Title: Geodesic motion in relativistic astrophysics

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Abstract: Since the advent of (relativistic) astrophysics it has been one of the most important tasks to study the motion of freely falling particles, both from a purely academic and an observational point of view. In this presentation I review the solution methods for the equations of motion of particle-like objects and light within a wide variety of spacetimes. Moreover, we take a closer look on the importance of special orbits for phenomena like black hole shadows or accretion discs.

Geodesic motion in relativistic astrophysics

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Perimeter Institute

University of Bremen

Terra Q

Research Training Group Graduiertenkolleg Models of Gravity

CENTER OF APPLIED SPACE TECHNOLOGY AND MICROGRAVITY
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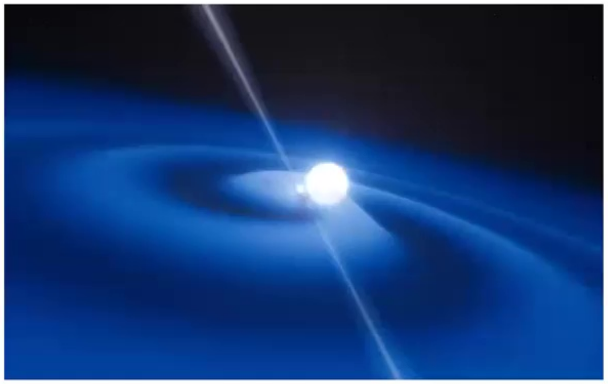
Strong gravitational fields



- Compact astronomical objects like black holes and neutron stars generate extremely strong gravitational fields.
- Informations about strong gravitational fields can substantially increase our knowledge about the nature of gravitation.

Images: artist's view of a black hole and of PSR J0348+0432.

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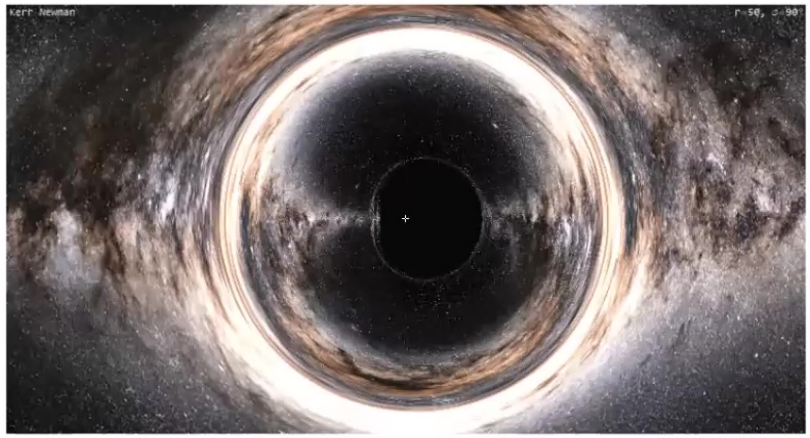
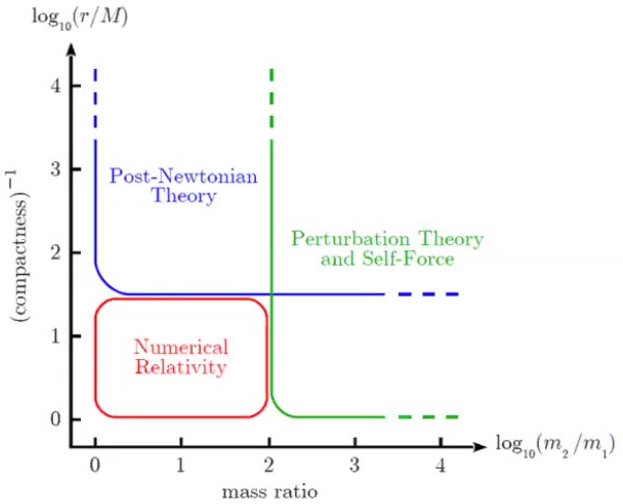
Compact objects



- Information about compact objects
  $\leftrightarrow$ electromagnetic radiation and gravitational waves
- Both require knowledge about the motion of objects/light

Binary motion

Generally described by the field equation



from Le Tiec 2014

Here

- Einstein(-Maxwell) or more general
- assume extreme mass ratios
- leading order geodesic term + some inner structure (charge/spin)
- exact analytical methods

