Title: Hyperboliclike orbits in a two-body system: review of recent results

Speakers: Donato Bini

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Abstract: I will review some recent results obtained for the scattering angle in PN theory, including radiation reaction effects.

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Hyperboliclike orbits in a two-body system: review of recent results

Donato Bini @ IAC.CNR.IT

Based on works done in collaboration with:

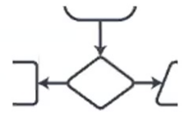
T. Damour, A. Geralico

The 24° Capra meeting, June 10 , 2021

PN, PM, MPM, SF, EOB, EFT, etc.

Capturing information from....

TF has computed the 2-body H up to 6PN



Computation of the nonlocal (tail-related) parts of the EOB potentials (up to 6PN) is easy! Beyond the 6PN the nonlocality becomes technically more involved.

TuttiFrutti

and leading to....

6PN nonlocal part of the two-body Hamiltonian

6PN local part of the two-body Hamiltonian

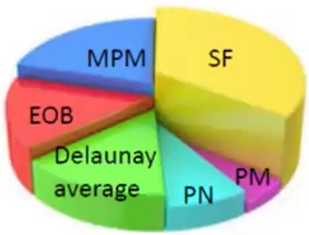
Actually TF approach computes three GI quantities

Hamiltonian averaging→ all the nonlocal part of H is fixed. (Ellipticlike motion)

1SF, Redshift→ all the local linear in ν part of H is fixed. (Ellipticlike motion)

Scattering angle→ all the local part of H is fixed modulo 2 parameters @5PN and 4 more parameters at 6PN. (Hyperboliclike motion)

Key tools:
1st law of the two body dynamics
Mass structure of the scattering angle



Intermediate results: II

Determining the local part of the EOB potential @ O(ν^1)

$$a_{4+5+6\text{PN,loc,f}} = \left[\left(\frac{2275}{512}\pi^2 - \frac{4237}{60}\right)\nu + \left(\frac{41}{32}\pi^2 - \frac{221}{6}\right)\nu^2\right]u^5 + \left[\left(-\frac{1026301}{1575} + \frac{246367}{3072}\pi^2\right)\nu + a_{6,f}^{(\nu)}\right]u^6 + \left[\left(-\frac{2800873}{262144}\pi^4 + \frac{608698367}{1769472}\pi^2 - \frac{1469618167}{907200}\right)\nu + a_{7,f}^{(\nu)}\right]u^7,$$

$$\bar{d}_{4+5+6\text{PN,loc,f}} = \left[\left(\frac{1679}{9} - \frac{23761}{1536}\pi^2\right)\nu + \left(-260 + \frac{123}{16}\pi^2\right)\nu^2\right]u^4 + \left(\frac{331054}{175}\nu - \frac{63707}{512}\nu\pi^2 + \bar{d}_{5,f}^{(\nu)}\right)u^5 + \left[\left(\frac{229504763}{98304}\pi^2 + \frac{135909}{262144}\pi^4 - \frac{99741733409}{6350400}\right)\nu + \bar{d}_{6,f}^{(\nu)}\right]u^6,$$

$$q_{4,4+5+6\text{PN,loc,f}} = \left(20\nu + q_{43}^{(\nu)}\right)u^3 + \left[\left(-\frac{93031}{1536}\pi^2 + \frac{1580641}{3150}\right)\nu + q_{44,f}^{(\nu)}\right]u^4 + \left[\left(\frac{81030481}{65536}\pi^2 - \frac{3492647551}{423360}\right)\nu + q_{45,f}^{(\nu)}\right]u^5,$$

$$q_{6,4+5+6\text{PN,loc,f}} = \left(-\frac{9}{5}\nu + q_{62}^{(\nu)}\right)u^2 + \left(\frac{123}{10}\nu + q_{63,f}^{(\nu)}\right)u^3 + \left[\left(-\frac{9733841}{327680}\pi^2 - \frac{112218283}{294000}\right)\nu + q_{64,f}^{(\nu)}\right]u^4,$$

$$q_{8,5+6\text{PN,loc,f}} = q_{82}(\nu)u^2 + \left(-\frac{7447}{560}\nu + q_{83}^{(\nu)}\right)u^3,$$

$$q_{10,6\text{PN,loc,f}} = q_{10,2}(\nu)u^2.$$

Parametrize the local part of the EOB potential at all orders ν^n (n>1) by using uknown coefficients!

