Title: New spin-orbit and spin-squared post-Newtonian results from first-order self-force

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Abstract: The scattering angle function exhibits a simple dependence on the mass ratio, which has been recently used to obtain new post-Newtonian (PN) results for arbitrary mass ratios from first-order self-force calculations. In this talk, I will present results for the spin-orbit coupling at fourth subleading PN order (5.5PN), including both local and nonlocal contributions, and the spin-squared coupling at third subleading PN order (5PN) for aligned spins. The spin-orbit results are missing one coefficient at second order in the mass ratio, and the spin-squared results are missing one coefficient at first order in the mass ratio. The latter could be determined from a self-force calculation of the spin-precession invariant for circular orbits in Schwarzschild to linear order in the spin of the small object. I will also discuss implications regarding the first law of binary mechanics with spin quadrupole and its relation to tidal invariants.

New spin-orbit and spin-squared post-Newtonian results from first-order self-force

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Outline

- Spin-orbit dynamics through fourth-subleading PN order (5.5PN)
  [Antonelli,Kavanagh,MK,Steinhoff,Vines 2003.11391, 2010.02018] [MK (in prep)]
- Spin$_1$-spin$_2$ and spin-squared dynamics at third-subleading PN order (5PN)
  [Antonelli,... 2010.02018] [MK,Kavanagh,Steinhoff,Vines,... (in prep)]
- First law of binary mechanics with spin quadrupole

Nonlocal part of the 5.5PN SO Hamiltonian

- Total action split into local and nonlocal-in-time pieces $S_{\text{tot}}^{\leq n} = S_{\text{loc}}^{\leq n} + S_{\text{nonloc}}^{\leq n}$.
  Nonlocal Hamiltonian

$$\boxed{\delta H_{\text{nonloc}}^{4+5.5\text{PN}}(t) = -\frac{GM}{c^3}\text{Pf}_{2s/c}\int \frac{d\tau}{|\tau|}\mathcal{F}_{1.5\text{PN}}^{\text{split}}(t,t+\tau) + 2\frac{GM}{c^3}\mathcal{F}_{1.5\text{PN}}^{\text{split}}(t,t)\ln\left(\frac{r}{s}\right)}$$

[Damour, Jaranowski, Schäfer 1401.4548, 1502.07245] [Bini, Damour, Geralico 2003.11891, 2007.11239]

- Time-symmetric GW energy flux

$$\mathcal{F}_{1.5\text{PN}}^{\text{split}}(t,t') = \frac{G}{c^5}\left[\frac{1}{5}I_{ij}^{(3)}(t)I_{ij}^{(3)}(t') + \frac{16}{45c^2}J_{ij}^{(3)}(t)J_{ij}^{(3)}(t')\right]$$

- SO part of effective Hamiltonian in the effective-one-body (EOB) formalism

$$H_{\text{eff}}^{\text{SO}} = \frac{1}{c^3 r^3}\boldsymbol{L}\cdot\left[g_S(r,p_r)\boldsymbol{S} + g_{S^*}(r,p_r)\boldsymbol{S}^*\right]$$

- Write nonlocal part of the gyro-gravitomagnetic factors with unknown coefficients in p_r expansion, and match the Delaunay average

$$\left\langle \delta H_{\text{nonloc}}^{4+5.5\text{PN}}\right\rangle = \left\langle \delta H_{\text{nonloc}}^{\text{EOB}}\right\rangle$$

Spin-orbit scattering angle through N^{4}LO

- Relate local Hamiltonian to scattering angle [Damour 1609.00354]

$$\chi^{\text{loc}} = -2\int_{r_0}^{\infty} \frac{\partial p_r(E^{\text{loc}}, L, r)}{\partial L}\mathrm{d}r - \pi$$