Geodesic motion also covers (to leading order):
propagation of electromagnetic signals & clocks in (strong) gravity fields

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Spacetimes

Assume basic symmetries

- stationarity $\to$ conserved energy
- axial symmetry $\to$ conserved angular momentum

Metric in Boyer Lindquist coordinates

$$g = g_{00}(dx^0)^2 + 2g_{0\phi}dx^0 d\phi + g_{\phi\phi}d\phi^2 + g_{\theta\theta}d\theta^2 + g_{rr}dr^2$$

Equations from energy and angular momentum conservation

$$u^0 = \frac{Eg_{\phi\phi} - Lg_{0\phi}}{g_{0\phi}^2 - g_{00}g_{\phi\phi}} \qquad u^\phi = \frac{Lg_{00} + Eg_{0\phi}}{g_{00}g_{\phi\phi} - g_{0\phi}^2}$$

From normalisation $g_{\mu\nu}u^\mu u^\nu = -\epsilon$ we then find

$$-\epsilon = g_{rr}(u^r)^2 + g_{\theta\theta}(u^\theta)^2 + \frac{E^2 g_{\phi\phi} + 2ELg_{0\phi} + L^2 g_{00}}{g_{\phi\phi}g_{00} - g_{0\phi}^2}$$

Equations of geodesic motion

Equation separates $\leftrightarrow$ existence of a fourth constant

$$\frac{g_{ab}}{g_{\phi\phi}g_{00} - g_{0\phi}^2} = \frac{f_{ab}(r) + g_{ab}(\theta)}{F(r) + G(\theta)},$$
$$g_{rr} = (F(r) + G(\theta)) f_r(r),$$
$$g_{\theta\theta} = (F(r) + G(\theta)) g_\theta(\theta)$$

Example Kerr:
$F(r) + G(\theta)$
$= r^2 + a^2 \cos^2\theta$

for $ab = 00, 0\phi, \phi\phi$ and some functions $f_{ab}, g_{ab}, f_r, g_\theta, F, G$.

define a new parameter λ ('Mino time') by $d\tau = (F(r) + G(\theta))d\lambda$

$$\left(\frac{dr}{d\lambda}\right)^2 = R(r)\,, \qquad \frac{d\phi}{d\lambda} = \Phi_r(r) + \Phi_\theta(\theta)\,,$$
$$\left(\frac{d\theta}{d\lambda}\right)^2 = \Theta(\theta)\,, \qquad \frac{dt}{d\lambda} = T_r(r) + T_\theta(\theta)$$

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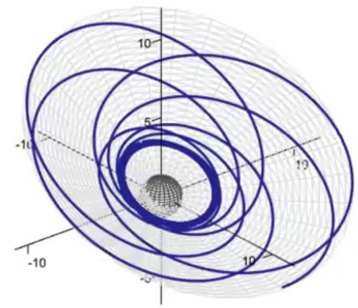
Separability

Existence of fourth constant rather special

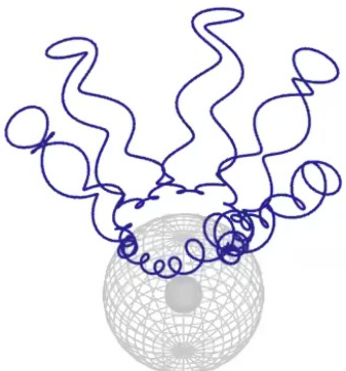
- Kerr-Newman-NUT-de Sitter
- some higher and lower dimensional spacetimes
- some regular 'black holes'
- some parametrised spacetimes
- ...



Generally problematic

- existence of 'hairs'
- perturbations from the environment
- black hole imposters; etc.

→ generally chaotic motion appears



Analytical solution methods

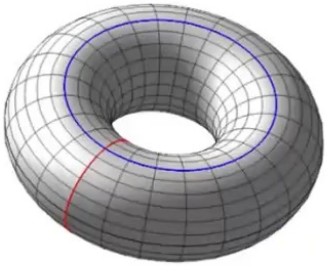
ODE $\frac{dx}{dy} = R(x, \sqrt{P(x)})$ most commonly $\frac{dr}{d\lambda} = \sqrt{P(r)}$

Algebro-geometric methods (special cases)

- P of order 1 or 2: elementary functions; Kepler problem
- P of order 3 or 4: elliptic functions; Kerr-Newman-NUT spacetimes

General approach Sharp 1979, Lämmerzahl&Hackmann 2015

- Reduce ODE to a standard form by some substitutions
- write down the solution!