Almost all of them will be determined by using the scattering angle along hyp orbits

What remains....

$$A(u;\nu) \;=\; 1 - 2u + \nu a^{\nu^1}(u) + \nu^2 a^{\nu^2}(u) + \nu^3 a^{\nu^3}(u) + \ldots$$

$$\bar{D}(u;\nu) \;=\; 1 + \nu \bar{d}^{\nu^1}(u) + \nu^2 \bar{d}^{\nu^2}(u) + \nu^3 \bar{d}^{\nu^2}(u) + \ldots$$

Reminder of EOB potentials

List of the f-route EOB potentials in p_r-gauge.

$a_5^{\text{loc,f}}$	$\left(-\frac{4237}{60} + \frac{2275}{512}\pi^2\right)\nu + \left(\frac{41}{32}\pi^2 - \frac{221}{6}\right)\nu^2$
$a_6^{\text{loc,f}}$	$\left(-\frac{1026301}{1575} + \frac{246367}{3072}\pi^2\right)\nu + a_6^{\nu^2}\nu^2 + 4\nu^3$
$a_7^{\text{loc,f}}$	$\left(-\frac{2800873}{262144}\pi^4 + \frac{608698367}{1769472}\pi^2 - \frac{1469618167}{907200}\right)\nu + a_7^{\nu^2}\nu^2 + a_7^{\nu^3}\nu^3$
$\bar{d}_4^{\text{loc,f}}$	$\left(\frac{1679}{9} - \frac{23761}{1536}\pi^2\right)\nu + \left(\frac{123}{16}\pi^2 - 260\right)\nu^2$
$\bar{d}_5^{\text{loc,f}}$	$\left(\frac{331054}{175} - \frac{63707}{512}\pi^2\right)\nu + \bar{d}_5^{\nu^2}\nu^2 + \left(-\frac{205}{16}\pi^2 + \frac{1069}{3}\right)\nu^3$
$\bar{d}_6^{\text{loc,f}}$	$\left(\frac{229504763}{98304}\pi^2 + \frac{135909}{262144}\pi^4 - \frac{99741733409}{6350400}\right)\nu + \bar{d}_6^{\nu^2}\nu^2 + \left(\frac{45089}{72} - \frac{44489}{1536}\pi^2 + \bar{d}_5^{\nu^2} - 15a_6^{\nu^2}\right)\nu^3 - 48\nu^4$
$q_{43}^{\text{loc,f}}$	$20\nu - 83\nu^2 + 10\nu^3$
$q_{44}^{\text{loc,f}}$	$\left(\frac{1580641}{3150} - \frac{93031}{1536}\pi^2\right)\nu + \left(-\frac{2075}{3} + \frac{31633}{512}\pi^2\right)\nu^2 + \left(640 - \frac{615}{32}\pi^2\right)\nu^3$
$q_{45}^{\text{loc,f}}$	$\left(\frac{81030481}{65536}\pi^2 - \frac{3492647551}{423360}\right)\nu + q_{45}^{\nu^2}\nu^2 + \left(-\frac{14}{3}\bar{d}_5^{\nu^2} + \frac{36677}{1152}\pi^2 - \frac{474899}{216}\right)\nu^3 + \left(\frac{1435}{32}\pi^2 - \frac{7375}{6}\right)\nu^4$
$q_{62}^{\text{loc,f}}$	$-\frac{9}{5}\nu - \frac{27}{5}\nu^2 + 6\nu^3$
$q_{63}^{\text{loc,f}}$	$\frac{123}{10}\nu - \frac{69}{5}\nu^2 + 116\nu^3 - 14\nu^4$
$q_{64}^{\text{loc,f}}$	$\left(-\frac{9733841}{327680}\pi^2 - \frac{112218283}{294000}\right)\nu + \left(\frac{156397}{1280}\pi^2 - \frac{21996581}{21000}\right)\nu^2 + \left(\frac{6977}{6} - \frac{29665}{256}\pi^2\right)\nu^3 + \left(\frac{287}{8}\pi^2 - \frac{3640}{3}\right)\nu^4$
$q_{82}^{\text{loc,f}}$	$\frac{6}{7}\nu + \frac{18}{7}\nu^2 + \frac{24}{7}\nu^3 - 6\nu^4$
$q_{83}^{\text{loc,f}}$	$-\frac{7447}{560}\nu - \frac{963}{56}\nu^2 - \frac{117}{10}\nu^3 - 147\nu^4 + 18\nu^5$
$q_{10,2}^{\text{loc,f}}$	$-\frac{11}{21}\nu - \frac{11}{7}\nu^2 - \frac{20}{7}\nu^3 - \frac{5}{3}\nu^4 + 6\nu^5$

