Title: Transforming to the highly regular gauge for use in second-order self-force calculations

Speakers: Samuel Upton

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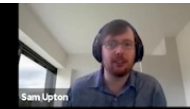
Abstract: With the publication of the first second-order self-force results, it has become even more clear of the need for fast and efficient calculations to avoid the computational expense encountered when using current methods in the Lorenz gauge. One ingredient for efficient calculation of second-order self-force data will be the use of the highly regular gauge (1703.02836 and 2101.11409) with its weaker divergences along the worldline of the small object. In this talk, we will present steps towards transforming the current Lorenz gauge data into the highly regular gauge to be used for quasicircular orbits in Schwarzschild spacetime. The end result will be a source that can be used as an input into the second-order Einstein equations (see talks by Andrew Spiers and Benjamin Leather). In particular, this will allow us to solve the second-order Teukolsky equation using a point particle source instead of requiring the use of a puncture scheme.



Transforming to the highly regular gauge for use in second-order self-force calculations

Samuel Upton
Supervisor: Adam Pound
With: Leanne Durkan, Benjamin Leather, Andrew Spiers, Niels Warburton and Barry Wardell

School of Mathematical Sciences
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Capra 24
June 2021

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Introduction

- Second-order self-force crucial for EMRI detection with LISA
- One difficult aspect is the strong divergence of the second-order singular field, $h^{\mathrm{S2}}_{\mu\nu}$, near the worldline of the small object $\sim m^2/r^2$
  - Appears in a generic gauge, including Lorenz gauge
- In the highly regular gauge, however, $h^{\mathrm{S2}}_{\mu\nu} \sim m^2/r$ [Pound, 2017, arXiv:1703.02836; SDU & Pound, 2021, arXiv:2101.11409]
  - Weaker divergence allows distributional definition of second-order EFEs
  - This allows definition of the stress-energy tensor through second order – the *Detweiler stress-energy tensor*
- We want to transform existing Lorenz gauge data for quasicircular orbits in Schwarzschild [Pound et al., 2020, arXiv:1908.07419; Warburton's talk] into the highly regular gauge to take advantage of these benefits
- This will be used as input into the second-order Teukolsky equation, as discussed by Spiers & Leather

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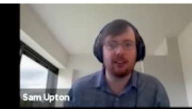

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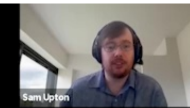


Benefits of the Highly Regular Gauge

- Allows the second-order EFEs to be well-defined as distributions as most singular term in source $\delta^2 G^{\mu\nu}[h^{\mathrm{S1}}, h^{\mathrm{S1}}] \sim m^2/r^2$
  - In a generic gauge, including Lorenz, $\delta^2 G^{\mu\nu}[h^{\mathrm{S1}}, h^{\mathrm{S1}}] \sim m^2/r^4$ which is 'trickier' to handle
- This gives a well-defined second-order stress-energy tensor

$$T^{\mu\nu}_{2,\mathrm{HR}} = -\frac{m}{2}\int u^\mu u^\nu \left(g^{\alpha\beta} - u^\alpha u^\beta\right) h^{\mathrm{R1}}_{\alpha\beta}\, \delta^4(x,z)\, d\tau$$

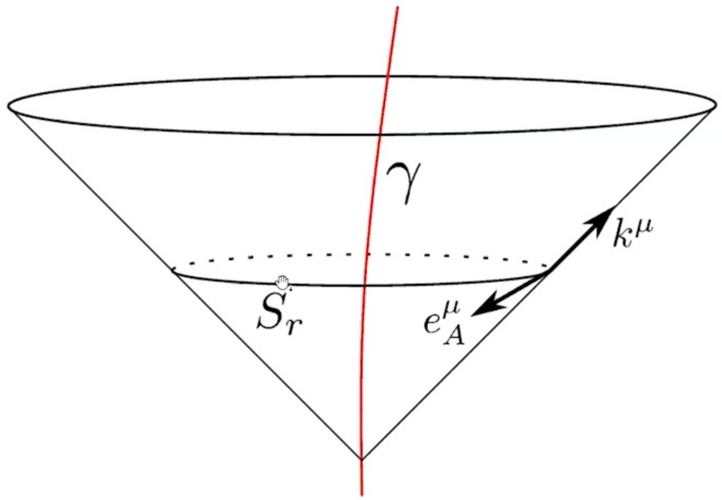
whose form is valid under smooth gauge transformation just with a change of the regular field

