Title: Pseudospectrum and black hole quasi-normal mode (in)stability

Speakers: Rodrigo Panosso Macedo

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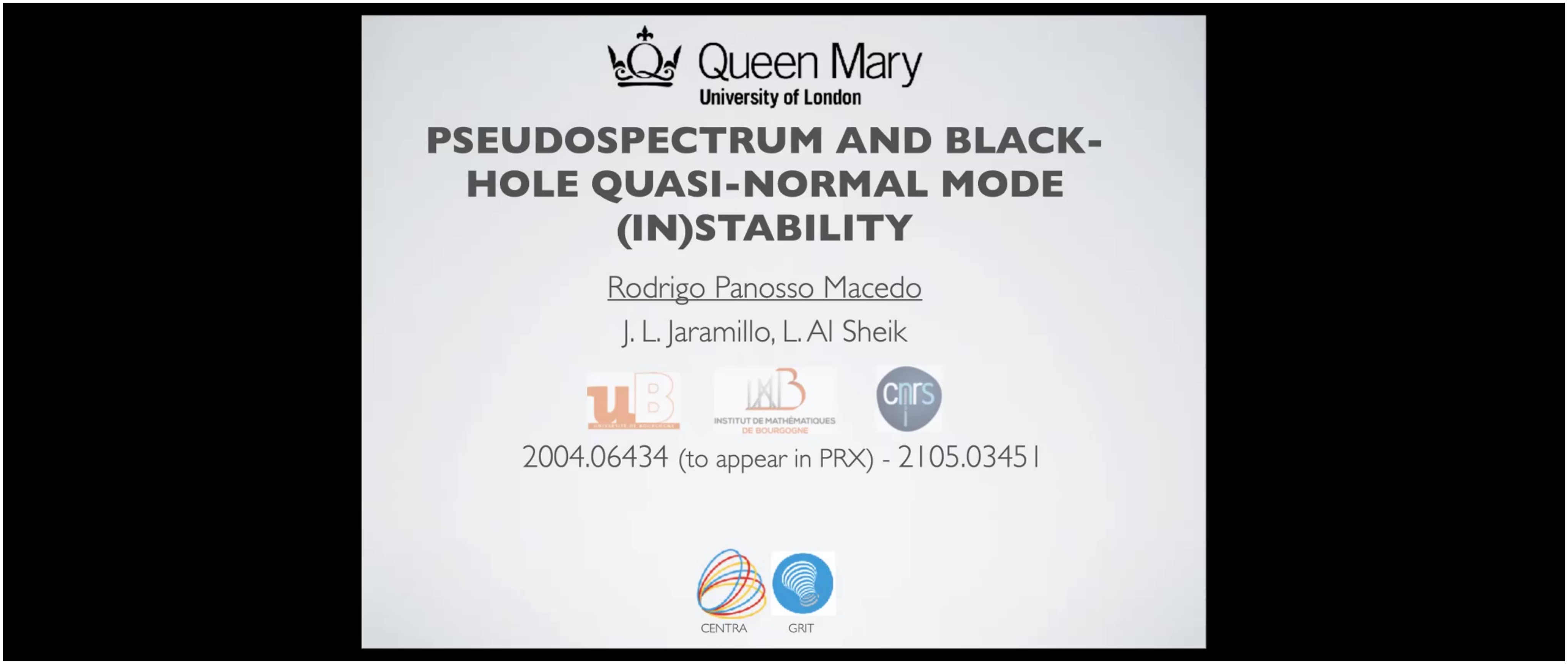
Abstract: We study the stability of quasi-normal modes (QNM) in asymptotically flat black hole spacetimes by means of a pseudospectrum analysis. The construction of the Schwarzschild QNM pseudospectrum reveals: i) the stability of the slowest decaying QNM under perturbations respecting the asymptotic structure, reassessing the instability of the fundamental QNM discussed by Nollert (1996); ii) the instability of all overtones under small scale perturbations of sufficiently high frequency, that migrate to a universal class of QNM branches along pseudospectra boundaries, shedding light on Nollert & Price's analysis (1996).