Final results

We have determined the local 5PN (first) and 6PN (later) EOB Hamiltonian modulo

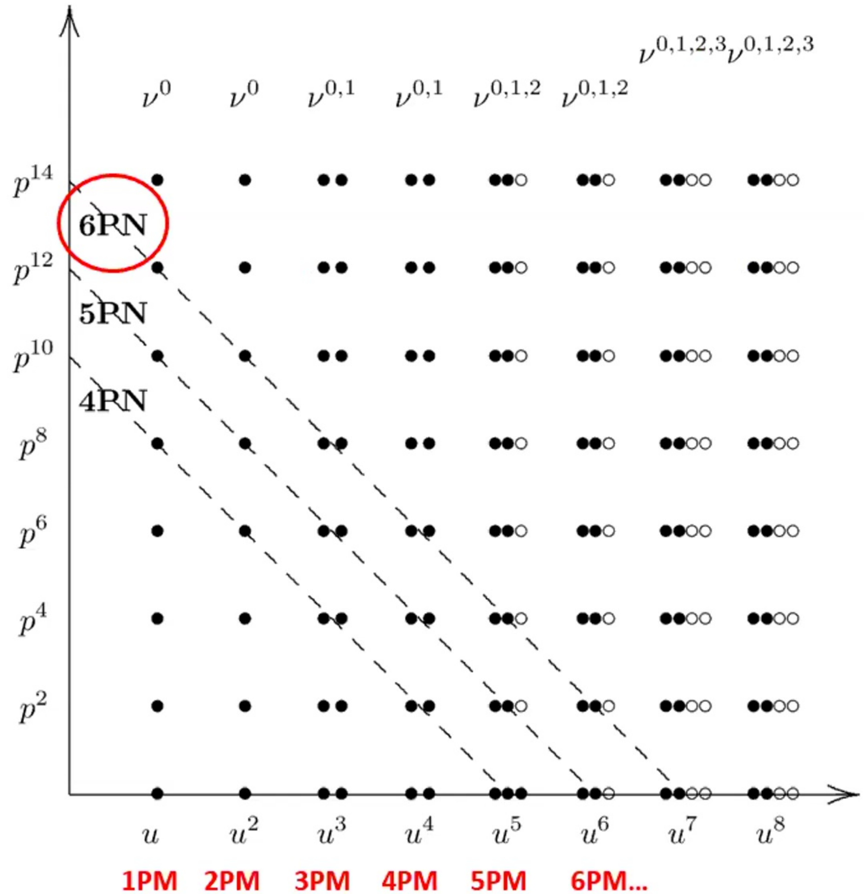
2 unknown at 5PN and 4 more unknown at 6PN.