- Replace (E, L) with (v, b) [Vines,Steinhoff,Buonanno 1812.00956]
- Match to the ansatz

$$\begin{aligned}\frac{M}{E}\chi_{\text{SO}}^{\text{loc}} = \frac{a_1}{b}\Bigg\{ & \left(\frac{GM}{v^2 b}\right)(-4v) + \pi\left(\frac{GM}{v^2 b}\right)^2\left[\cdots + \left(-\frac{21}{4} + \frac{3\delta}{4}\right)v^3\right] \\ & + \left(\frac{GM}{v^2 b}\right)^3\left[\cdots + (-50 + 10\delta + \boldsymbol{X_{35}^{\nu}}\nu)v^5 + \boldsymbol{X_{37}^{\nu}}\nu v^7 + \boldsymbol{X_{39}^{\nu}}\nu v^9\right] \\ & + \pi\left(\frac{GM}{v^2 b}\right)^4\left[\cdots + \left(-\frac{1365}{32} + \frac{315\delta}{32} + \boldsymbol{X_{47}^{\nu}}\nu + \boldsymbol{X_{47}^{\delta\nu}}\delta\nu\right)v^7 + \left(\boldsymbol{X_{49}^{\nu}}\nu + \boldsymbol{X_{49}^{\delta\nu}}\delta\nu\right)v^9\right] \\ & + \left(\frac{GM}{v^2 b}\right)^5\left[\cdots + \left(-336 + 84\delta + \boldsymbol{X_{59}^{\nu}}\nu + \boldsymbol{X_{59}^{\delta\nu}}\delta\nu + \boldsymbol{X_{59}^{\nu^2}}\nu^2\right)v^9\right]\Bigg\} + 1 \leftrightarrow 2\end{aligned}$$

(...) known from test scattering angle [Bini, Geralico, Vines 1707.09814], and lower PM orders,
1 unknown at NNLO, **3 unknowns at N^{3}LO**, **6 unknowns at N^{4}LO**

Spin-orbit scattering angle through N^{4}LO

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Redshift and spin precession frequency

- First law for spinning eccentric BBH (for aligned spins to linear order in spin)

$$\boxed{\mathrm{d}E = \Omega_r \mathrm{d}I_r + \Omega_\phi \mathrm{d}L + \sum\nolimits_i \left(z_\mathrm{i}\mathrm{d}m_\mathrm{i} + \Omega_\mathrm{i}\mathrm{d}S_\mathrm{i}\right)}$$

[Le Tiec, Blanchet, Whiting 1111.5378] [Blanchet, Buonanno, Le Tiec 1211.1060]
[Le Tiec 1506.05648] [Antonelli,Kavanagh,MK,Steinhoff,Vines 2010.02018]

- Redshift and spin precession frequency from (local + nonlocal) Hamiltonian

$$z_\mathrm{i} = \left\langle \frac{\partial H}{\partial m_\mathrm{i}} \right\rangle, \qquad \Omega_\mathrm{i} = \left\langle \frac{\partial H}{\partial S_\mathrm{i}} \right\rangle$$

- Express z_1 and Ω_1 in terms of gauge-independent variables, expand to linear order in the mass ratio, then express in terms of Kerr-geodesic variables (e, u_p).
- Match PN to 1SF results and check that all logarithms and Euler gamma cancel

$$z_1 = \cdots + q\left[\cdots + s(\dots) + a(\cdots + e^2(\dots) + e^4(\dots))\right]$$
$$\Omega_1 = \cdots + q\left[\cdots + e^2(\dots)\right]$$

q: mass ratio, s: spin of the small body, a: background Kerr spin

[Akcay,Barack,Bini,Damour,Dempsey,Detweiler,Dolan,Geralico,Harte,Hopper,Kavanagh, Le Tiec,Munna,Nolan,Ottewill,Pound,Sago,Shah,van de Meent,Warburton,Wardell,...]