Queen Mary
University of London
PSEUDOSPECTRUM AND BLACK-
HOLE QUASI-NORMAL MODE
(IN)STABILITY
Rodrigo Panosso Macedo
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INSTITUT DE MATHÉMATIQUES
DE BOURGOGNE
2004.06434 (to appear in PRX) - 2105.03451
CENTRA
GRIT

QUASINORMAL MODES

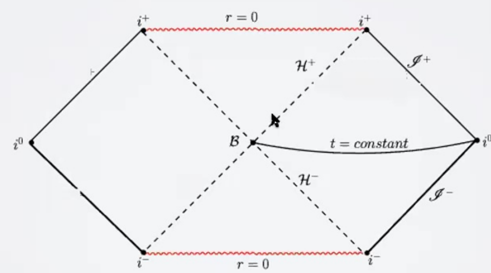
- **Wave equation on (spherically sym.) black-hole background**

$$-\Psi_{,\bar{t}\bar{t}} + \Psi_{,xx} - \mathcal{P}\Psi = 0$$

- **Phenomenological approach:**

  Ansatz for time dependence (Fourier modes)

$$\Psi = e^{-i\omega\bar{t}}\psi(x) \qquad \Longrightarrow \qquad \left[\partial^2_{xx} - \mathcal{P}\right]\psi = -\omega^2\psi$$



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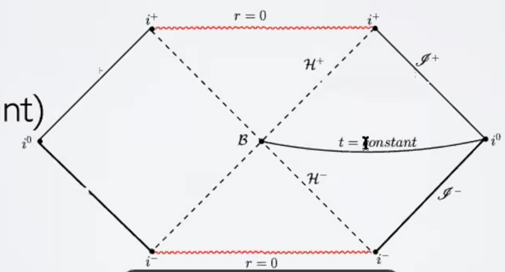
$\mathcal{B}$: Horizon
(bifurcation point)
$x \to -\infty$



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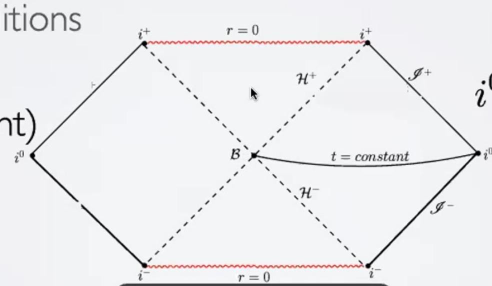
Boundary conditions

$\mathcal{B}$: Horizon (bifurcation point) $x \to -\infty$



i^0 : spacelike infinity $x \to +\infty$

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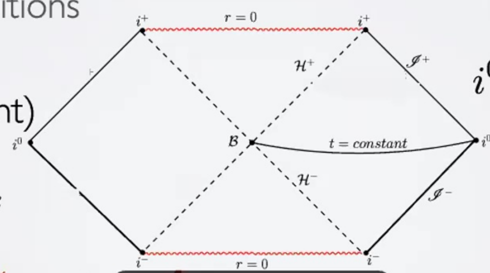

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$\psi \sim e^{i\omega x}$

- **Comparison to *normal* modes:** eigenvalue problem

NORMAL OPERATORS: SPECTRAL THEOREMS

Def. *Given matrix* L *and its adjoint* $L^\dagger$*. Then* L *is* *normal* *iff* $[L, L^\dagger] = 0$

Ex. *Symmetric, hermitian, orthogonal, unitary…*

Spectral Theo. (matrices): L *is normal iff is unitarily diagonalisable*

Obs. *Theorem extends to normal operators in Hilbert (or Banach) space*

Hermitian Physics (self adjoint operators):
Eigenvectors are orthogonal and form complete basis
Eigenvalues are stable $L \to L + \epsilon\delta L \Rightarrow \lambda \to \lambda + \epsilon\delta\lambda$

