Title: Â Lorenz-gauge reconstruction for Teukolsky solutions with sources in electromagnetism

Speakers: Stephen Green

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Abstract: Reconstructing a metric or vector potential that corresponds to a given solution to the Teukolsky equation is an important problem for self-force calculations. Traditional reconstruction algorithms do not work in the presence of sources, and they give rise to solutions in a radiation gauge. In the electromagnetic case, however, Dolan (2019) and Wardell and Kavanagh (2020) very recently showed how to reconstruct a vector potential in Lorenz gauge, which is more convenient for self-force. Their algorithm is based on a new Hertz-potential 2-form. In this talk, I will first show that the electromagnetic Teukolsky formalism takes a simplified form when expressed in terms of differential forms and the exterior calculus. This formalism makes the new Lorenz-gauge construction much more transparent, and it enables an extension to nonzero sources. In particular, I will derive a corrector term, related to the charge current, which when added to the vector potential gives a solution to the Maxwell equations with nonzero source. I will conclude by discussing prospects for extending to the gravitational case.

Lorenz-gauge reconstruction in electromagnetism with sources

Stephen R. Green
Albert Einstein Institute Postdam

with V. Toomani, S. Hollands, …

Capra Meeting
Perimeter Institute / online
June 9, 2021

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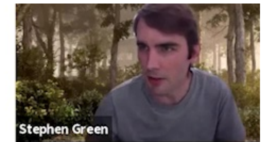

Status of field reconstruction in Kerr

- Starting from Teukolsky solutions, use a Hertz potential to generate corresponding A_a or h_{ab}.

Maxwell

	Radiation gauge	Lorenz gauge
no sources		
sources		

Einstein

	Radiation gauge	Lorenz gauge
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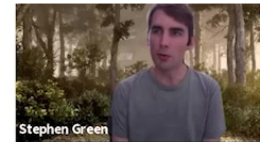
sources		

2



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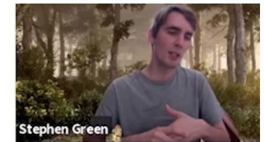

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- Starting from Teukolsky solutions, use a Hertz potential to generate corresponding A_a or h_{ab}.

Maxwell

	Radiation gauge	Lorenz gauge
no sources	✓ CCK (1970s)	✓ Dolan (2019), Wardell, Kavanagh (2020)
sources	✓ Toomani, Hollands (2021)	This talk

Einstein

	Radiation gauge	Lorenz gauge
no sources	✓ CCK (1970s)	✗
sources	✓ Green, Hollands, Zimmerman (2020)	✗

2

Preliminaries: exterior calculus

- Differential p-form $\omega_{a_1 \ldots a_p} = \omega_{[a_1 \ldots a_p]}$
    - totally antisymmetric
- Wedge product $(\alpha \wedge \beta)_{ab} = 2\alpha_{[a}\beta_{b]}$
    - antisymmetrized product
- Exterior derivative $(d\omega)_{ba_1 \ldots a_p} = (p+1)\nabla_{[b}\omega_{a_1 \ldots a_p]}$
    - antisymmerized derivative; $d^2 = 0$
- Hodge dual $(*\alpha)_{abc} = \epsilon_{abc}{}^{d}\alpha_d$
    - $** = \pm 1$
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- **Electromagnetism**
  - 2-form field F_{ab}
  - Maxwell equations

$$dF = 0$$

$$\delta F = j$$

  - Maxwell potential 1-form A_a

$$F = dA$$

Automatically $dF = d^2A = 0$

3

Bivector basis

- Decompose F_{ab}

Maxwell scalars

$$F = \phi_0 X + \phi_1 Y + \phi_2 Z + \text{c.c.}$$

$\bar{m} \wedge n$

$n \wedge l - \bar{m} \wedge m$

$l \wedge m$

$(l, n, \bar{m}, m)$ is NP tetrad 

- X, Y, Z are "anti-self-dual": $\quad * X = iX$

- $\bar{X}, \bar{Y}, \bar{Z}$ are "self-dual": $\quad * \bar{X} = - i\bar{X}$

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Exterior calculus and Teukolsky