Determining N^{4}LO SO unknowns from self-force results

- Scattering angle ansatz

$$\frac{M}{E}\chi_{\text{SO}}^{\text{loc}} = \frac{a_1}{b}\Bigg\{ \left(\frac{GM}{v^2 b}\right)(-4v) + \pi \left(\frac{GM}{v^2 b}\right)^2 \left[\cdots + \left(-\frac{21}{4} + \frac{3\delta}{4}\right) v^3\right]$$
$$+ \left(\frac{GM}{v^2 b}\right)^3 \left[\cdots + (-50 + 10\delta + X_{35}^{\nu}\nu) v^5 + X_{37}^{\nu}\nu v^7 + X_{39}^{\nu}\nu v^9\right]$$
$$+ \pi \left(\frac{GM}{v^2 b}\right)^4 \left[\cdots + \left(-\frac{1365}{32} + \frac{315\delta}{32} + X_{47}^{\nu}\nu + X_{47}^{\delta\nu}\delta\nu\right) v^7 + \left(X_{49}^{\nu}\nu + X_{49}^{\delta\nu}\delta\nu\right) v^9\right]$$
$$+ \left(\frac{GM}{v^2 b}\right)^5 \left[\cdots + \left(-336 + 84\delta + X_{59}^{\nu}\nu + X_{59}^{\delta\nu}\delta\nu + X_{59}^{\nu^2}\nu^2\right) v^9\right]\Bigg\} + 1 \leftrightarrow 2$$

- Need both redshift and spin precession to solve for the unknowns

$$z_1 = \cdots + q\left[\cdots + s(\dots) + a\left(X_{39}^{\nu},\, X_{49}^{\nu} - X_{49}^{\delta\nu},\, X_{59}^{\nu} - X_{59}^{\delta\nu}\right)\right]$$
$$\Omega_1 = \cdots + q\left[X_{39}^{\nu},\, X_{49}^{\nu} + X_{49}^{\delta\nu},\, X_{59}^{\nu} + X_{59}^{\delta\nu}\right]$$

$a_1 \to s$: spin of the small body, $a_2 \to a$: background Kerr spin

Determining N^{4}LO SO unknowns from self-force results

$$
\begin{aligned}
\frac{\chi_{\text{SO}}^{\text{N}^4\text{LO,loc}}}{E/M} &= \frac{a_+}{b}\Bigg\{\left(\frac{GM}{v^2b}\right)^3\frac{26571}{1120}\nu\frac{v^9}{c^9}+\pi\left(\frac{GM}{v^2b}\right)^4\left(\frac{80823}{8192}\pi^2-\frac{403129}{4800}\right)\nu\frac{v^9}{c^9}+\left(\frac{GM}{v^2b}\right)^5\Bigg[\frac{16}{3}\frac{v}{c}+(-192+32\nu)\frac{v^3}{c^3}\\
&\quad+(-2016+1032\nu-16\nu^2)\frac{v^5}{c^5}+\left(-2240+\frac{150220}{27}\nu-\frac{2755\pi^2}{36}\nu-168\nu^2\right)\frac{v^7}{c^7}\\
&\quad+\left(-336+\frac{402799}{270}\nu-\frac{4135\pi^2}{144}\nu+X_{59}^{\nu^2+}\nu^2\right)\frac{v^9}{c^9}\Bigg]\Bigg\}\\
&+\frac{\delta a_-}{b}\Bigg\{\pi\left(\frac{GM}{v^2b}\right)^4\left(\frac{97585\pi^2}{8192}-\frac{533669}{4800}\right)\nu\frac{v^9}{c^9}+\left(\frac{GM}{v^2b}\right)^5\Bigg[\frac{4}{3}\frac{v}{c}+(-48+4\nu)\frac{v^3}{c^3}+(-504+109\nu)\frac{v^5}{c^5}\\
&\quad+\left(-560+\frac{21995}{54}\nu+\frac{80\pi^2}{9}\nu\right)\frac{v^7}{c^7}+\left(-84+\frac{2477\pi^2}{16}\nu-\frac{285673}{240}\nu\right)\frac{v^9}{c^9}\Bigg]\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
g_S^{5.5\text{PN,loc}} &= \left[-\frac{\nu^4}{64}-\frac{413\nu^3}{512}+\nu^2\left(-\frac{3X_{59}^{\nu^2+}}{32}-\frac{583\pi^2}{192}+\frac{235111}{2304}\right)+\left(\frac{62041\pi^2}{3072}-\frac{11646877}{57600}\right)\nu\right]\frac{1}{r^4}\\
&+\left[\frac{3\nu^4}{8}-\frac{8259\nu^3}{128}+\left(\frac{198133}{384}-\frac{1087\pi^2}{128}\right)\nu^2+\left(\frac{3612403}{6400}-\frac{22301\pi^2}{512}\right)\nu\right]\frac{p_r^2}{r^3}\\
&+\left[-\frac{107\nu^4}{64}-\frac{73547\nu^3}{512}+\frac{31913\nu^2}{256}+\frac{8337\nu}{256}\right]\frac{p_r^4}{r^2}+\left[\frac{1577\nu^4}{320}-\frac{11397\nu^3}{512}-\frac{2553\nu^2}{256}-\frac{893\nu}{256}\right]\frac{p_r^6}{r}\\
&+\left[\frac{189\nu^4}{64}+\frac{945\nu^3}{512}+\frac{99\nu^2}{256}-\frac{27\nu}{128}\right]p_r^8,
\end{aligned}
$$