  - Can be combined with $T_1^{\mu\nu}$ to form the *Detweiler stress-energy tensor*

$$\epsilon T_1^{\mu\nu} + \epsilon^2 T_2^{\mu\nu} = \epsilon m \int_\gamma \tilde{u}^\mu \tilde{u}^\nu \frac{\delta^4(x-z)}{\sqrt{-\tilde{g}}}\, d\tilde{\tau} + \mathcal{O}\left(\epsilon^3\right)$$

  - This is the stress-energy tensor of a point particle in $\tilde{g}_{\mu\nu} = g_{\mu\nu} + h^{\mathrm{R}}_{\mu\nu}$ (confirming conjecture in [Detweiler, 2012, arXiv:1107.2098])

Gauge Conditions for the Highly Regular Gauge



Based on image by Adam Pound

- Highly regular gauge conditions are

$$h^{\mathrm{HR}}_{\mu\nu} k^\nu = 0 \qquad h^{\mathrm{HR}}_{\mu\nu} e^\mu_A e^\nu_B \Omega^{AB} = 0$$

where k^μ is a future-directed null vector and Ω_{AB} is metric on S^2

- 'Inherited' from lightcone gauge it is constructed from [Poisson, 2005, arXiv:gr-qc/0501032]

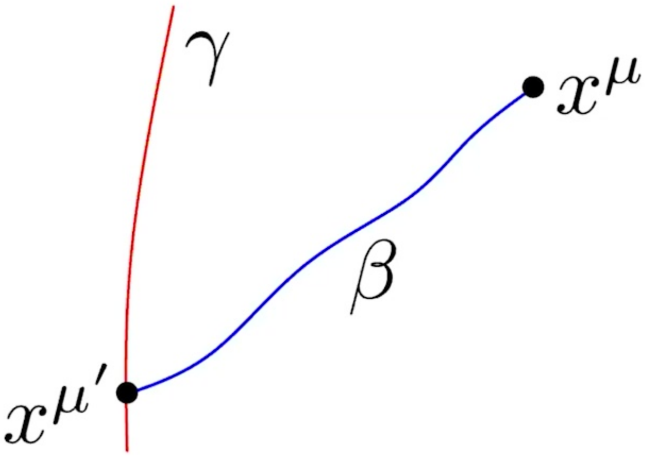
From Lorenz to Highly Regular Gauge

- To take advantage of the highly regular gauge, we need to transform the first-order singular field from the Lorenz gauge
- We use the standard transformation law for a first-order quantity

$$h_{\mu\nu}^{\mathrm{S1,HR}} = h_{\mu\nu}^{\mathrm{S1,Lor}} + \mathcal{L}_{\xi_1} g_{\mu\nu}$$

  where the covariant form of $h_{\mu\nu}^{\mathrm{S1,Lor}}$ is given in [Heffernan et al., 2012, arXiv:1204.0794; Pound & Miller, 2014, arXiv:1403.1843]
- Take the RHS and impose the gauge conditions from previous slide while expanding in distance from worldline, λ

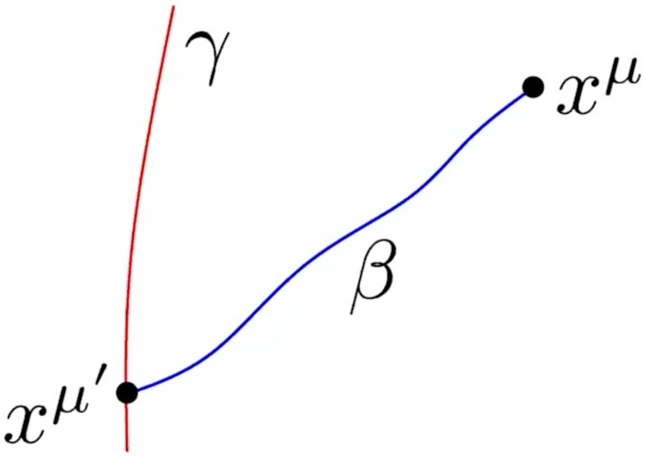
Gauge Vector for Transformation



- In collaboration with Spiers, we find the gauge vector is given by

$$\xi_\mu = g_\mu{}^{\mu'}\left[2M\ln(\rho)u_{\mu'} + \frac{M}{\rho}P_{\mu'\nu'}\sigma^{\nu'}\right] + \mathcal{O}(\lambda^2)$$