NON-NORMAL OPERATORS: NO SPECTRAL THEOREMS

If $[L, L^\dagger] \neq 0$

Completeness more difficult to study
Eigenvectors not necessarily orthogonal
Spectral Instabilities

Non-Hermitian Physics (non-self adjoint operator):

$$\epsilon < 1 \qquad L(\epsilon) = L + \epsilon \delta L, \quad |\delta L| = 1$$

$$|\lambda_i(\epsilon) - \lambda_i| \leq \kappa_i \, \epsilon$$

Eigenvalue Condition Number: $\kappa_i = \dfrac{||r_i|| \, ||l_i||}{|r_i \cdot l_i|}$

$L \, r_i = \lambda_i \, r_i$ (Right eigenvector)

$L^\dagger \, l_i = \bar{\lambda}_i \, l_i$ (Left eigenvector)

self adjoint operator:

$L = L^\dagger \Rightarrow r_i \parallel l_i \Rightarrow \kappa_i = 1$

Small perturbation in operator; small displacement in eigenvalue

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non-self adjoint operator:

$L \neq L^\dagger \overset{?}{\Rightarrow} r_i \nparallel l_i \Rightarrow \kappa_i > 1$

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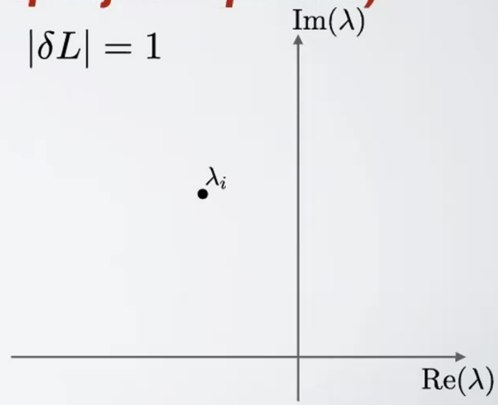
<u>Spectra:</u>

$\sigma(L) : ||L - \lambda_i \mathbb{I}|| = 0$ (Eigenvalue)