$$
\begin{aligned}
g_{S^*}^{5.5\text{PN,loc}} &= \left[-\frac{5\nu^4}{128}-\frac{37\nu^3}{32}+\nu^2\left(-\frac{X_{59}^{\nu^2+}}{8}-\frac{41\pi^2}{16}+\frac{29081}{384}\right)+\left(\frac{23663\pi^2}{3072}-\frac{55}{2}\right)\nu-\frac{567}{128}\right]\frac{1}{r^4}\\
&+\left[\frac{57\nu^4}{64}-\frac{163\nu^3}{2}+\left(\frac{77201}{192}-\frac{41\pi^2}{4}\right)\nu^2+\left(\frac{34677}{160}-\frac{4829\pi^2}{384}\right)\nu-\frac{9}{16}\right]\frac{p_r^2}{r^3}\\
&+\left[-\frac{459\nu^4}{128}-\frac{4635\nu^3}{32}+\frac{4045\nu^2}{128}+\frac{107\nu}{12}+\frac{2525}{384}\right]\frac{p_r^4}{r^2}+\left[\frac{5359\nu^4}{640}-\frac{797\nu^3}{160}-\frac{293\nu^2}{128}+\frac{77\nu}{24}+\frac{1185}{128}\right]\frac{p_r^6}{r}\\
&+\left[\frac{315\nu^4}{128}+\frac{105\nu^3}{32}+\frac{351\nu^2}{128}+\frac{63\nu}{32}+\frac{231}{128}\right]p_r^8.
\end{aligned}
$$

N^3LO spin$_1$-spin$_2$

- Scattering angle ansatz

$$\frac{M}{E}\chi_{a_1a_2} = \frac{a_1a_2}{b^2}\Bigg\{\left(\frac{GM}{v^2b}\right)(4+4v^2) + \pi\left(\frac{GM}{v^2b}\right)^2\left(3+\frac{45}{2}v^2+\frac{9}{2}v^4\right) + \left(\frac{GM}{v^2b}\right)^3\left[\cdots + (440 + \boldsymbol{X^{\nu}_{34}}\nu)\,v^4 + (40 + \boldsymbol{X^{\nu}_{36}}\nu)\,v^6\right] + \pi\left(\frac{GM}{v^2b}\right)^4\left[\cdots + \left(\frac{9975}{16} + \boldsymbol{X^{\nu}_{46}}\nu\right)v^6\right]\Bigg\}$$