- At level of forms, Teukolsky decoupling operator is simply $\mathcal{S} \equiv \zeta^{-2} d \zeta^{2}$,

$\zeta = \Psi_2^{-1/3}$

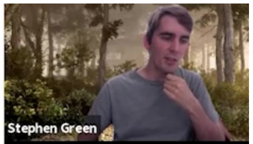
Teukolsky equations

Fackerell-Ipser equation

$$
\begin{aligned}
\zeta^{-2} d\zeta^{2} \delta F^{A} = {} & X\left[\frac{1}{2}\mathcal{O}(\phi_0)\right] \\
& + Y\left[\frac{1}{2}\Theta^{a}\Theta_{a}\phi_1 + (B+B')\cdot\Theta\phi_1 + 2(B\cdot B')\phi_1 + 2\Psi_2\phi_1\right] \\
& + Z\left[\frac{1}{2}\mathcal{O}'(\phi_2)\right] \\
& - \bar{X}\zeta^{-2}\left\{\eth'^{2}(\zeta^{2}\phi_0) + \text{þ}^{2}(\zeta^{2}\phi_2) - 2[\text{þ}\,\eth' - \tau'\,\text{þ} + (\text{þ}\,\tau') - (\eth'\,\rho)](\zeta^{2}\phi_1)\right\} \\
& - \bar{Y}\zeta^{-2}\left\{(\text{þ}'\,\eth' + \bar{\tau}\,\text{þ}')(\zeta^{2}\phi_0) + (\text{þ}\,\eth + \bar{\tau}'\,\text{þ})(\zeta^{2}\phi_2) - [\eth\,\eth' + \text{þ}\,\text{þ}' + \ldots](\zeta^{2}\phi_1)\right\} \\
& - \bar{Z}\zeta^{-2}\left\{\text{þ}'^{2}(\zeta^{2}\phi_0) + \eth^{2}(\zeta^{2}\phi_2) - 2[\text{þ}'\,\eth + \ldots](\zeta^{2}\phi_1)\right\}
\end{aligned}
$$

(related to)
Teukolsky-Starobinski identities

5



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$\mathcal{O}$

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Exterior calculus and Teukolsky

- With $F = dA \equiv \mathscr{T}A$, the sourced Maxwell equation is

$$j = \delta F = \delta d A \equiv \mathscr{E} A$$

- In terms of operators on forms, the (electromagnetic) Wald identity is trivial:

$$\begin{array}{ccccccc} \zeta^{-2} d \zeta^{2} & \circ & \delta d & = & \zeta^{-2} d \zeta^{2} \delta & \circ & d \\ \mathscr{S} & & \mathscr{E} & & \mathscr{O} & & \mathscr{T} \end{array}$$

- This (and its adjoint) can now be used in reconstruction.

Lorenz-gauge reconstruction

- Source-free case (Dolan, Wardell, Kavanagh) easy using exterior calculus:
  - Hertz 2-form $\mathcal{H} = \Phi_0 X + \Phi_2 Z$

$$\longrightarrow \quad A = \delta\zeta\mathcal{H}$$

  - Automatically Lorenz gauge $\delta A = \delta^2\zeta\mathcal{H} = 0$
  - Field strength

$$\begin{aligned} F &= dA \\ &= d\delta\zeta\mathcal{H} \\ &= \underbrace{\zeta^{-1} d\zeta^2\delta\mathcal{H}}_{\zeta\mathcal{O}} + \underbrace{X \text{Ł}_\xi \Phi_0 - Z \text{Ł}_\xi \Phi_2 + Y(C' \cdot \Theta\Phi_0 + C \cdot \Theta\Phi_2)}_{\text{anti-self-dual}} \end{aligned}$$