Gauge Vector for Transformation



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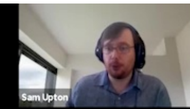
$$\xi_\mu = g_\mu{}^{\mu'}\left[2M\ln(\rho)u_{\mu'} + \frac{M}{\rho}P_{\mu'\nu'}\sigma^{\nu'}\right] + \mathcal{O}(\lambda^2)$$

- Perform a coordinate expansion in terms of $\Delta x^{\mu'}$

$$\begin{aligned}\xi_\mu = {} & \left(2M\ln(\rho_0)u_{\mu'} - \frac{M}{\rho_0}P_{\mu'\nu'}\Delta x^{\nu'}\right) + \lambda\Big(2M\ln(\rho_0)\Gamma^{\alpha'}_{\mu'\nu'}\Delta x^{\nu'}u_{\alpha'} \\ & - \frac{M}{\rho_0}\Gamma^{\alpha'}_{\mu'\beta'}\Delta x^{\beta'}\Delta x^{\nu'}P_{\alpha'\nu'} + \frac{M}{2\rho_0^3}\Gamma^{\gamma'}_{\alpha'\beta'}\Delta x^{\alpha'}\Delta x^{\beta'}\Delta x^{\nu'}\Delta x^{\iota'}P_{\mu'\nu'}P_{\gamma'\iota'} \\ & + \frac{M}{\rho_0^2}\Gamma^{\iota'}_{\alpha'\beta'}\Delta x^{\alpha'}\Delta x^{\beta'}\Delta x^{\nu'}P_{\iota'\nu'}u_{\mu'} - \frac{M}{2\rho_0}\Gamma^{\nu'}_{\alpha'\beta'}\Delta x^{\alpha'}\Delta x^{\beta'}P_{\mu'\nu'}\Big) + \mathcal{O}(\lambda^2)\end{aligned}$$



Mode Decomposition of Gauge Vector

- To perform the mode decomposition we follow [Heffernan et al., 2012, arXiv:1204.0794; Warburton & Wardell, 2014, arXiv:1311.3104; Wardell & Warburton, 2015, arXiv:1505.07841]
- Introduce a rotated coordinate system, $(\theta, \phi) \to (\alpha, \beta)$, so that small object is always at north pole
  - All but lowest m modes vanish when evaluated here
- Near north pole adopt a quasi-Cartesian coordinate system, (w_1, w_2), calculate gauge vector components here and then transform back to (α, β) to perform mode decomposition
  - The (w_1, w_2) coordinates avoid the coordinate singularity in (α, β) at $\alpha = 0$
- We combine this with the mode decomposed first-order singular field provided in [Wardell & Warburton, 2015, arXiv:1505.07841] to find $h_{\mu\nu}^{\mathrm{S1},\mathrm{HR},\ell m}$

Scalar and Vector Sector

- Scalar sector

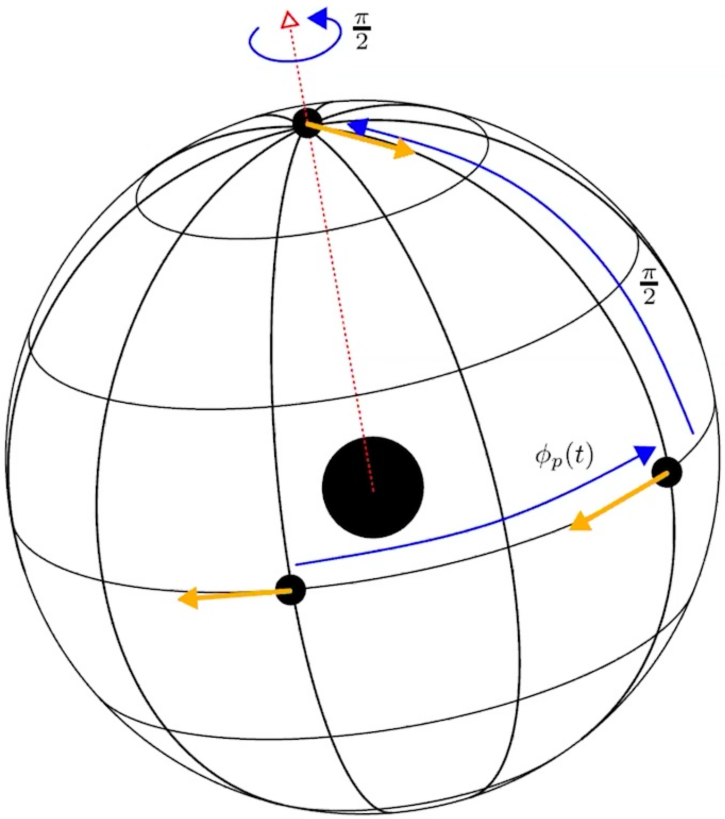
$$\xi_{t/r}(\alpha,\beta) = \sum_{\ell=0}^{\infty} \sum_{m'=-\ell}^{\ell} \xi_{t/r}^{\ell m'} Y^{\ell m'}(\alpha,\beta)$$