Note that thinking of all possible terms entering the 6PN real Hamiltonian, we have determined 147 terms of the 151 total number!

@5PN instead the real Hamiltonian contains 97 coefficients of which we have determined 95!

The missing terms are second order in ν in the EOB potentials: 1SF cannot help!
One needs 2SF which is the next challenge in SF

What is needed to complete the 2 body dynamics at a fixed PM level



Each vertical column of dots describes the PN expansion (keyed by powers of p^2) of an energy-dependent function parametrizing the scattering angle.

The various columns at a given PM level correspond to increasing powers of ν.

D. Bini, T. Damour and A.Geralico,
``Sixth post-Newtonian local-in-time dynamics of binary systems,"
Phys. Rev. D 102, no.2, 024061 (2020)
[arXiv:2004.05407 [gr-qc]].

Schematic representation of the irreducible information contained, at each PM level (keyed by a power of u = GM/r), in the local dynamics.

The next challenge(s)…

→ Improve the present knowledge of the nonlocal part of χ (N^nLO).

This implies the evaluation of the nl part of the Hamiltonian averaged along hyperbolic orbits (that is of the flux-split integrals expanded in large eccentricity) and can be done both in the time domain and in the frequency domain (with difficulties in both cases…)

→ Compute rr contributions to χ (presently @ 2PN)

This implies the use of the rr χ expression for time sym scattering obtained in 2012

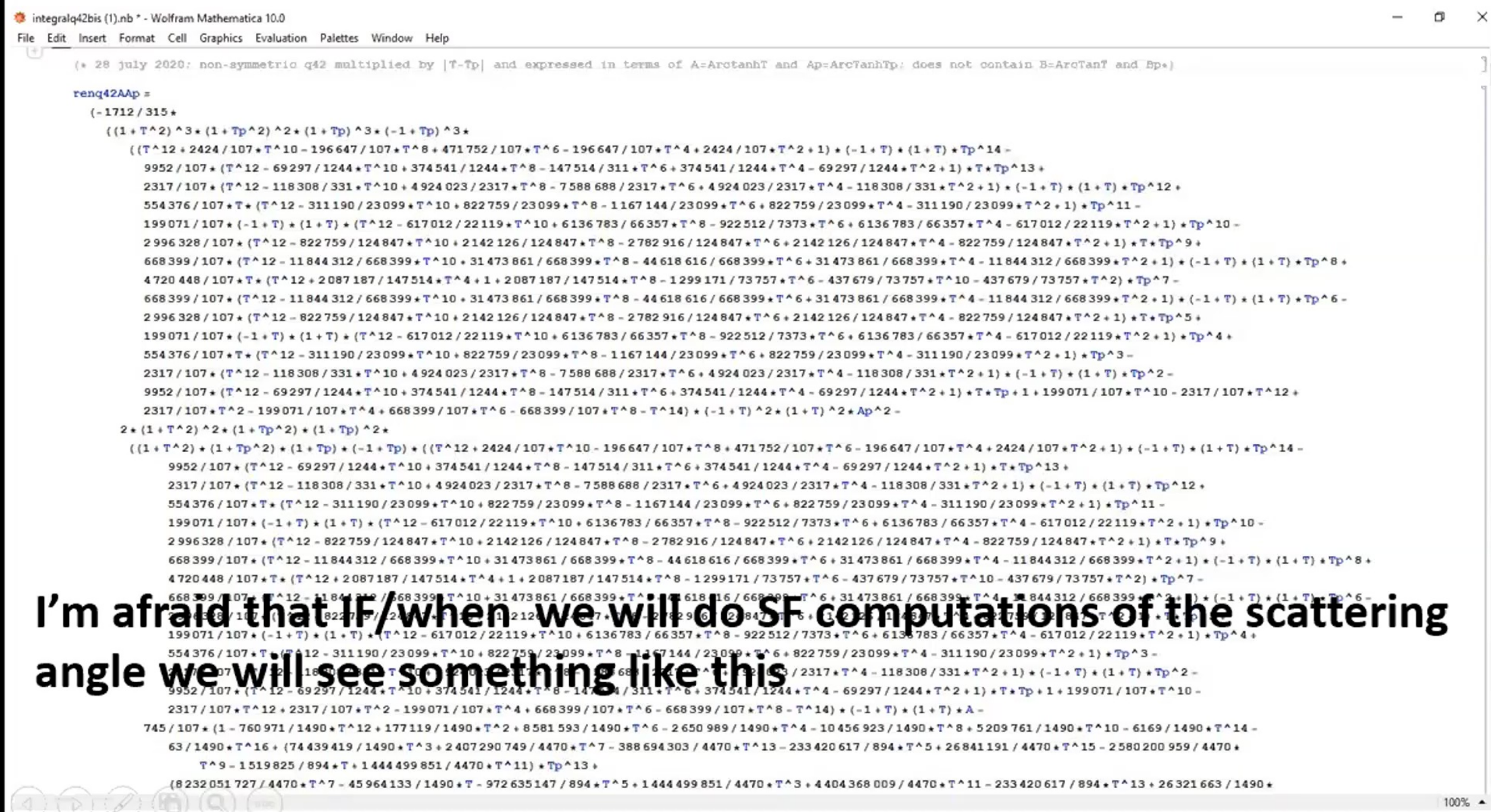
PHYSICAL REVIEW D **86**, 124012 (2012) BD and T. Damour