Available standard forms include

- Jacobi functions sn, cn, dn; integrals F, E, Π
- Weierstrass functions $\wp$, $\wp'$; integral σ
- included in all major CAS systems (Mathematica, Maple)

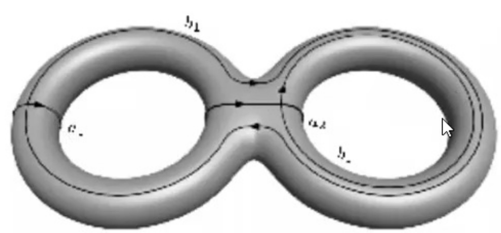
Hyperelliptic functions

Generally P of order $2g+1$ or $2g+2$
(Examples $g=2$: Kerr-Newman-NUT-de Sitter spacetimes)

Hackmann&Lämmerzahl 2008, Enolskii+ 2011, Lämmerzahl&Hackmann 2015

- functions need to have $2g$ periods (for $g \geq 1$)
- Kleinian sigma functions/Riemann theta functions
- dimensional reduction necessary
  (restriction to subalgebra, theta-divisor)

Explicit analytical solutions possible in all Petrov type D spacetimes with separable equations of motions!



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Spinning particles

MPD equations, pole-dipole approximation (from $\nabla^\mu T_{\mu\nu} = 0$)

$$\frac{Dp_a}{d\tau} = -\frac{1}{2} R_{abcd} u^b S^{cd} \,,$$

$$\frac{DS^{ab}}{d\tau} = p^a u^b - p^b u^a \,,$$

$+$ Spin Supplementary Condition (SSC) $\leftrightarrow$ choice of reference worldline

Particular setup Hackmann+ 2014

- Kerr spacetime, equatorial plane, (anti-)aligned spins
- SSC $S^{ab} p_b = 0$ (ZAMO observer)
- to all orders in the spin

$\rightarrow$ results in ODE's of hyperelliptic type, explicitly analytical solvable

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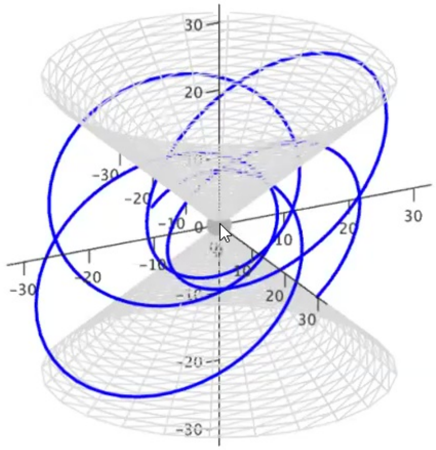
Periodic motion



For bound orbits outside the horizons:

- The radial motion is periodic, $r \in [r_\mathrm{p}, r_\mathrm{a}]$
- The polar motion is periodic, $\theta \in [\theta_\mathrm{min}, \theta_\mathrm{max}]$

From $\left(\frac{dr}{d\lambda}\right)^2 = R$, $\left(\frac{d\theta}{d\lambda}\right)^2 = \Theta$:

- Radial period Λ_r: $r(\lambda + \Lambda_r) = r(\lambda)\,, \quad \Lambda_r = 2\int_{r_\mathrm{p}}^{r_\mathrm{a}} \frac{dr}{\sqrt{R}}$, $\Upsilon_r = \frac{2\pi}{\Lambda_r}$
- Polar period Λ_θ: $\theta(\lambda + \Lambda_\theta) = \theta(\lambda)\,, \quad \Lambda_\theta = 2\int_{\theta_\mathrm{min}}^{\theta_\mathrm{max}} \frac{d\theta}{\sqrt{\Theta}}$, $\Upsilon_\theta = \frac{2\pi}{\Lambda_\theta}$