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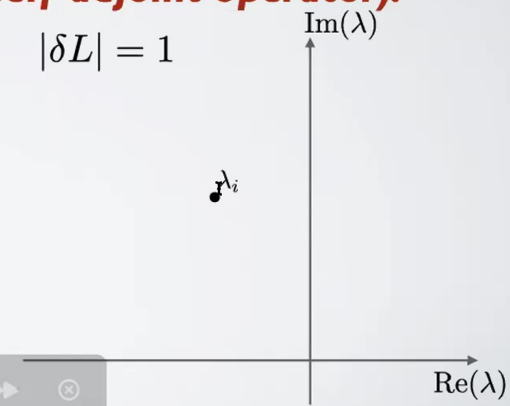
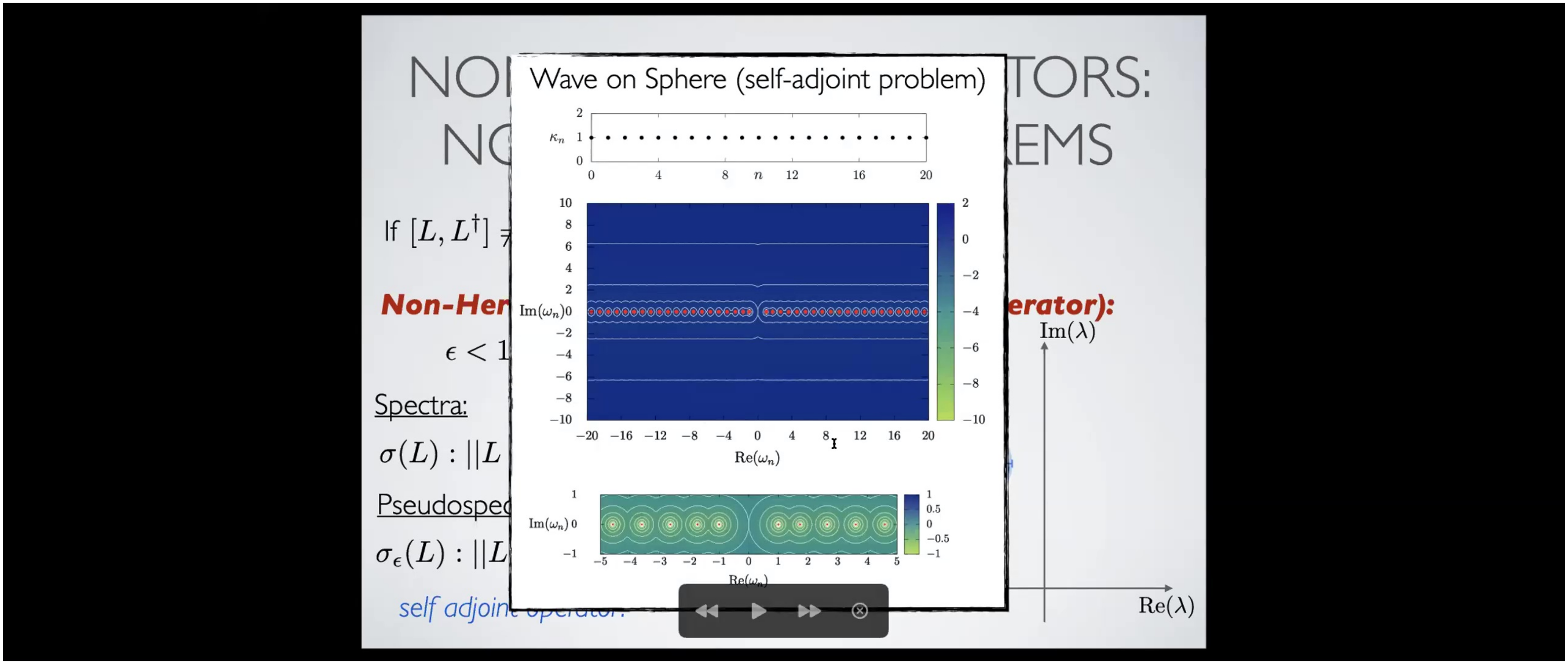


<u>Spectra:</u>

$\sigma(L) : ||L - \lambda_i \mathbb{I}|| = 0$ (Eigenvalue)

<u>Pseudospectra:</u>

$\sigma_\epsilon(L) : ||L - \lambda_i \mathbb{I}|| < \epsilon$


Wave on Sphere (self-adjoint problem)
κ_n
n
$\mathrm{Im}(\omega_n)$
$\mathrm{Re}(\omega_n)$
If $[L, L^\dagger]$
Non-Her
erator):
$\epsilon < 1$
Spectra:
$\sigma(L) : ||L$
Pseudospec
$\sigma_\epsilon(L) : ||L$
$\mathrm{Im}(\lambda)$
$\mathrm{Re}(\lambda)$

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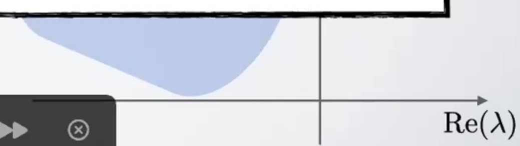
Spectral Instabilities

Take away message:

- The <u>unperturbed</u> operator contains <u>all</u> pieces of information to assess potential spectral instabilities
- Tools to measure: condition number and pseudospectra

$\sigma_\epsilon(L) : ||L - \lambda_i \mathbb{I}|| < \epsilon$

non-self adjoint operator:



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Non-Hermitian Physics (non-self adjoint operator):

Flow of energy, particles and information to external degrees of freedom out of the Hilbert space

Y. Ashida, Z. Gong, M. Ueda 2006.01827 (2020)

Table 1. A wide variety of classical and quantum systems described by non-Hermitian matrices/operators together with their physical origins of non-Hermiticity, presented in order of appearance in the present review.

