tensor have been derived, e.g.

$$
K_{\lambda(\beta}R^{*\lambda}{}_{\gamma\delta)\alpha} = Y^{\lambda}{}_{(\beta}g_{\gamma\delta)}\xi_{\alpha;\lambda} - g_{\alpha(\beta}Y^{\lambda}{}_{\gamma}\xi_{\delta);\lambda} + \frac{1}{2}Y_{\alpha(\beta}Y^{*\lambda}{}_{\gamma}G_{\delta)\lambda},
$$

  which can have other potential applications !

Current status

- Use a more general Ansatz for $D^{\rm tf}_{\alpha\beta\gamma}$, e.g.

$$D^{\rm tf}_{\alpha\beta\gamma} = \Lambda_1\left(\lambda_{\alpha\beta\gamma} + 2g_{\gamma[\beta}\nabla^{\lambda}K_{\alpha]\lambda}\right) + \Lambda_2\left(\xi_{[\alpha}Y^{*}_{\beta]\gamma} - \frac{1}{3}g_{\gamma[\alpha}\xi^{\delta}Y^{*}_{\beta]\delta}\right) + \Lambda_3\left(\xi^{\lambda}\epsilon_{\lambda\delta\gamma[\alpha}Y^{\delta}{}_{\beta]} - \frac{2}{3}\xi^{\lambda}Y^{*}_{\lambda[\alpha}g_{\beta]\gamma}\right)$$
$$+\Lambda_4\left(\xi^{\lambda}\epsilon_{\lambda\alpha\beta\delta}Y^{\delta}{}_{\gamma} + \frac{4}{3}\xi^{\lambda}Y^{*}_{\lambda[\alpha}g_{\beta]\gamma}\right) + \Lambda_5\left(R_{\alpha\beta\mu\nu}\xi^{\mu}Y^{*\nu}{}_{\gamma} + g_{\gamma[\alpha}T^{(5)}_{\beta]}\right)$$
$$+\Lambda_6\left(R_{\alpha\beta\mu\nu}\xi^{\lambda}\epsilon_{\lambda\delta\gamma}{}^{\mu}Y^{\delta\nu} + g_{\gamma[\alpha}T^{(6)}_{\beta]}\right) + \Lambda_7\left(R_{\alpha\beta\mu\nu}\xi_{\lambda}\epsilon^{\lambda\mu\nu\delta}Y_{\delta\gamma} + g_{\gamma[\alpha}T^{(7)}_{\beta]}\right)$$
$$+\Lambda_8\left(\xi^{\mu}R_{\mu\gamma}{}^{\nu}{}_{[\alpha}Y^{*}_{\beta]\nu} + g_{\gamma[\alpha}T^{(8)}_{\beta]}\right) + \Lambda_9\left(\xi^{\mu}R_{\mu}{}^{\nu}{}_{\gamma[\alpha}Y^{*}_{\beta]\nu} + g_{\gamma[\alpha}T^{(9)}_{\beta]}\right)$$
$$+\Lambda_{10}\left({}^{*}R_{\alpha\beta\delta\gamma}\nabla^{\delta}\mathcal{Z} + g_{\gamma[\alpha}T^{(10)}_{\beta]}\right) + \Lambda_{11}\left(\xi_{[\alpha}R_{\beta]\gamma\mu\nu}Y^{*\mu\nu} + g_{\gamma[\alpha}T^{(11)}_{\beta]}\right) + \dots$$

  ... and try to solve the constraint equation in a non-trivial way !
- At this point, our results are still not conclusive (but we suspect at least one non-trivial invariant to exist)
- Several highly non-trivial identities on KY tensor have been derived, e.g.

$$K_{\lambda(\beta}R^{*\lambda}{}_{\gamma\delta)\alpha} = Y^{\lambda}{}_{(\beta}g_{\gamma\delta)}\xi_{\alpha;\lambda} - g_{\alpha(\beta}Y^{\lambda}{}_{\gamma}\xi_{\delta);\lambda} + \frac{1}{2}Y_{\alpha(\beta}Y^{*\lambda}{}_{\gamma}G_{\delta)\lambda},$$

  which can have other potential applications !

Conclusion and perspectives

What remains to be done

- Find the quasi-invariant(s) !
- Evaluate the quasi-invariants in Kerr spacetime + physical significance
- Involution ? ($\leftrightarrow$ complete integrability ?)

	$\mathcal{S}$	E	ℓ	$\mathcal{Q}_Y$	$\mathcal{Q}_R$	$\mathcal{N}$
μ^2	0	0	0	$\mathcal{O}(\mathcal{S}^2)$	$\mathcal{O}(\mathcal{S}^2)$	$\mathcal{O}(\mathcal{S}^2)$
$\mathcal{S}$		0	0	0	0	0
E			0	0	?	?
ℓ				0	?	?
$\mathcal{Q}_Y$					?	?
$\mathcal{Q}_R$						?

Perspectives

- Chaos/integrability at linear order ?
- Extensions at $\mathcal{O}(\mathcal{S}^2)$ and/or including the quadrupolar moment
- Relation with Grassmann-odd spinning particle [Kubiznak et al. 2011]
- Generalization to Einstein-Maxwell ?
- Relation with Hamilton-Jacobi ? [Witzany 2019] Separation of EOMs ?
- Link with PN (integrability at 2PN) ? [Tanay, Stein and Gálvez Ghersi 2020]
- ...