Gravitational radiation reaction along general orbits in the effective one-body formalism

(N^nLO) Time domain computations

One needs to compute flux split integrals (with a singularity line). Expand in large eccentricity and work in the time domain.

$$\int_{[-1,1]\times[-1,1]} dT\,dT' \frac{\mathcal{G}(T,T')}{|T-T'|},$$



I'm afraid that IF/when we will do SF computations of the scattering angle we will see something like this

(N^nLO) Time domain computations

The arising integrals correspond to very large expressions (numerical computation and then recovery of the analytical results by using the PSLQ algorithm; analytical computations in terms of harmonic polylogs).

We reached the NNNLO level, $O(G^7)$ of accuracy (in the expansion in large eccentricity).

At NNLO we found large integrands , difficult to evaluate even numerically (fruitful collaboration with HEP people: **S. Laporta** and **P. Mastrolia**).

Q_{20}	524.767292180212584342735955703101758476141999557369011937728711238498839830097712093907037158196060831706238995205677052067946783744966475134730111010455883184170170829347212071124106113165 8613485679

PSLQ

Q_{20}	$\frac{25883}{1800} + \frac{22333}{140}\mathrm{K} - \frac{625463}{3360}\pi - \frac{361911}{560}\pi \ln 2 + \frac{99837}{160}\pi\zeta(3)$

Noticeably, in the intermediate results the Catalan constant and $\zeta(3)$ enter (enriching the transcendental structure of the result). However, the first cancels out in the scattering angle while BOTH cancel out in the periastron advance, leading to a rational coefficient only (at NNLO).

Frequency domain computations

Expressions involving BesselK Functions, with the integration variable appearing either in the argument or in the order.

$$\begin{aligned}\mathcal{K}_{\rm N}(u) &= -\frac{32}{5}\frac{\nu^2}{\bar{a}_r^2}\frac{p^2}{u^4}e^{-i\pi p}\left\{u^2(p^2+u^2+1)(p^2+u^2)K_{p+1}^2(u)\right.\\ &-2u\left[\left(p-\frac{3}{2}\right)u^2+p(p-1)^2\right](p^2+u^2)K_p(u)K_{p+1}(u)\\ &\left.+2\left[\frac{1}{2}u^6+\left(2p^2-\frac{3}{2}p+\frac{1}{6}\right)u^4+\left(\frac{5}{2}p^4-\frac{7}{2}p^3+p^2\right)u^2+p^4(p-1)^2\right]K_p^2(u)\right\}\end{aligned}$$