Systems / Processes	Physical origin of non-Hermiticity	Theoretical methods
Photonics	Gain and loss of photons	Maxwell equations [12, 13]
Mechanics	Friction	Newton equation [14, 15]
Electrical circuits	Joule heating	Circuit equation [16]
Stochastic processes	Nonreciprocity of state transitions	Fokker-Planck equation [17, 18]
Soft matter and fluid	Nonlinear instability	Linearized hydrodynamics [19–21]
Nuclear reactions	Radiative decays	Projection methods [4–6]
Mesoscopic systems	Finite lifetimes of resonances	Scattering theory [22, 23]
Open quantum systems	Dissipation	Master equation [24, 25]
Quantum measurement	Measurement backaction	Quantum trajectory approach [26–31]

NON-NORMAL OPERATORS: NO SPECTRAL THEOREMS

Completeness more difficult to study

Gravity (Black hole theory)

Fields/Energy leaking away from the system:

at far distances and through the black-hole horizon

How to geometrically impose the notions of far distant (wave zone) and black-hole horizon?

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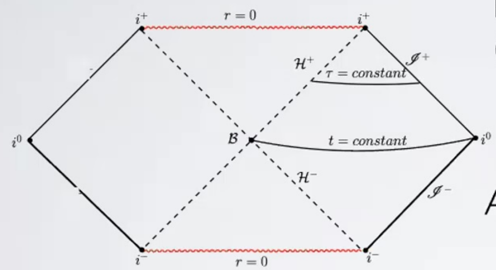
Open quantum systems	Dissipation	Master equation [24, 25]
Quantum measurement		Quantum trajectory approach [26–31]

HYPERBOLOIDAL SLICES

- **New coordinate system: spacelike hypersurfaces of constant time extending between black-hole horizon $\mathcal{H}^+$ (horizon-penetrating) and future null infinity $\mathscr{I}^+$**