1 unknown at NNLO, 2 unknowns at N^3LO

- Need 1SF spin precession to $\mathcal{O}(ae^2)$to solve for unknowns

$$\Omega_1 = \cdots + q\left[\cdots + a\left(\cdots + e^2(\dots)\right)\right]$$

- Solution

$$\boldsymbol{X^{\nu}_{34}} = -66, \qquad \boldsymbol{X^{\nu}_{36}} = -\frac{1093}{5}, \qquad \boldsymbol{X^{\nu}_{46}} = \frac{615}{256}\left(3\pi^2 - 416\right)$$

[Antonelli,Kavanagh,MK,Steinhoff,Vines 2010.02018]

N^3LO spin squared

$$\frac{M}{E}\chi_{a^2} = \frac{\mathfrak{h}_1^2}{b^2}\Bigg\{ \left(\frac{GM}{v^2 b}\right)(2+2v^2) + \pi\left(\frac{GM}{v^2 b}\right)^2 \left[\frac{3}{2} + \left(\frac{87}{8} - \frac{21\delta}{8}\right)v^2 + \left(\frac{69}{32} - \frac{21\delta}{32}\right)v^4\right]$$
$$+ \left(\frac{GM}{v^2 b}\right)^3 \left[\cdots + (220 - 80\delta + \boldsymbol{X^\nu_{34}}\nu)\, v^4 + (20 - 8\delta + \boldsymbol{X^\nu_{36}}\nu)\, v^6\right]$$
$$+ \pi\left(\frac{GM}{v^2 b}\right)^4 \left[\cdots + \left(\frac{20625}{64} - \frac{8775\delta}{64} + \boldsymbol{X^\nu_{46}}\nu + \boldsymbol{X^{\nu\delta}_{46}}\nu\delta\right)v^6\right] \Bigg\} + 1 \leftrightarrow 2$$

- 1SF redshift known to $\mathcal{O}(a^2 e^4)$ and $\mathcal{O}(s^2 e^0)$, but not enough to solve for unknowns [Bini,Geralico 1907.11080] [Bini,Geralico,Steinhoff 2003.12887]

$$z_1 = \cdots + q\left[\cdots + s^2 C_{1ES^2}(\ldots) + \underbrace{a^2\left(\cdots + e^2(\ldots) + e^4(\ldots)\right)}_{\boldsymbol{X^\nu_{36}},\ \boldsymbol{X^\nu_{46}} - \boldsymbol{X^{\delta\nu}_{46}}}\right]$$

$$\Omega_1 = \cdots + q\left[\cdots + \underbrace{s\,(\text{unavailable})}_{\boldsymbol{X^\nu_{36}},\ \boldsymbol{X^\nu_{46}} + \boldsymbol{X^{\delta\nu}_{46}}}\right]$$

- Assuming Ω_1 is known to $\mathcal{O}(s)$ for circular orbits in Schwarzschild,

$$\psi_1 \equiv \frac{\Omega_1}{\Omega_\phi} = \cdots + qs\left(y^{3/2} - 3y^{5/2} + 0 + Cy^{9/2}\right)$$

$$\boldsymbol{X^\nu_{36}} = -\frac{1041}{10}, \quad \boldsymbol{X^\nu_{46}} = \frac{115245\pi^2}{32768} - \frac{35955}{64} - \frac{15C}{32}, \quad \boldsymbol{X^{\nu\delta}_{46}} = \frac{945\pi^2}{32768} + \frac{2235}{64} - \frac{15C}{32}$$