Expansion in large eccentricity involves derivatives with respect to the order [unknown beyond the third order].

$$K_p(u) = K_0(u) + \frac{1}{2}p^2\frac{\partial^2 K_\nu(u)}{\partial\nu^2}\bigg|_{\nu=0}$$

Various tricks
Use of various integral transform, like the Mellin transform.

$$\begin{aligned}\int_0^\infty du\mathcal{I}_{\rm 1PN}^{\rm NLO}(u) &= -\frac{96}{5\pi}\int_{-\infty}^\infty dv\arctan\left(\tanh\frac{v}{2}\right)\cosh v\left[\sinh v(g_{K_0\cos}(5;v)+2g_{K_1\cos}(6;v))\right.\\ &\left.+(\cosh^2 v-2)(g_{K_0\sin}(6;v)+g_{K_1\sin}(5;v))\right]\\ &= \int dv\arctan\left(\tanh\left(\frac{v}{2}\right)\right)\frac{\sinh v}{\cosh^4 v}\left(-\frac{4032}{5}+\frac{2448}{\cosh^2 v}\right)\\ &= \frac{2048}{25},\end{aligned}$$

$$g(s) \equiv \mathfrak{M}\{f(x);s\} = \int_0^\infty x^{s-1}f(x)dx$$

$$\frac{dg}{ds}\bigg|_{s=1} = \int_0^\infty dx f(x)\ln(x)$$

The status of the art

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http://dx.doi.org/10.1080/10652469.2016.1164156

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Higher derivatives of the Bessel functions with respect to the order

Yu. A. Brychkov

$$
\begin{aligned}
\left.\frac{\partial^2 K_\nu(z)}{\partial \nu^2}\right|_{\nu=n} &= \left.\frac{\partial^2 K_\nu(z)}{\partial \nu^2}\right|_{\nu=-n} = \frac{1}{2}[\ln z - \ln(-z)] \\
&\times \left[\ln z + \ln\left(-\frac{z}{4}\right) + 2\mathrm{C}\right] K_n(z) + \frac{n!}{2}\left(\frac{z}{2}\right)^{-n} \sum_{k=0}^{n-1} \frac{(z/2)^k}{k!(n-k)} \\
&\times \{\psi(n-k) - [\ln z - \ln(-z)] + \mathrm{C}\} K_k(z) + \frac{z}{4}\sum_{k=0}^{n}\binom{n}{k} \\
&\times \frac{(\frac{3}{2})_k}{[(k+1)!(k+1)]^2}(-2z)^k[\ln z - \ln(-z)][zK_{n-k}(z) - 2(n-k)K_{n-k-1}(z)] \\
&\times {}_3F_4\left(\begin{matrix} k+1, k+1, k+\frac{3}{2}; z^2 \\ k+2, k+2, k+2, k+2 \end{matrix}\right) \\
&- (-1)^n \frac{\pi^{3/2}}{2}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{z}{2}\right)^{-k} I_{n-k}(z) G^{4,1}_{4,6}\left(-z^2 \,\middle|\, \begin{matrix} 0, \frac{1}{2}, \frac{1}{2}, 1 \\ 0, 0, 0, 0, \frac{1}{2}, k \end{matrix}\right) \\
&- \frac{\sqrt{\pi}}{2}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{z}{2}\right)^{-k}\{(-1)^k K_{n-k}(z) + (-1)^n[\ln z - \ln(-z)] I_{n-k}(z)\} \\
&\times G^{3,1}_{3,5}\left(-z^2 \,\middle|\, \begin{matrix} \frac{1}{2}, -\frac{1}{2}, 1 \\ 0, 0, 0, -\frac{1}{2}, k \end{matrix}\right). \qquad (7.3)
\end{aligned}
$$

There are some followups to the work of Brychkov on the n-derivative of the Bessel functions by L.Gonzalez-Santander: https://arxiv.org/abs/1711.06849, https://arxiv.org/pdf/1808.05608 .
In particular the first one has some analytical closed-form results and the second one has an efficient numerical technique to evaluate them (in case one uses PSLQ for fitting numerical results).