GHP-Lie derivative

Lorenz-gauge reconstruction

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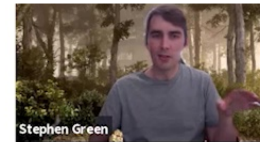
&= \underbrace{\zeta^{-1} d\zeta^2}_{\zeta\mathcal{O}} \delta\mathcal{H} + \underbrace{X \text{Ł}_\xi \Phi_0 - Z \text{Ł}_\xi \Phi_2 + Y (C' \cdot \Theta\Phi_0 + C \cdot \Theta\Phi_2)}_{\text{anti-self-dual}}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{O}\mathcal{H} = {} & \frac{1}{2} X \mathcal{O}(\Phi_0) + \frac{1}{2} Z \mathcal{O}'(\Phi_2) \\
& - \bar{X} \zeta^{-2} \left[\text{ð}'^2 (\zeta^2 \Phi_0) + \text{þ}^2 (\zeta^2 \Phi_2) \right] - \bar{Z} \zeta^{-2} \left[\text{þ}'^2 (\zeta^2 \Phi_0) + \text{ð}^2 (\zeta^2 \Phi_2) \right] \\
& - \bar{Y} \zeta^{-2} \left[(\text{þ}' \, \text{ð}' + \bar{\tau} \, \text{þ}') (\zeta^2 \Phi_0) + (\text{þ} \, \text{ð} + \bar{\tau}' \, \text{þ}) (\zeta^2 \Phi_2) \right]
\end{aligned}
$$



Lorenz-gauge reconstruction

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$\zeta\mathcal{O}$

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\end{aligned}
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- If $(\Phi_0, -\Phi_2)$ satisfy TSI, then self-dual part of $\mathcal{O}\mathcal{H}$ vanishes, and F is anti-self-dual. Thus, $F = -i * F = -i * dA$, and

$$\delta F = *d*F = i d^2 A = 0 \quad \longrightarrow \quad \text{solution}$$

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\begin{aligned}
\mathcal{O}\mathcal{H} = \frac{1}{2}X\mathcal{O}(\Phi_0) + \frac{1}{2}Z\mathcal{O}'(\Phi_2) \\
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- Consistent with inversion relations $\text{Ł}_\xi\Phi_0 = \phi_0$

$$\text{Ł}_\xi\Phi_2 = -\phi_2$$

Inclusion of sources

- Suppose we have a Maxwell solution (F, j).

- Hertz potential stays the same

$$\mathcal{H} = \Phi_0 X + \Phi_2 Z$$

- Potential picks up a corrector 1-form

$$A = \delta\zeta\mathcal{H} + G$$

- Inversion relations

$$\text{Ł}_\xi \Phi_0 = \phi_0$$
$$\text{Ł}_\xi \Phi_2 = -\phi_2$$

$$\text{Ł}_\xi G_n = \zeta j_n$$
$$\text{Ł}_\xi G_l = -\zeta j_l$$
$$\text{Ł}_\xi G_m = -\zeta j_m$$
$$\text{Ł}_\xi G_{\bar{m}} = \zeta j_{\bar{m}}$$

- Similar (but more involved) calculation to source-free case.

Conclusions

- Nonzero charge current j can be included as a source for Lorenz gauge reconstruction in electromagnetism.
  - Corrector is related to j via time derivatives.
- Differential forms notation very useful in calculations.
  - **Next steps:** Other Lorenz gauge constructions. (Barry's talk)
  - **Next steps:** Gravitational case. Can the formalism be adapted? Take bivector components of Weyl tensor?
  - **Next steps:** Point particle. Reconstruct field.

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  - **Next steps:** Point particle. Reconstruct field.
- If successful, this should greatly simplify self-force calculations, by enabling use of Teukolsky equation in producing Lorenz-gauge metric perturbations.