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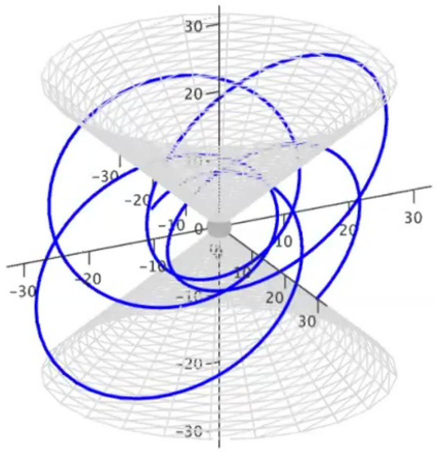
Fundamental Frequencies



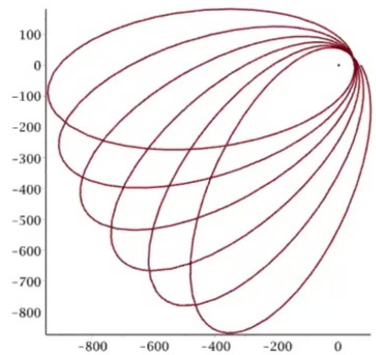
- φ, t, and τ are not periodic
- can be expressed as a linear function in $\lambda +$ periodic oscillations
- Ansatz: $\varphi(\lambda) = \Upsilon_\varphi \lambda + \Phi^r_{osc} + \Phi^\theta_{osc}$
  Υ_φ infinite $\lambda-$average
- Analogously: $\tau(\lambda) = \Upsilon_\tau \lambda +$ osc.;
  $t(\lambda) = \Upsilon_t \lambda +$ osc.

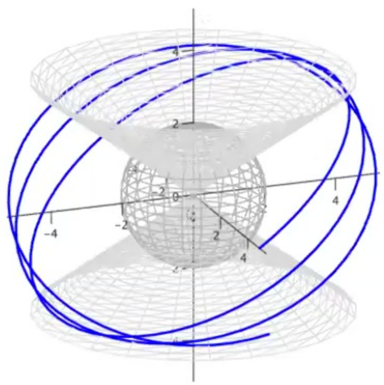
Schmidt 2002, Drasco&Hughes 2004, Fujita&Hikida 2009, Hackmann&Lämmerzahl 2012

A note of care Warburton+ 2013

- parametrisation by fundamental frequencies is not unique

Periapsis precession and Lense-Thirring effect





Periapsis precession

- mismatch of radial and angular frequency wrt coordinate time
- $\dot{\omega} = \Omega_r - \Omega_\varphi = \frac{\Upsilon_r}{\Upsilon_t} - \frac{\Upsilon_\varphi}{\Upsilon_t}$
  $= (2\pi - \Lambda_r \Upsilon_\varphi)/P_r$
- $P_r = \Lambda_r \Upsilon_t$ anomalistic period

Lense-Thirring effect

- mismatch of polar and angular frequency wrt coordinate time
- $\dot{\Omega} = \Omega_\theta - \Omega_\varphi = (2\pi - \Lambda_\theta \Upsilon_\varphi)/P_\theta$
- $P_\theta = \Lambda_\theta \Upsilon_t$ draconitic period

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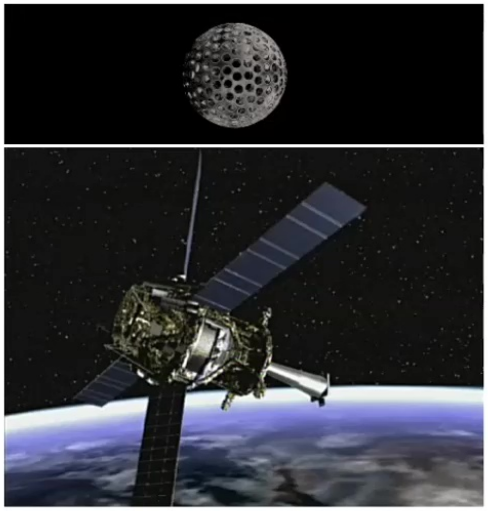
Frame dragging effects effect

Frame dragging

- induced by rotation
- also called gravitomagnetism



Experimental tests

- Spin-orbit coupling (Lense-Thirring effect): LAGEOS/LARES
- Spin-spin coupling (Schiff effect): Gravity Probe B

There is a less well known effect to test: gravitomagnetic clock effect

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