- Vector sector

$$\xi_A(\alpha,\beta) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \left(\xi_+^{\ell m'} Y_A^{\ell m'}(\alpha,\beta) + \xi_-^{\ell m'} X_A^{\ell m'}(\alpha,\beta) \right)$$

where $Y_A^{\ell m} := D_A Y^{\ell m}$ and $X_A^{\ell m} := -\epsilon_A{}^C D_C Y^{\ell m}$

Rotation Back to (θ, ϕ) Coordinates



- To convert to (θ, ϕ) coordinates, we use the Wigner D-matrix
  - Uses Euler angles (α, β, γ) to represent rigid body rotations around $z - y - z$ axes
  - In our case this is $(\phi_p(t), \pi/2, \pi/2)$
- Mode coefficients can be written as

$$f_{\ell m} = \sum_{m'=-\ell}^{\ell} D^{\ell}_{mm'}(\alpha, \beta, \gamma) f_{\ell m'}$$

Constructing the Metric Perturbations

- Write the perturbations in the Carter tetrad, $\{l^\mu, n^\mu, m^\mu, \bar{m}^\mu\}$, in terms of spin-weighted spherical harmonics, ${}_sY_{\ell m}$ [Spiers et al. (in preparation)]

$$\Delta h^{1\ell m}_{ab} = 2\delta_{(a}\xi^{\ell m}_{b)} \qquad \Delta h^{1\ell m}_{m\bar{m}} = \frac{1}{r^2}\Delta h^{1\ell m}_{\circ}$$

$$\Delta h^{1\ell m}_{am} = -\frac{\lambda_1}{\sqrt{2}r}\left(\Delta h^{1\ell m}_{a+} + i\Delta h^{1\ell m}_{a-}\right) \qquad \Delta h^{1\ell m}_{a\bar{m}} = \frac{\lambda_1}{\sqrt{2}r}\left(\Delta h^{1\ell m}_{a+} - i\Delta h^{1\ell m}_{a-}\right)$$

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where $\lambda_s := \sqrt{(\ell+|s|)!/(\ell-|s|)!}$

- Each m increases the spin weight by 1 and each $\bar{m}$ reduces it by 1
- Components are given in terms of the gauge vector by

$$\Delta h^{1\ell m}_{a+} = \xi^{\ell m}_{a} + \delta_a\xi^{\ell m}_{+} - 2r_a r^{-1}\xi^{\ell m}_{+} \qquad \Delta h^{1\ell m}_{a-} = \delta_a\xi^{\ell m}_{-} - 2r_a r^{-1}\xi^{\ell m}_{-}$$

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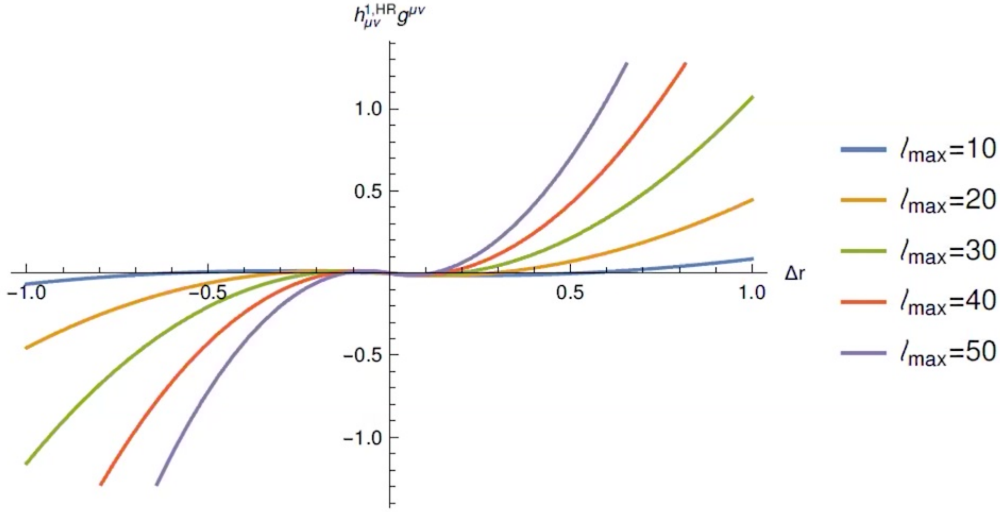
- Can get l and n components by contracting in the relevant basis vector