Personal communication by R. Gozzo

MejerG functions

Radiation-Reaction?

B-D 2012

$$\delta^{(\mathrm{RR})}\chi = \frac{1}{2}\left(\frac{\partial \chi^{(\mathrm{conserv})}(\tilde{E}, j)}{\partial \tilde{E}}\delta^{(\mathrm{RR})}E + \frac{\partial \chi^{(\mathrm{conserv})}(\tilde{E}, j)}{\partial j}\delta^{(\mathrm{RR})}j\right)$$

Known at 2PN
and recently used @ 3PM by T. Damour to solve the paradoxical situation concerning the HE limit of χ.

Obtained in the case of first order rr effects, and in absence of CM total momentum recoil

Passing from the conservative case to the case of presence of rr one uses the parametric eqs of the orbit and a variation of the constants method. Actually the constants which vary are the energy, the angular momentum and the angular dispacement of the apsidal line. The latter, however, does not contribute due to symmetry reasons up to the level in which the above mentioned limitations apply.

$$\dot{x}^i = \frac{\partial \mathcal{H}(\mathbf{x}, \mathbf{p})}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial \mathcal{H}(\mathbf{x}, \mathbf{p})}{\partial x^i} + \mathcal{F}_i$$

In EOB rr force modifies Hamilton eqs.

The quantity $c_\lambda(t)$ corresponds in our above simplified treatment to the direction of the vector $\mathbf{A}(t)$. We found above that the direction of $\mathbf{A}(t)$ did not include a secular change under the influence of $\mathcal{F}$, because of symmetry reasons linked, finally, to the time-odd character of $\mathcal{F}$. This fact has a correspondent in $c_\lambda(t)$. Indeed, Ref. [61] found that there were no secular changes in $c_\lambda(t)$ [and $c_\ell(t)$] precisely because $dc_\lambda(t)/dt$ is an odd function of ϕ, around the periastron, and remarked that this was linked to the time-odd character of $\mathcal{F}$.

We are currently generalizing this result!

Blumlein, Maier, Marquard, Schaefer et al 2020

Complete agreement with the recent work by J. Blumlein, A. Maier, P. Marquard, G. Schafer, arXiv: 2010.13672 @5PN.

See Eqs. 63-64 there, with the determination of the π^2 part of the 5PN missing coefficients in TF.

$$\bar{d}_5 = r_{\bar{d}_5} + \frac{306545}{512}\pi^2$$

$$a_6 = r_{a_6} + \frac{25911}{256}\pi^2$$

Conclusions and plans for future works

- Can TF method be applied to reach 7PN? → In principle yes...but one finds 6 more undetermined parameters..2+4+6
- Push forward the knowledge of the nonlocal in time dynamics and obtain more information about the nonlocal part of the scattering angle...
- 5PN h-coordinate-based potential Hamiltonian [See the conservative radiation Foffa-Sturani pioneering works, as well as the recent works by J. Blumlein, et al.]
- Scattering angle from 1SF? [L. Barack et al. @Capra 2020, S. Hopper, 2018, O. Long, L. Barack: e-Print: 2105.05630 [gr-qc]]
- Full scattering angle @ 4PM from amplitudes, including rr effects? [Potential graviton contributions recently obtained by Z. Bern et al., 2021. How to include rad gravitons?] New and promising avenues from amplitudes [G. Veneziano et al., R. Porto et al., J. Parra Martinez et al., F. Vernizzi et al., etc.]
- Extension of results to spinning bodies or in general to bodies with a given multipolar structure [See e.g. A. Buonanno, J. Steinhoff, J. Vines, etc.]

Thanks for your kind attention!