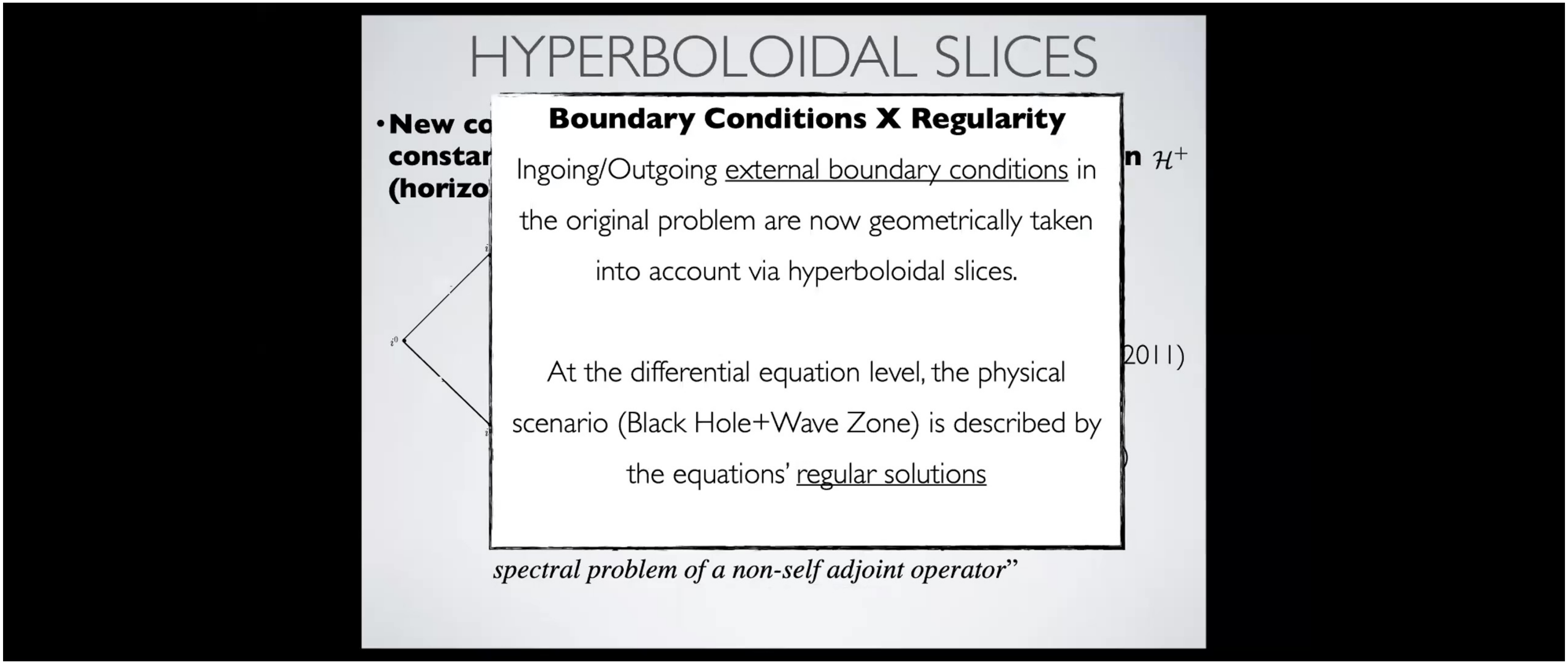


B. Schmidt considered this picture (private conversation)

"On relativistic stellar oscillations"
Gravity Research Foundation essay (1993)

A. Zenginoglu : PRD 83 127502 (2011)

"Black hole perturbation theory is typically studied on time surfaces that extend between the bifurcation sphere and spatial infinity. From a physical point of view, however, it may be favourable to employ time surfaces that extend between the future event horizon and future null infinity. *This framework resolves problems regarding the representation of quasinormal mode eigenfunctions* (…)"


HYPERBOLOIDAL SLICES
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(horizo
n $\mathcal{H}^+$
Boundary Conditions X Regularity
Ingoing/Outgoing external boundary conditions in the original problem are now geometrically taken into account via hyperboloidal slices.
At the differential equation level, the physical scenario (Black Hole+Wave Zone) is described by the equations' regular solutions
2011)
spectral problem of a non-self adjoint operator”

SPECTRAL INSTABILITY

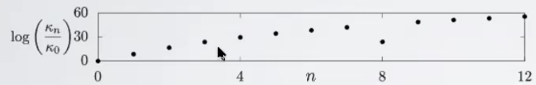
- Condition Number + Pseudospectral tools applied to Schwarzschild

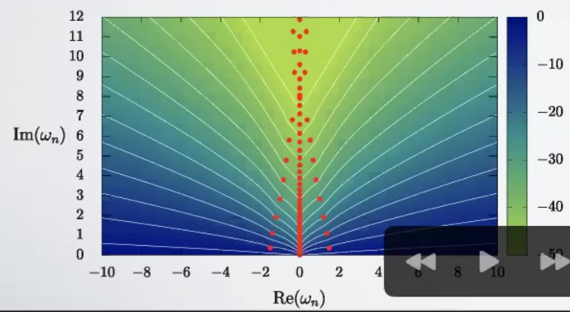
$$\partial_\tau u = iLu \qquad L\hat{u} = \omega\hat{u}$$

$$L = \frac{1}{i}\left(\begin{array}{c|c} 0 & 1 \\ \hline L_1 & L_2 \end{array}\right)$$

$$L_1 = (1+\sigma)^{-1}\left(\partial_\sigma\left[\sigma^2(1-\sigma)\partial_\sigma\right] - \underbrace{\left[\ell(\ell+1) + (1-s^2)\sigma\right]}_{q_\ell}\right)$$

$$L_2 = (1+\sigma)^{-1}\left((1-2\sigma^2)\partial_\sigma - 2\sigma\right) \qquad \sigma \in [0,1]$$