First law of binary mechanics with spin quadrupole

- Redshift to linear order in spin follows from the definition of the Lagrangian $\mathcal{L}$ and $\int \mathrm{d}t\,\mathcal{L} \sim -m_\mathrm{i} \int \mathrm{d}t\, \mathrm{d}\tau_\mathrm{i}/\mathrm{d}t$

$$z_\mathrm{i} \equiv \left\langle \frac{\mathrm{d}\tau_\mathrm{i}}{\mathrm{d}t} \right\rangle = -\left\langle \frac{\partial \mathcal{L}}{\partial m_\mathrm{i}} \right\rangle = \left\langle \frac{\partial H}{\partial m_\mathrm{i}} \right\rangle$$

- Nonminimal part of the action for spin quadrupole [Levi, Steinhoff 1501.04956]

$$\mathcal{L}_{ES^2} = \frac{C_{ES^2}}{2m} \frac{\mathcal{E}_{\mu\nu}}{\sqrt{-u^2}} S^\mu S^\nu$$

$\mathcal{E}_{\mu\nu} \equiv R_{\mu\alpha\nu\beta} u^\alpha u^\beta$ is the electric component of the Riemann tensor.

- Define the quadrupole $Q \equiv C_{ES^2} S^2/2m$ to remove the mass dependence of the nonminimal part, such that $z = \langle \partial H/\partial m \rangle$.

- Define the eigenvalues $M^2 \mathcal{E}(u) = \mathrm{diag}[\Lambda_1^E, \Lambda_2^E, -(\Lambda_1^E + \Lambda_2^E)]$, leading to

$$\frac{\partial H}{\partial Q} = -\frac{\partial \mathcal{L}}{\partial Q} = -\frac{z_1 \Lambda_2^E}{M^2}$$

for aligned spins.

First law of binary mechanics with spin quadrupole

$$\mathrm{d}E = \Omega_r \mathrm{d}I_r + \Omega_\phi \mathrm{d}L + \sum_{\mathrm{i}} \left(z_{\mathrm{i}} \mathrm{d}m_{\mathrm{i}} + \Omega_{\mathrm{i}} \mathrm{d}S_{\mathrm{i}} - \frac{z_{\mathrm{i}} \Lambda_2^E}{M^2} \mathrm{d}Q_{\mathrm{i}} \right), \qquad Q_{\mathrm{i}} \equiv \frac{C_{\mathrm{i}ES^2} S_{\mathrm{i}}^2}{2m_{\mathrm{i}}}$$

- Redshift from PN Hamiltonian agrees with 1SF [Bini,Geralico,Steinhoff 2003.12887]

$$z_1 = \left(\frac{\partial H}{\partial m_1} \right)_{Q_1} = s^0 + s^2 + q \left[s^0 + s + s^2 C_{1ES^2} \right] + \ldots$$

- Spin precession invariant from H agrees with the test spin result $\psi_1^s = s \frac{3y^{5/2}}{\sqrt{1-3y}}$

$$\psi_1 = s^0 + s + q \left[s^0 + s \right] + \ldots$$

- Eigenvalue Λ_2^E calculated from H agrees with 1SF results for λ_2^E
[Dolan,Nolan,Ottewill,Warburton,Wardell 1406.4890][Bini,Damour 1409.6933][Bini,Geralico 1806.03495]

$$\lambda_2^E = \frac{m_2^2}{M^2} \Lambda_2^E = -\frac{m_2^2}{z_1} \frac{\partial H}{\partial Q_1} = y^3 + 3y^4 + 9y^5 + q \left(-y^3 - \frac{3y^4}{2} - \frac{23y^5}{8} \right)$$

Conclusions

- Calculated spin-orbit coupling at 5.5PN (local and nonlocal), except for one coefficient at second order in the mass ratio.
- Calculated spin-spin coupling at 5PN for aligned spins, except for one coefficient that requires the 1SF precession frequency for circular orbits in Schwarzschild to linear order in spin.
- Motivated a first law with spin quadrupole that agrees with available results for redshift, spin precession, and the eigenvalues of the tidal tensors.
- Future work: calculating 1SF precession frequency to linear order in spin, deriving the first law with spin quadrupole more rigorously, and extending the spin-spin results to precessing spins.