Trace Condition Check



Trace condition $h^{\mathrm{S1,HR}}_{m\bar{m}} - h^{\mathrm{S1,HR}}_{ln} = 0$ for $\theta = \pi/2$, $\phi = 0$, $M = 1$ and $r_0 = 10$

- Convergence on the small object but divergence away
  - When performing mode decomposition, we perform an expansion in Δr and truncate this at $\mathcal{O}(\lambda)$ to match gauge vector expansion
  - This is the cause for the divergence when summing over ℓ for $\Delta r \neq 0$

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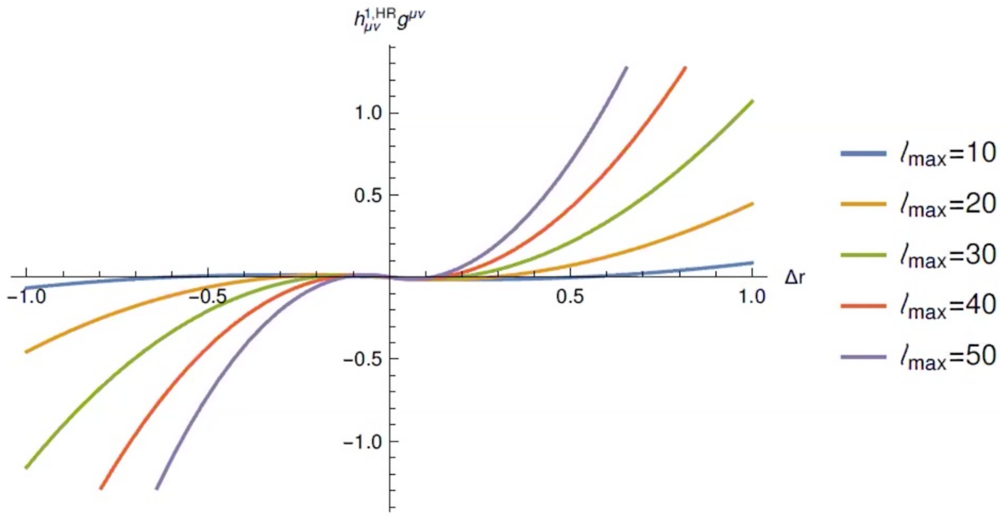

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- Convergence on the small object but divergence away
  - When performing mode decomposition, we perform an expansion in Δr and truncate this at $\mathcal{O}(\lambda)$ to match gauge vector expansion
  - This is the cause for the divergence when summing over ℓ for $\Delta r \neq 0$
  - We have started to recompute the integrals exactly instead of performing an expansion

Null Condition Check

- The null condition, $h^{\mathrm{S1,HR}}_{\mu\nu} k^{\nu} = 0$ gives four constraints, one for each of l, n, m and $\bar{m}$
- Unfortunately, none of these conditions are satisfied anywhere in spacetime, including on the small object
  - Not entirely clear why the gauge condition fails

Summary

- Discussed the benefits the highly regular gauge will bring to self-force calculations
  - With the weaker singularity near the worldline, we can define the second-order EFEs distributionally leading to the Detweiler stress-energy tensor
  - A well-defined source will allow us to use the second-order Teukolsky formalism, discussed by Spiers & Leather, instead of a puncture scheme
  - The weaker divergence in $\delta^2 G^{\mu\nu}$ would also be useful for any other calcuation using this as a source term
- Presented first steps towards the implementation of the highly regular gauge for quasicircular orbits in Schwarzschild
  - One gauge condition is satisfied on the small object while the other fails everywhere in spacetime