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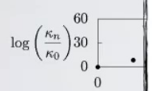
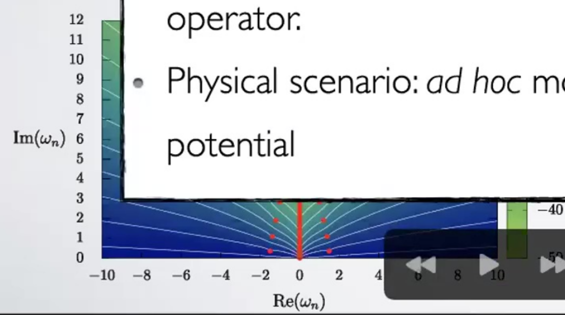
$$L\hat{u} = \omega\hat{u}$$

$$L_2 = (1+ \quad \ldots \quad - s^2)\sigma\left.\right]\Big)$$



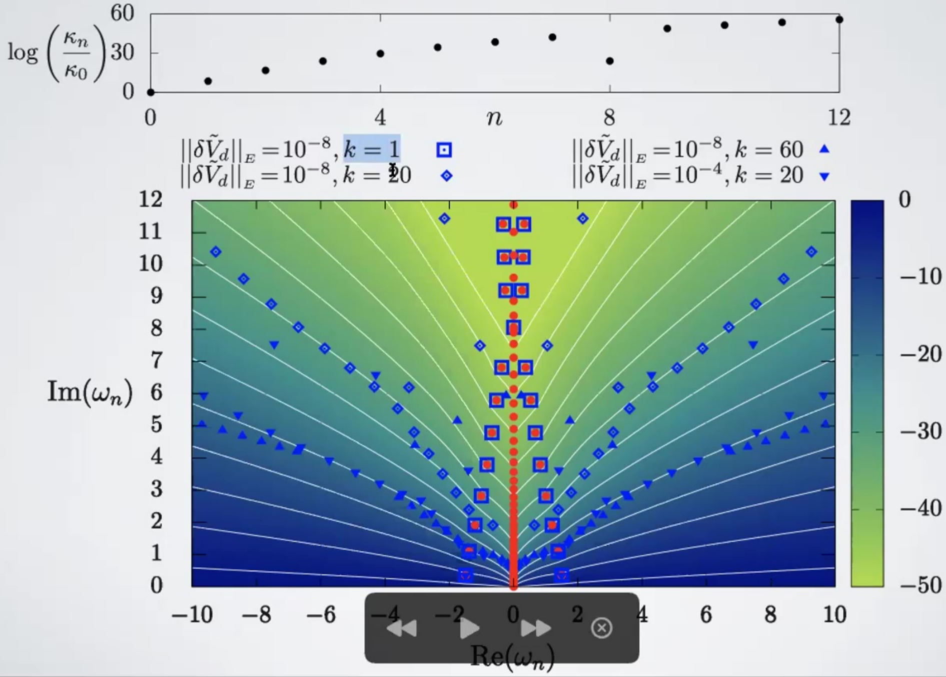
Partial Conclusions:

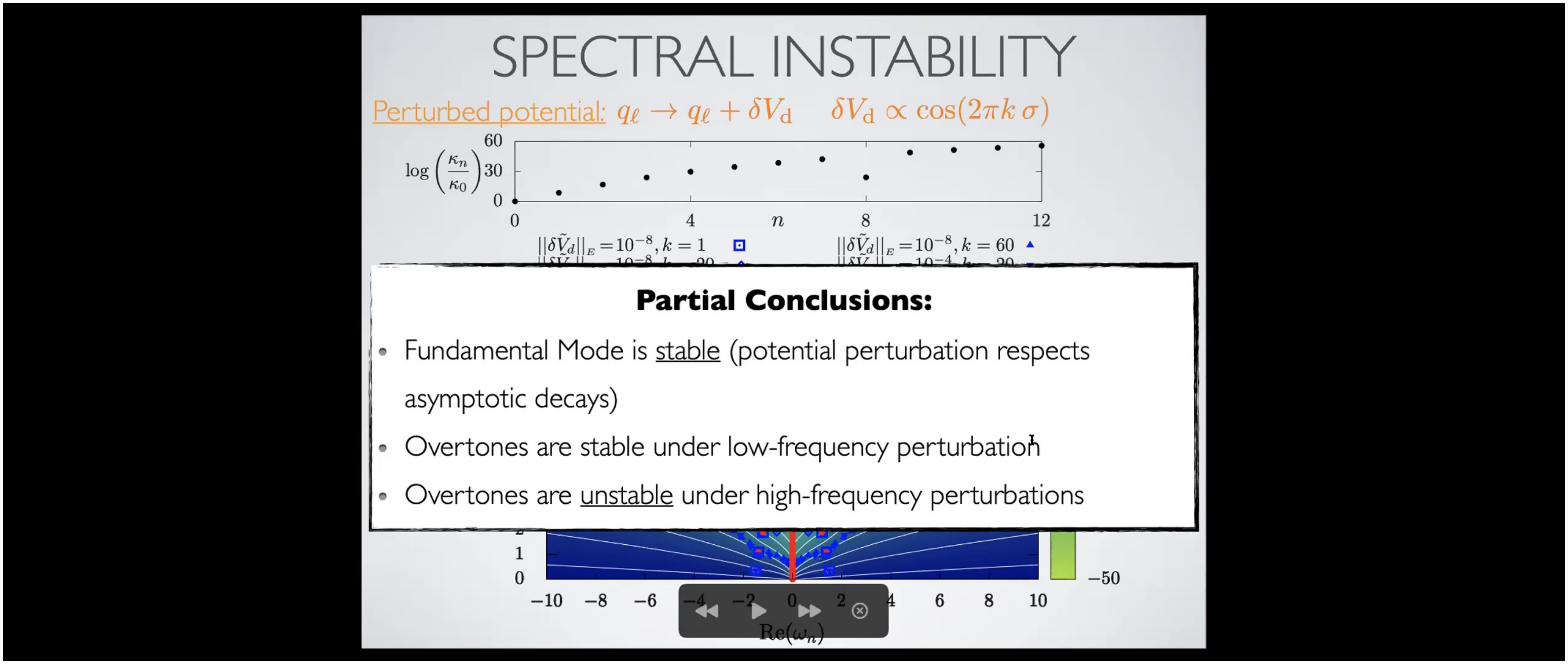
- Condition number and pseudospectra indicate potential instabilities under perturbation of the operator.
- Physical scenario: *ad hoc* modification of the potential



SPECTRAL INSTABILITY

Perturbed potential: $q_\ell \to q_\ell + \delta V_{\rm d}$ $\quad \delta V_{\rm d} \propto \cos(2\pi k\,\sigma)$

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$\log\left(\frac{\kappa_n}{\kappa_0}\right)$
n
$||\delta\tilde{V}_d||_E = 10^{-8}, k = 1$
$||\delta\tilde{V}_d||_E = 10^{-8}, k = 60$
Partial Conclusions:
Fundamental Mode is stable (potential perturbation respects asymptotic decays)
Overtones are stable under low-frequency perturbation
Overtones are unstable under high-frequency perturbations
$\mathrm{Re}(\omega_n)$

CONCLUSION

- **Hyperboloidal Framework for perturbation theory:**
  Field has reached a mature stage to offer a rich set of theoretical and numerical tools to expand studies in gravitational wave physics (mathematical relativity, theoretical physics, astrophysics and gravitational wave astronomy)

- **Fundamental aspects of black-hole physics:**
  Use of tools from theory of scattering resonances and non-self adjoint operators — pseudospectra — to assess structural aspects of QNM spectra.

- **Perspectives:**
  Astrophysics and Cosmology: Black hole spectroscopy; universality of compact object QNM
  Fundamental gravitational physics: (Sub)planckian-scale physics; QNMs and strong cosmic censorship; spacetime semiclassical limits
  Mathematical relativity: Formal statements of the numerical experiements